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# NONLINEARITY-DEFECT INTERACTION: SYMMETRY BREAKING BIFURCATION IN A NLS WITH A $\delta'$ IMPURITY

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We illustrate some new results and comment on perspectives of a recent research line, focused on the stability of stationary states of nonlinear NLS with point interactions. We describe in detail the case of a "$\delta'$" interaction, that provides a rich model endowed with a pitchfork bifurcation with symmetry breaking in the family of ground states. Finally, we give a direct proof of the stability of the ground states in the cases of a subcritical and critical (in the sense of the blow-up) nonlinearity power.

**Keywords:** nonlinear dynamics, quantum mechanics, solitons, symmetry breaking, pitchfork bifurcation.

## 1. A new range of application for point interactions

One of the most celebrated features of the point interactions (PI) lies in their capability of supplying exactly solvable models. For this reason, PI have been widely employed to construct toy models and for pedagogical purposes. Nonetheless, they prove useful also when used to model real physical systems: the more a physically relevant quantity (e.g. energy spectrum or time evolution) is explicitly known, the more information can be extracted. General theory and reference to physical applications with extended bibliography are in (see [10, 11]). In particular, PI fit well the needs of modeling the so-called defects, namely, small inhomogeneities in a medium where a wave propagates, under the hypothesis that the details of the internal structure of the inhomogeneity are not relevant, so that its action can be modeled as concentrated at a point. More precisely, the smallness of the inhomogeneity is to be evaluated with respect to the typical wavelength of the incoming waves, or equivalently, in the case of a quantum system, to the width of the wave function. In this paper we address the analysis of effects of the interaction between nonlinearity and point defects in the behaviour of solutions of nonlinear Schrödinger (NLS) equation. We prefer not to enter in a description of the vast field of application of the NLS equation, from the theory of integrable systems and inverse scattering to the propagation of amplitude envelope of waves. We cite just two relevant applications of the NLS as an effective model for real physical systems: dynamics of Bose-Einstein condensates (BEC) and laser beam propagation in nonlinear (Kerr) media. In both cases it is physically meaningful to consider the propagation of NLS waves in the presence of defects. In particular, the recent spectacular development of both theoretical research and experimental technology involving BEC (see [45] and references therein, and [13, 14, 16]) provides point interactions with a wide range of applications.

As widely known, in current experiments the formation of a BEC is induced in bounded region of spaces, usually delimited by magnetic and/or optical traps. In such situations, the condensate lies in a one-particle quantum state, whose corresponding wavefunction is characterized as the minimizer of the Gross-Pitaevskii functional. When the trap is removed, the wavefunction of the BEC spreads out according to the evolution prescribed by the cubic Schrödinger equation

$$i\partial_t\psi(t,x) \;=\; -\partial_x^2\psi(t,x)+\alpha|\psi(t,x)|^2\psi(t,x), \tag{1.1}$$

where we denoted by $\psi$ the wave function of the condensate, and the space variable $x$ belongs to $\mathbf{R}$, $\mathbf{R}^2$ or $\mathbf{R}^3$ according to the fact that we are modeling a *cigar-shaped* or a *disc-shaped* or a genuinely three-dimensional BEC. We recall that the nonlinearity carries the information that, even though its state is an actual one-particle state, the condensate consists of a large number of interacting particles (in the experimentally realized condensates, at least thousands); the fact that the nonlinearity is cubic means that the dynamical effects of the two-particle interactions overwhelm the effects of many-body collisions. The strength of the nonlinear term, given by the constant $\alpha$, is proportional to the scattering length of the two-body interaction between the particles. Here we do not summarize the progress in the derivation of (1.1) as an effective equation for a many-body quantum system. The interested reader is referred to [20–22] for the three-dimensional problem, to [34] for the two-dimensional case, and to [1, 2, 12] for the case of cigar-shaped condensates. In the following we focus on this last case, in which, on one hand, the nonlinearity is milder, while, on the other hand, the family of point interactions is richer.

A natural question in this context is the following: what happens when a wave (i.e. a condensate) is sent against a defect? One would guess (and it has been shown for some models, see e.g. [19, 29, 32, 44]) that the incoming wave splits into a reflected wave, a transmitted wave and a captured component. Similar results have been proven for propagation on graphs also (see [4]), in the case of a repulsive vertex, where no capture occurs. Indeed, it seems reasonable to conjecture that a capture can occur only if a nonlinear stationary state exists. Since equation (1.1) is dispersive, the presence of a nonlinear stationary state (or more than one) must be related to the defect. This is the reason why such possible stationary states are called *defect modes*. Even though at this stage it is an unproven fact, it is plausible to link the persistence of a captured wave with some sort of stability (in a sense to be made precise) of the defect mode. For this reason the interest in determining the stability of the defect modes lies not only in the problem itself, but extends to models of reality too.

As a short review on results on stability and instability of defect modes in the presence of a power nonlinearity $|\psi|^{2\mu}\psi$, we recall results proved in [26, 27, 40], where the effects of a $\delta$-like defect are analysed. The first cited work deals with an attractive defect, and shows that, for any frequency $\omega$ above the proper frequency of the unique bound state of the delta potential, there is a unique defect mode that oscillates at frequency $\omega$. It turns out that the wavefunction of such a defect mode is nothing but the nonlinear deformation of the linear bound state. The stability (more exactly, the *orbital* stability, see Definition 2.2) of such a mode depends on $\mu$ and $\omega$: if $\mu \leqslant 2$, then the defect mode is stable for any $\omega$; if $\mu > 2$, then it becomes unstable at high frequencies. References [26,40] extend the analysis to a repulsive delta-like defect. The situation becomes more involved in the case of a more singular defect, for instance, the so-called $\delta'$ defect. The following sections are devoted to this case. For more details see also the comprehensive review [7] and the forthcoming paper [6].

The established theoretical framework for the study of stability is provided by Weinstein and Grillakis-Shatah-Strauss theory (see [30, 31, 49, 50]) or, alternatively, by Lions concentration-compactness method (see [41, 42] and [17] for a review). The occurrence of bifurcation in the ground state has been investigated in [33] and more recently in [28, 35, 36, 43, 46].

