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BIFURCATION CONDITION FOR OPTIMAL SETS OF THE AVERAGE DISTANCE FUNCTIONAL

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Consider the quasi-static irreversible evolution of a connected network, which minimizes the average distance functional. We look for conditions forcing a bifurcation, thus changing the topology. We would give here a sufficient conditions. Then we will give an explicit example of sets satisfying the bifurcation condition, and analyze this special case. Proofs given here will be somewhat sketchy, and this work is based on [9], in which more details can be found.

Keywords: optimal transport, Euler scheme, minimizing movements, average distance.

1. Introduction

Historically, the so-called "minimizing movement theory" was introduced by De Giorgi in [8] to study evolution processes with some kind of variational structure. In this paper we will consider the general quasi-static, rate independent, evolution for connected networks related to an average distance functional, and our main goal is to analyze whether optimal sets exhibit a bifurcation.

Given a domain Ω, $S \subset \Omega$ with $\dim_{\mathcal{H}} S = 1$, consider first these problems: given the system

$$\begin{cases} -\Delta_p u = 1 & in\ \Omega \backslash S \\ u = 0 & on\ S \end{cases},$$

we aim to minimize (among all S satisfying some length constraints) the associated energy, namely

$$F_p(S) := (1 - \frac{1}{p}) \int_\Omega ||\nabla u||^p dx.$$

If we let $p \to \infty$, then the energy F_p Γ-converges to the so called average distance, i.e.

$$F(S) := \int_\Omega \mathrm{dist}(x, S) dx.$$

This will be our main functional in the paper. As this energy operates on Hausdorff one-dimensional, connected, compact sets with limited Hausdorff length, we denote

$$A_l := \left\{ \mathcal{X} \subseteq \Omega : \mathcal{X} \text{ compact, connected and } \mathcal{H}^1(\mathcal{X}) \leqslant l \right\}, \qquad A := \bigcup_{j \geqslant 0} A_j. \tag{1.1}$$

Both A_l and A depend on the domain Ω, but to simplify notations, when there will be no risk of confusion, we will omit this dependence.

The average distance functional has a sort of monotonicity:

Proposition 1.1. *Given a domain Ω, for any $S_1, S_2 \in A$, with $S_1 \subseteq S_2$, we have $F(S_1) \geqslant F(S_2)$.*

Proof. The proof is straightforward, by writing the thesis explicitly (in the integral form):

$$F(S_1) = \int_\Omega \mathrm{