## 2. Results

### 2.1. The $\delta'$ "potential"

The so-called $\delta'$-defect, with strength $-\gamma$, located (just to be definite) at zero, is defined imposing the boundary condition

$$\psi(0+) - \psi(0-) = -\gamma\psi'(0+) = -\gamma\psi'(0-) \tag{2.1}$$

to the solutions to (1.1) (see [9, 23]). The parameter $\gamma$ is real; when positive, the defect is called *attractive*, otherwise *repulsive*. More formally, one defines a *linear Hamiltonian operator* $H_\gamma$ as the operator that acts as $-\partial_x^2$ on the domain $D(H_\gamma)$ made of functions in $H^2(\mathbb{R}^-) \oplus H^2(\mathbb{R}^+)$ satisfying (2.1). Note that the only continuous elements of the domain of $H_\gamma$ have a vanishing derivative at the origin. The operator $H_\gamma$ is a self-adjoint operator with the following spectral features: singular continuous spectrum is empty, absolutely continuous spectrum is given by the positive halfline and point spectrum is empty in the repulsive case, and coincides with $\{-4/\gamma^2\}$ in the attractive case. In the last case the corresponding (non-normalized) eigenfunction is

$$\varphi_\gamma(x) \ = \ \epsilon(x)e^{-\frac{2}{\gamma}|x|},$$

where we denoted the sign function by $\epsilon$. Notice that $\varphi_\gamma$ is odd. The quadratic form $F_\gamma$ associated to $H_\gamma$ is defined on the domain $Q := H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-)$ (we stress that $Q$ is independent of $\gamma$) and reads

$$F_\gamma(\psi) \ = \ \|\psi'\|^2 - \gamma^{-1}|\psi(0+) - \psi(0-)|^2,$$

where we made the following abuse of notation

$$\|\psi'\|^2 := \lim_{\varepsilon\to 0+}\int_\varepsilon^{+\infty}|\psi'(x)|^2dx + \lim_{\varepsilon\to 0+}\int_{-\infty}^{-\varepsilon}|\psi'(x)|^2dx,$$

that will be extensively repeated.

At variance with the delta potential, the Schrödinger operator with a $\delta'$ interaction cannot be derived from a form sum, because the $\delta'$ is not small with respect to the laplacian. Nevertheless it can be obtained as the norm-resolvent limit of the sum of three $\delta$ potentials (see [18, 24]) with a fine tuned rescaling, defined as follows

$$[H_\gamma+\nu]^{-1} \ = \ \lim_{\varepsilon\to 0}\left[-\partial_x^2 - \left(\frac{1}{\gamma}+\frac{1}{2\varepsilon}\right)\delta(x-\varepsilon) - \left(\frac{1}{\varepsilon}+\frac{\gamma}{2\varepsilon^2}\right)\delta(x) - \left(\frac{1}{\gamma}+\frac{1}{2\varepsilon}\right)\delta(x-\varepsilon) + \nu\right]^{-1}$$

for any $-\nu$ in the resolvent set of $H_\gamma$ (see Figure 1).
Moreover, since any delta potential, in its turn, can be approximated by a strong limit of rescaled regular potentials, then it is possible to interpret a $\delta$-prime potential as the suitable limit of rescaled, well-behaved potentials. Let us remark that if $\psi$ belongs to the *operator* domain of $H_\gamma$, then the form associated to $H_\gamma$ has the expression

$$(\psi, H_\gamma\psi) \ = \ \|\psi'\|_2^2 - \gamma|\psi'(0+)|^2,$$

which explains the questionable name of $\delta'$.

### 2.2. Combining nonlinearity and defect

Once constructed the operator $H_\gamma$, the evolution in the presence of both a generic power nonlinearity and a defect is defined by

$$i\partial_t\psi(t,x) \ = \ H_\gamma\psi(t,x) + \alpha|\psi(t,x)|^{2\mu}\psi(t,x). \tag{2.2}$$

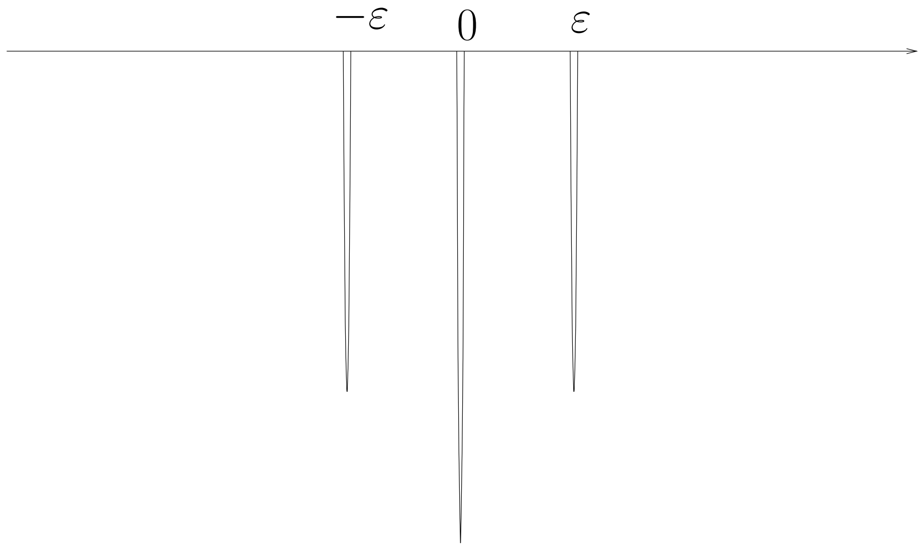


Fig. 1. A regular approximation for an attractive $\delta$-prime potential centred at zero. To obtain an approximation for a repulsive $\delta$-prime, one must reverse the central well.

For such equation it is possible to prove global well-posedness if $\mu < 2$ (see [3]), local well-posedness if $\mu \geqslant 2$ (see [6]), and to provide examples of blow-up for this last case (see [8]). However, until the solution exists, $L^2$-norm and energy

$$\mathcal{E}(\psi) = \frac{1}{2}\|\psi'\|^2 - \frac{1}{2\gamma}|\psi(0+) - \psi(0-)|^2 - \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2}$$

are conserved by time evolution.

Thanks to the existence of a conserved energy it is possible to introduce a notion of *nonlinear ground state*: intuitively, one would define it as a minimizer of the energy among the wavefunctions endowed with the same $L^2$-norm, as this is the definition that naturally extends the more familiar notion of linear ground state.

As in the linear case, it is meaningful to search for *stationary states* of (2.2), i.e. solutions of the form

$$\psi(x,t) = e^{i\omega t}\psi_\omega(x) \ . \tag{2.3}$$

The amplitudes $\psi_\omega$ are solutions of the stationary equation

$$H_\gamma\psi_\omega + \omega\psi_\omega - \lambda|\psi_\omega|^{2\mu}\psi_\omega \ = \ 0. \tag{2.4}$$

This leads to the introduction of the so-called *action functional*

$$S_\omega(\psi) \ = \ \mathcal{E}(\psi) + \frac{\omega}{2}\|\psi\|^2, \tag{2.5}$$

defined on the energy domain $Q$. It is immediate indeed that Euler-Lagrange equations for $S_\omega$ are given just by (2.4). Note that the action (and the energy as well) is not bounded from below on $Q$. To overcome this problem, a *ground state* $\psi_\omega$ is usually defined as a minimizer of $S_\omega$ constrained on the *Nehari manifold*

$$I_\omega(\psi) = S'_\omega(\psi)\psi = (\psi, H_\gamma\psi - \lambda|\psi|^{2\mu}\psi + \omega\psi) \ = \ 0.$$

The above set is a codimension one manifold that obviously contains all stationary points of $S_\omega$, and it tuns out that on it the action is bounded from below.

The relation between the constrained variational problem for $\mathcal{E}$ and $S_\omega$ is a byproduct of the Grillakis-Shatah-Strauss theory on stability of stationary states (see [30], [31]) applied to minimizers of $S_\omega$: a minimizer $\psi_\omega$ of the action on the Nehari manifold is a local minimizer of the energy among the function with the same $L^2$-norm $\|\psi_\omega\|$ *if and only if it is stable* (in the sense of Definition (2.2)).

The following preliminary result is obtained through variational techniques (we remove the subscript $\omega$ from $\psi_\omega$ when not needed to avoid ambiguity):

**Theorem 2.1.** *For any $\omega > \frac{4}{\gamma^2}$ there exists at least one minimizer of $S_\omega$ among all functions on the Nehari manifold. Furthermore, the minimizer solves the stationary Schrödinger equation with defect:*

$$H_\gamma \psi + \omega\psi - \lambda|\psi|^{2\mu}\psi \;=\; 0. \tag{2.6}$$

*For $\omega \leqslant \frac{4}{\gamma^2}$, equation* (2.6) *admits no solutions in $D(H_\gamma)$.*

The line of the proof is standard, except that: first, the functional space of reference $Q$ is larger than $H^1(\mathbb{R})$; second, the problem is one-dimensional, so that one must cope with a lack of compactness when passing from weak convergence in $Q$ to strong convergence in $L^p$; third, the boundary condition to be reconstructed is non standard. A complete proof is in [6].

An important point about Theorem 2.1 is that, in order to find the ground states, it suffices to determine which one among the solutions of (2.6) has least action. This can be made directly, as the solutions to equation (2.6) can be explicitly found. It has been said, however, that the variational analysis provides information beyond the one obtainable through the direct ODE approach; for example, the minimum is constrained to a finite codimension (one in this case) manifold, an information which is important for stability issues.

### 2.3. Symmetry breaking

Equation (2.6) can be rephrased as follows:

$$-\partial_x^2\psi + \omega\psi - \lambda|\psi|^{2\mu}\psi \;=\; 0, \tag{2.7}$$

with $\psi \in H^2(\mathbb{R}^+) \oplus H^2(\mathbb{R}^-)$ satisfying the boundary condition (2.1).
The only solutions to (2.7) that vanish at infinity are constructed by gluing together two pieces of a solitary wave for the NLS, namely

$$\psi_{\omega,\pm}^{x_1,x_2}(x) = \begin{cases} \pm\lambda^{-\frac{1}{2\mu}}(\mu+1)^{\frac{1}{2\mu}}\omega^{\frac{1}{2\mu}}\cosh^{-\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_1)], & x<0 \\ \lambda^{-\frac{1}{2\mu}}(\mu+1)^{\frac{1}{2\mu}}\omega^{\frac{1}{2\mu}}\cosh^{-\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_2)], & x>0 \end{cases}$$

where the parameters $x_1$ and $x_2$ are to be adjusted so that (2.1) is satisfied. Now, it is immediately seen by (2.5) that due to contribution of the point interaction energy, one has

$$S_\omega(\psi_{\omega,-}^{x_1,x_2}) \;<\; S_\omega(\psi_{\omega,+}^{x_1,x_2})$$

so we can restrict the search for minimizers to the functions $\psi_{\omega,-}^{x_1,x_2}$, i.e. solutions of (2.6) that *change sign* at the origin (and only there).

For any such function, the boundary condition (2.1) translates into the system

$$\begin{cases} t_1^{2\mu} - t_1^{2\mu+2} = t_2^{2\mu} - t_2^{2\mu+2} \\ t_1^{-1} + t_2^{-1} = \gamma\sqrt{\omega} \end{cases}, \quad 0 \leqslant t_i = |\tanh(\mu\sqrt{\omega}x_i)| \leqslant 1, \tag{2.8}$$

whose solutions can be depicted as the intersection of the full and the dashed lines in Figure 2. One immediately finds that for $\frac{4}{\gamma^2} < \omega \leqslant \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$ the unique solution is given by $t_1 = t_2 = \frac{2}{\gamma\sqrt{\omega}}$,

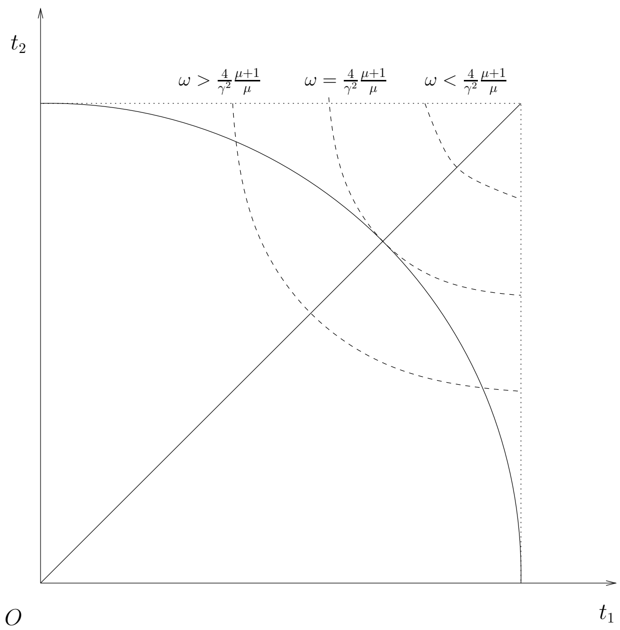


Fig. 2. The full lines represent the solutions to the first equation in (2.8): they consist of the line $0 \leqslant t_1 = t_2 \leqslant 1$, and of a curve, that is concave if $\mu$ is not too small. The dashed lines represent the solutions to the second equation of (2.8): they consist of a family of hyperbola parametrized by $\omega$

that corresponds to an antisymmetric stationary state $\psi_\omega^{y,-y}$, where

$$y = x_1 = -x_2 = \frac{1}{2\mu\sqrt{\omega}} \log \frac{\gamma\sqrt{\omega}+2}{\gamma\sqrt{\omega}-2}.$$

At $\omega = \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$ two new solutions arise, giving birth to two new branches of stationary states that persist for $\omega > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$; they correspond to the couple of asymmetric stationary states $\psi_\omega^{y_1,-y_2}$, $\psi_\omega^{y_2,-y_1}$, with both $y_1$ and $y_2$ positive but, except in the cubic case $\mu = 1$, not in explicit form. A direct computation yields, for these values of $\omega$,

$$S_\omega(\psi_\omega^{y_1,-y_2}) \;=\; S_\omega(\psi_\omega^{y_2,-y_1}) \;<\; S_\omega(\psi_\omega^{y,-y}) \,.$$

We conclude that with the growth of the frequency $\omega$ there exist *two* branches of asymmetric ground states which bifurcate from the branch of (anti)symmetric ones. We are then in the presence of a spontaneous symmetry breaking of the set of ground states.

### 2.4. Stability: a pitchfork bifurcation

The study of the stability for such a system can be made by applying the Grillakis-Shatah-Strauss theory (see [30, 31]). This theory provides sufficient conditions for the *orbital stability* of stationary states, which is stability "up to the symmetries". Roughly speaking, the notion of orbital stability coincides with the ordinary Ljapunov stability *for orbits instead of states*, where orbits are to be understood with respect to a symmetry group. In our case the symmetry group is $U(1)$, corresponding to the well known phase invariance of the NLS, which persists in the presence of point perturbation too. So, a stationary state $\psi_\omega$ is said to be *orbitally stable* if at any time a solution to (2.2) remains arbitrarily close *to the orbit* $\{e^{i\theta}\psi_\omega,\ \theta \in [0, 2\pi)\}$, provided that it started sufficiently close to it. More rigorously,

**Definition 2.2.** A stationary state $\psi_\omega$ is called *orbitally stable* if for any $\varepsilon > 0$ there exists a $\sigma > 0$ s.t.

$$\inf_{\theta\in[0,2\pi)} \|\psi_0 - e^{i\theta}\psi_\omega\|_Q \leqslant \sigma \Rightarrow \sup_{t>0} \inf_{\theta\in[0,2\pi)} \|\psi_t - e^{i\theta}\psi_\omega\|_Q \leqslant \varepsilon,$$

where $\psi_t$ is the solution corresponding to the initial condition $\psi_0$.

A stationary state is called *orbitally unstable* if it is not orbitally stable.

The Grillakis-Shatah-Strauss theory (see [30, 31]) carries out a deep investigation of the orbital stability of stationary states of (infinite dimensional) hamiltonian systems with symmetries, generalizing previous work by the same authors and independently by Michael Weinstein (see [48–50]).

They succeeded in giving sufficient conditions for stability and instability by studying second-order approximation of the action (linearization) around a stationary state, and carefully controlling the nonlinear remainders exploiting symmetries and conservation laws. In the present situation, as it is well known, one gets a hamiltonian system from NLS equation passing to real variables $(\eta,\rho) = (\mathrm{Re}\psi, \mathrm{Im}\psi)$. We confine ourself to a brief operative summary of the method, and so we omit the (however important) connection with hamiltonian systems referring to the original literature for details.

Neglecting higher order terms, one has for the action expanded around the stationary state $\psi_\omega$ (we omit other superscripts for simplicity)

$$S_\omega(\psi_\omega + \eta + i\rho) \cong S_\omega(\psi_\omega) + \frac{1}{2}\left(S''(\psi_\omega)\begin{pmatrix}\eta\\ \rho\end{pmatrix}, \begin{pmatrix}\eta\\ \rho\end{pmatrix}\right).$$

The Hessian operator $S''_\omega(\psi_\omega)$ can be represented in matrix form as (we implicitly introduced the representation of a function $\eta + i\rho$ as the real vector function $(\eta,\rho)$)

$$S''_\omega(\psi_\omega) \;:=\; \begin{pmatrix} L_1 & 0\\ 0 & L_2\end{pmatrix},$$

where $L_1$ and $L_2$ are two selfadjoint operators with $D(L_1) = D(L_2) = D(H_\gamma)$ given by

$$\begin{aligned} L_1 &= H_\gamma + \omega - \lambda(2\mu+1)|\psi_\omega|^{2\mu}\\ L_2 &= H_\gamma + \omega - \lambda|\psi_\omega|^{2\mu}. \end{aligned}$$

Now, were $S''_\omega(\psi_\omega)$ a positive operator, the (linear) stability of $\psi_\omega$ would be immediately established, as the situation would be analogous to what happens for a classical particle in a potential well. Unfortunately, this cannot be the case. First of all, the operator $S''_\omega(\psi_\omega)$ is endowed with a non trivial kernel that consists of the linear span of $(0,\psi_\omega)$, due to the symmetry. Second, recall that every ground state $\psi_\omega$ is a minimizer only on the constraint provided by the Nehari manifold, which has codimension one. On the space orthogonal to the Nehari manifold, $S''_\omega(\psi_\omega)$ is surely negative, as

$$(\psi_\omega, S''_\omega\psi_\omega) \;<\; 0.$$

It follows that there exists a cone on which $S''_\omega(\psi_\omega)$ is actually negative.

Nevertheless, it is possible that the dynamical constraints given by the conservation laws prevent the wave function from further evolving far inside that cone, finally forcing the solution to remain close to the orbit of the ground state. The Grillakis-Shatah-Strauss theory establishes that this is the case if a certain number of conditions are satisfied. In its easiest version, such a set of conditions can be collected as follows

$i)$ *Spectral conditions:*

(1) $\mathrm{Ker}L_2 \;=\; \mathrm{Span}\{\psi_\omega\}$,

(2) $L_2 \geqslant 0$,

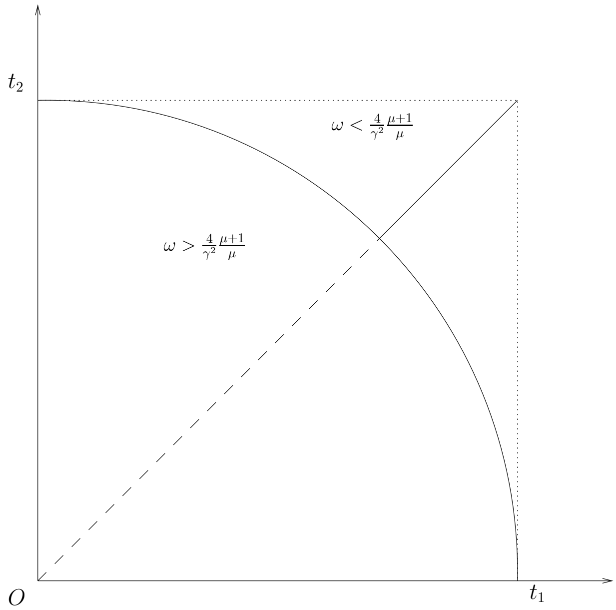


Fig. 3. Bifurcation diagram for $\mu \leqslant 2$. The full line denotes stable stationary states, while the dashed line represents unstable stationary states. Notice that ground states are always stable

(3) $\mathrm{Ker} L_1 = \{0\}$,

(4) $L_1$ has exactly *one* negative eigenvalue.

$ii)$ *Vakhitov-Kolokolov's criterion* (see [47]):

$$\frac{d\|\psi_\omega\|_2}{d\omega} > 0,$$

that, since $\frac{dS_\omega(\psi_\omega)}{d\omega} = \frac{1}{2}\|\psi_\omega\|_2^2$, is equivalent to

$$\frac{d^2 S_\omega(\psi_\omega)}{d\omega^2} > 0. \tag{2.9}$$

In the case of interest, conditions $i)$ and $ii)$ are verified except for the the stationary states in the branch $\psi_\omega^{y,-y}$ with $\omega > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, where a more sophisticated version of conditions $i)$ and $ii)$ is needed, again provided by the Grillakis-Shatah-Strauss theory (see [31]).

The results on stability can be summed up as follows.

**Theorem 2.3.** *For any* $\mu > 0$

(1) *If* $\omega < \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, *then the unique (up to a phase) ground state* $\psi_\omega^{y,-y}$ *is orbitally stable.*

(2) *If* $\omega > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, *then the stationary state* $\psi_\omega^{y,-y}$ *is orbitally unstable.*

*For* $0 \leqslant \mu \leqslant 2$, $\omega > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, *the two ground states* $\psi_\omega^{y_1,-y_2}$, $\psi_\omega^{y_2,-y_1}$ *are orbitally stable.*

*For* $2 < \mu < \mu^* < 2.5$ *there exist* $\omega_1 > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$ *and* $\omega_2 > \omega_1$, *such that, if* $\frac{4}{\gamma^2}\frac{\mu+1}{\mu} < \omega < \omega_1$, *then* $\psi_\omega^{y_1,-y_2}$ *and* $\psi_\omega^{y_2,-y_1}$ *are orbitally stable; if* $\omega > \omega_2$, *then* $\psi_\omega^{y_1,-y_2}$ *and* $\psi_\omega^{y_2,-y_1}$ *are orbitally unstable.*

*For* $\mu > \mu^*$, *there exist* $\omega_1 > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, $\omega_2 > \omega_1$, *such that, if* $\frac{4}{\gamma^2}\frac{\mu+1}{\mu} < \omega < \omega_1$ *or* $\omega > \omega_2$, *then* $\psi_\omega^{y_1,-y_2}$ *and* $\psi_\omega^{y_2,-y_1}$ *are orbitally unstable.*

The bifurcation diagrams for the system are portrayed in Figures 3 and 4.

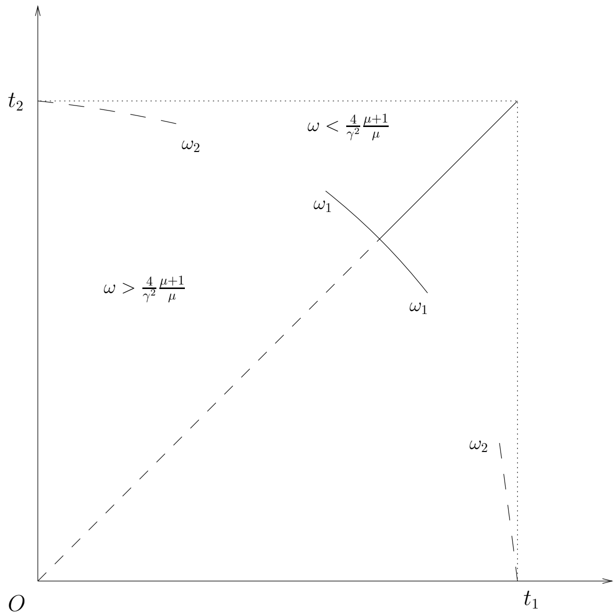


Fig. 4. Bifurcation diagram for $2 < \mu < \mu^*$. We have no results for the interval $(\omega_1, \omega_2)$, but we conjecture that it is always possible to choose $\omega_1 = \omega_2$

## 3. Proof of stability

The content of this section is technical. Here we we give a proof of the stability of all ground states in the case $\mu \leqslant 2$. Under such a restriction, every ground state satisfies the Vakhitov-Kolokolov's criterion. The proof we present here differs from the one given in [6], as it does not use the Grillakis-Shatah-Strauss theory and so it does not refer to linearization. We decided to include in this report such a technical part in order to convey some information on the method of proofs and on the techniques employed. An analogous analysis is given for the case of a NLS with $\delta$ interaction in [27], and both are inspired by [25].
In order to proceed we need some preliminary definitions and results.

First, the definition of orbital stability can be reformulated using the notion of *orbital neighbourhood*.

**Definition 3.1.** The set

$$U_\eta(\phi) := \{\psi \in Q, \text{ s.t. } \inf_{\theta \in [0,2\pi)} \|\psi - e^{i\theta}\phi\|_Q \leqslant \eta\}$$

is called the *orbital neighbourhood with radius $\eta$ of the function $\phi$.*

It is convenient to introduce a function that associates to any frequency $\omega > \frac{4}{\gamma^2}$ the value of the minimum attained by $S_\omega$ evaluated on functions in the Nehari manifold corresponding to that frequency. Namely,

$$d \ : \ \left(\frac{4}{\gamma^2}, +\infty\right) \to \mathbb{R}$$
$$\omega \longmapsto d(\omega) := \min\{S_\omega(\phi), \phi \in Q, I_\omega(\phi) = 0\}.$$

It is then important to stress other points that we did not mention explicitly so far.

**Remark 3.2.**

(1) In the energy space $Q$ the following norm is defined:

$$\|\phi\|_Q^2 := \|\phi\|_2^2 + \lim_{\varepsilon\to 0+}\int_{\varepsilon}^{+\infty} |\phi'|^2\,dx + \lim_{\varepsilon\to 0+}\int_{-\infty}^{-\varepsilon} |\phi'|^2\,dx.$$

(2) For any $\theta \in [0, 2\pi)$ and any $\phi \in Q$, one has $S_\omega(e^{i\theta}\phi) = S_\omega(\phi)$. As a consequence, if $\psi_\omega$ is a ground state, then all the functions $e^{i\theta}\psi_\omega$ in its orbit are ground states too. In the proof of Theorem (3.4) we will make the phase explicit by denoting

$$\psi_\omega^{x_1,x_2,\theta} := e^{i\theta}\psi_\omega^{x_1,x_2,0}.$$

The result we need to go through the proof are summarized in the following Proposition. Their proof is contained in [6].

**Proposition 3.3.**

(1) For any function $\phi$ in the Nehari manifold one has $S_\omega(\phi) = \widetilde{S}(\phi)$, where $\widetilde{S}$ is the functional defined by

$$\widetilde{S}(\phi) := \frac{\lambda\mu}{2\mu+2}\|\phi\|_{2\mu+2}^{2\mu+2}.$$

(2) Any minimizer $\psi_\omega$ of the functional $S_\omega$ on the Nehari manifold minimizes also the functional $\widetilde{S}$ on the region $I_\omega \leqslant 0$.

(3) Following Fibich and Wang (see [25]) we recall that the map

$$\widehat{\omega} : Q \longrightarrow \mathbb{R}, \qquad \phi \mapsto d^{-1}\left(\frac{\lambda}{2}\frac{\mu}{\mu+1}\|\phi\|_{2\mu+2}^{2\mu+2}\right)$$

is well-defined. Notice that $\widehat{\omega}$ maps a function $\phi$ into the frequency of a ground state having the same $L^{2\mu+2}$-norm as $\phi$. Such a ground state may not be unique, but the $L^{2\mu+2}$-norm always is.

(4) If $\psi_\omega$ minimizes $S_\omega$ on the Nehari manifold $I_\omega = 0$, then $\psi_\omega$ minimizes $S_\omega$ on the set $\{\phi \in Q,\ \|\phi\|_{2\mu+2} = \|\psi_\omega\|_{2\mu+2}\}$.

(5) For any $\omega > 0$, the function $\chi_{[0,+\infty)}\psi_\omega^{0,0}$ minimizes the functional

$$S_\omega^0(\phi) := \frac{1}{2}\|\phi'\|_2^2 + \frac{\omega}{2}\|\phi\|_2^2 - \frac{\lambda}{2\mu+2}\|\phi\|_{2\mu+2}^{2\mu+2}$$

among the functions in $Q$ that satisfy

$$I_\omega^0(\phi) := \|\phi'\|_2^2 + \omega\|\phi\|_2^2 - \lambda\|\phi\|_{2\mu+2}^{2\mu+2} = 0.$$

(6) If $\mu \leqslant 2$, then any ground state satisfies the Vakhitov-Kolokolov's condition (2.9).

Now we can prove the

**Theorem 3.4.** *If $1 \leqslant \mu \leqslant 2$, then any ground state is stable.*

*Proof.* We specialize to the case with $\omega > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, namely, beyond the frequency of bifurcation. In fact, for $\omega < \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$, this proof can be easily adapted and one recovers essentially the argument given in [27].

Fix $\omega_0 > \frac{4}{\gamma^2}\frac{\mu+1}{\mu}$ and suppose that the stationary solution $e^{i\omega_0 t}\psi_\omega^{y_1,-y_2,0}$ is orbitally unstable. This means that there exists $\varepsilon_0 > 0$ and a sequence $\varphi_k \in U_{\frac{1}{k}}(\psi_\omega^{y_1,-y_2,0})$ such that

$$\sup_{t\geqslant 0}\inf_{\theta\in[0,2\pi)}\|\varphi_k(t) - \psi_{\omega_0}^{y_1,-y_2,\theta}\|_Q \geqslant \varepsilon_0,$$

where $\varphi_k(t)$ is the solution to equation (2.2) with initial data $\varphi_k$.

With no loss of generality, we assume

$$\varepsilon_0 \leqslant \inf_{\theta\in[0,2\pi)} \|\psi_{\omega_0}^{y_1,-y_2,0} - \psi_{\omega_0}^{y_2,-y_1,\theta}\|_Q = \|\psi_{\omega_0}^{y_1,-y_2,0} - \psi_{\omega_0}^{y_2,-y_1,0}\|_Q. \tag{3.1}$$

Let $t_k$ be the smallest positive time for which

$$\inf_{\theta\in[0,2\pi)} \|\varphi_k(t_k) - \psi_{\omega_0}^{y_1,-y_2,\theta}\|_Q = \frac{\varepsilon_0}{2}, \tag{3.2}$$

and let us use the notation $\xi_k = \varphi_k(t_k)$. By conservation laws,

$$\begin{aligned} S_{\omega_0}(\xi_k) &= \mathcal{E}(\xi_k) + \frac{\omega_0}{2}\|\xi_k\|_2^2 = \mathcal{E}(\varphi_k) + \frac{\omega_0}{2}\|\varphi_k\|_2^2 \\ &\longrightarrow \mathcal{E}(\psi_{\omega_0}^{y_1,-y_2,0}) + \frac{\omega_0}{2}\|\psi_{\omega_0}^{y_1,-y_2,0}\|_2^2 = S_{\omega_0}(\psi_{\omega_0}^{y_1,-y_2,0}) = d(\omega_0). \end{aligned} \tag{3.3}$$

where we used the fact that, by construction, the sequence $\varphi_k$ converges to $\psi_{\omega_0}^{y_1,-y_2,0}$ strongly in $Q$, that implies the convergence of the energy and of the $L^2$-norm.

Let us denote $\omega_k = \widehat{\omega}(\xi_k)$. We recall the following result from [25], used in [27] also:

$$S_{\omega_k}(\xi_k) - S_{\omega_k}(\phi_0) \geqslant \frac{1}{4} d''(\omega_0)(\omega_k - \omega_0)^2 \tag{3.4}$$

where we denoted $\omega_k = \omega(\xi_k)$. The fact that the Vakhitov-Kolokolov's condition is satisfied (see [6] ), together with (3.4) and (3.3), implies $\omega_k \to \omega_0$, and therefore, by the definition of the function $\widehat{\omega}$, we have

$$\|\xi_k\|_{2\mu+2} = \left[\frac{2\mu+2}{\lambda\mu} S_{\omega_k}(\psi_{\omega_k})\right]^{\frac{1}{2\mu+2}} \longrightarrow \left[\frac{2\mu+2}{\lambda\mu} S_{\omega_0}(\psi_{\omega_0})\right]^{\frac{1}{2\mu+2}} = \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}. \tag{3.5}$$

We define the sequence $\zeta_k := \frac{\|\psi_{\omega_0}^{y_2,y_1,0}\|_{\mu+2}}{\|\xi_k\|_{\mu+2}}\xi_k$. By (3.5),

$$\|\zeta_k - \xi_k\|_Q = \left|\frac{\|\psi_{\omega_0}^{y_2,y_1,0}\|_{\mu+2}}{\|\xi_k\|_{\mu+2}} - 1\right| \|\xi_k\|_Q \longrightarrow 0. \tag{3.6}$$

As a consequence, $S_{\omega_0}(\zeta_k) - S_{\omega_0}(\xi_k) \to 0$, so $S_{\omega_0}(\zeta_k) \to S_{\omega_0}(\psi_{\omega_0})$. For this reason, and as $\|\zeta_k\|_{2\mu+2} = \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}$, point (4) in Proposition 3.3 implies that $\{\zeta_k\}$ is a minimizing sequence for the problem

$$\min\{S_{\omega_0}(\psi),\ \psi \in Q\backslash\{0\},\ \|\psi\|_{2\mu+2} = \|\psi_{\omega_0}^{y_2,-y_1,0}\|_{2\mu+2}\}.$$

By Banach-Alaoglu theorem there exists a subsequence, whose elements we denote by $\zeta_k$ too, that converges weakly in $Q$ and therefore in $L^{2\mu+2}$. Let us call $\zeta_\infty$ its weak limit.

First, notice that $\zeta_\infty \neq 0$. Indeed, were it zero, then weak convergence in $Q$ would imply $\zeta_\infty(0\pm) \to 0$, and therefore $S_{\omega_0}(\zeta_k) - S_{\omega_0}^0(\zeta_k) \to 0$, so

$$\lim_{k\to\infty} S_{\omega_0}^0(\zeta_k) = \lim_{k\to\infty} S_{\omega_0}(\zeta_k) = \frac{\lambda}{2}\frac{\mu}{\mu+1}\|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}^{2\mu+2}.$$

Then, employing the fact that $\phi_0 := \chi_{[0,+\infty)}\psi_{\omega_0}^{0,0}(0) \neq 0$, points (4) and (5) in Proposition 3.3 yield

$$\frac{\lambda}{2}\frac{\mu}{\mu+1}\|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}^{2\mu+2} = \lim S_{\omega_0}^0(\zeta_k) \geqslant S_{\omega_0}^0(\chi_+\phi_0) > S_{\omega_0}(\chi_+\phi_0) \geqslant \frac{\lambda}{2}\frac{\mu}{\mu+1}\|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}^{2\mu+2},$$

which is absurd. So it must be $\zeta_\infty \neq 0$.

We claim that $I_{\omega_0}(\zeta_\infty) = 0$. We proceed by contradiction.
Suppose indeed that $I_\omega(\zeta_\infty) < 0$. Then, by point (2) in Remark 3.2 and points (1) and (2) in Proposition 3.3 we know that the minimizers of the functional $\widetilde{S}$ on the region $I_{\omega_0} \leqslant 0$ are given

by $\psi_{\omega_0}^{y_1,-y_2,\theta}$ and $\psi_{\omega_0}^{y_2,-y_1,\theta}$, for all $\theta \in [0, 2\pi)$. Furthermore, all such functions lie on the set $I_{\omega_0} = 0$. As a consequence, recalling the definition of the functional $\widetilde{S}$, one obtains

$$\|\zeta_\infty\|_{2\mu+2} = \left[\frac{2(\mu+1)}{\lambda\mu}\widetilde{S}(\zeta_\infty)\right]^{\frac{1}{2\mu+2}} > \left[\frac{2(\mu+1)}{\lambda\mu}\widetilde{S}(\psi_{\omega_0}^{y_1,-y_2,0})\right]^{\frac{1}{2\mu+2}} = \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}.$$

But this is not possible, as $\zeta_\infty$ is the weak limit of functions having the same $L^{2\mu+2}$-norm as $\psi_{\omega_0}^{y_1,-y_2,0}$.

On the other hand, suppose that $I_{\omega_0}(\zeta_\infty) > 0$. By (3.3) and (3.5)

$$\lim_{k\to\infty} I_{\omega_0}(\xi_k) = 2\lim_{k\to\infty} S_{\omega_0}(\xi_k) - \frac{\lambda\mu}{\mu+1}\lim_{k\to\infty}\|\xi_k\|_{2\mu+2}^{2\mu+2}$$
$$= 2S_{\omega_0}(\psi_{\omega_0}^{y_1,-y_2,0}) - \frac{\lambda\mu}{\mu+1}\|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}^{2\mu+2} = I_{\omega_0}(\psi_{\omega_0}^{y_1,-y_2,0}) = 0.$$

Therefore, by (3.6),

$$\lim_{k\to\infty} I_{\omega_0}(\zeta_k) = 0.$$

From the following inequality (see [15])

$$\|u_n\|_p^p - \|u_n - u_\infty\|_p^p - \|u_\infty\|_p^p \longrightarrow 0, \quad \forall\, 1 < p < \infty. \tag{3.7}$$

one easily has

$$I_{\omega_0}(\zeta_k - \zeta_\infty) \longrightarrow -I_{\omega_0}(\zeta_\infty) < 0.$$

As a consequence, eventually in $k$ we obtain $I_{\omega_0}(\zeta_k - \zeta) < 0$ and then, using point (2) in Proposition 3.3

$$\|\zeta_k - \zeta_\infty\|_{2\mu+2} > \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}. \tag{3.8}$$

But from (3.7), and knowing that $\zeta_\infty \neq 0$, we have that the following inequality holds eventually in $k$

$$\|\zeta_k - \zeta_\infty\|_{2\mu+2} \leqslant \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2},$$

that contradicts (3.8).

We conclude that $I_{\omega_0}(\zeta_\infty)$ cannot be strictly positive and, as we already proved that it cannot be negative, it must vanish.

As a consequence, from point (2) in Proposition 3.3 again, we get $\|\zeta_\infty\|_{2\mu+2} \geqslant \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}$. But, since $\zeta_\infty$ is a weak limit, it must be

$$\|\zeta_\infty\|_{2\mu+2} = \|\psi_{\omega_0}^{y_1,-y_2,0}\|_{2\mu+2}.$$

This fact has the following relevant consequences:

- Owing to (3.7), the sequence $\{\zeta_n\}$ converges strongly to $\zeta_\infty$ in the topology of $L^{2\mu+2}$.
- The sequence $\{\zeta_k\}$ converges to $\zeta_\infty$ in the strong topology of $Q$. Indeed, by the convergence of $S_{\omega_0}(\zeta_k)$ to $S_{\omega_0}(\zeta_\infty)$, the weak convergence in $Q$, and the strong convergence of $\{\zeta_k\}$ in $L^{2\mu+2}$, we have

  $$\|\zeta_k'\|^2 + \omega_0\|\zeta_k\|^2 \longrightarrow \|\zeta_\infty'\|^2 + \omega_0\|\zeta_\infty\|^2. \tag{3.9}$$

  So the convergence is strong in the space $Q$ endowed with the norm given by (3.9), that is equivalent to the usual $Q$-norm.
- The sequence $\{\xi_k\}$ also converges to $\zeta_\infty$ in the strong topology of $Q$. Indeed, applying (3.6), we have

  $$\|\xi_k - \zeta_\infty\|_Q \leqslant \|\xi_k - \zeta_k\|_Q + \|\zeta_k - \zeta_\infty\|_Q \longrightarrow 0. \tag{3.10}$$
- The function $\zeta_\infty$ minimizes $S_{\omega_0}$ with the constraint $I_{\omega_0} = 0$, so, either $\zeta_\infty = \psi_{\omega_0}^{y_1,-y_2,\theta}$ or $\zeta_\infty = \psi_{\omega_0}^{y_2,-y_1,\theta}$ for some value of $\theta$ in $[0, 2\pi)$.

Let us suppose that $\zeta_\infty = \psi_{\omega_0}^{y_1,-y_2,\theta}$, for a certain value of $\theta$. By (3.10) we obtain $\xi_k \to \psi_{\omega_0}^{y_1,-y_2,\theta}$ strongly in $Q$, that contradicts inequality (3.2), and thus the assumption of the orbital instability of the stationary state $\psi_{\omega_0}^{y_1,-y_2,0}$ proves false.

On the other hand, consider the case with $\zeta_\infty = \psi_{\omega_0}^{y_2,-y_1,\theta}$ for some value of $\theta$. By (3.2) there exists a sequence $\theta_k$ such that

$$\|\xi_k - \psi_{\omega_0}^{y_1,-y_2,\theta_k}\|_Q \leqslant \frac{2}{3}\varepsilon_0. \tag{3.11}$$

Using elementary triangular identity, (3.1) and (3.11), we obtain, for any $\theta \in [0, 2\pi)$,

$$\|\xi_k - \psi_{\omega_0}^{y_2,-y_1,\theta}\|_Q \geqslant \|\psi_{\omega_0}^{y_1,-y_2,\theta_k} - \psi_{\omega_0}^{y_2,-y_1,\theta}\|_Q - \|\psi_{\omega_0}^{y_1,-y_2,\theta_k} - \xi_k\|_Q \geqslant \frac{\varepsilon_0}{3}.$$

This contradicts (3.10), so the proof is complete.

□

## 4. Perspectives

The interplay between nonlinearity and defects is, in our opinion, a promising and worth developing field. In particular, already in simple models highly non trivial behaviour can emerge. An enlightening example has been supplied by means of the $\delta'$ defect, in which the occurrence of a pitchfork bifurcation with symmetry breaking has been proved for the family of nonlinear ground states.

Such results have to be considered as the first achievements of our research project. Many non trivial variations on the theme could be given by studying the entire family of one-dimensional defects (a four parameters family, see [9]) and thus investigate the effect of various self-adjoint boundary conditions, in particular, of those that give rise to *two* bound states. We expect that, in the nonlinear problem, each of the two linear modes could be deformed into nonlinear modes for any frequency greater than the energy of the corresponding linear mode. Think, for instance, of a point interaction that, roughly speaking, is the sum of a $\delta$ and a $\delta'$ defect *at the same point*. It exhibits two bound states, one of which is even (as the ground state for a Dirac's delta), while the other is odd (as the ground state for a delta prime). A number of question then arises: how do the corresponding nonlinear mode interact? Does it exist a third family of stationary (possibly ground) states that does not preserve any parity symmetry?

However, all these steps are only preliminary to the problem of studying the detailed evolution of a travelling soliton that meets an impurity.

It remains completely open the problem of defining analogous models in higher dimension. We recall that in dimension two and three, the only point interaction is the delta interaction, and in dimension higher than three there are no point perturbations of the laplacian. For instance, in the three dimensional case a bare power nonlinearity seems to be too strong to be added to a Dirac's $\delta$ potential; so a different type of nonlinearity with a moderated behaviour at infinity should be considered. Conversely, in space dimension two the naïf power nonlinearity could be not necessarily in conflict with the domain of a delta interaction, but up to now no rigorous result exists on this problem.

Another related topic is given by quantum graphs (see [37–39] for the relevant definitions and analysis in the linear case). Also in the relatively simple case of a NLS on a star graph, the richer structure provides a larger number of nonlinear stationary states, for example two stationary states for a three edge star graph with a delta vertex, both attractive and repulsive, and the number increases with the number of edges (see [5]). In this respect, besides the determination of the ground state, it is an open interesting problem the analysis of stability of excited states, here explicitly known. Nothing is known for the a star graph with more general vertex conditions, for example the boundary condition of $\delta'$ type.

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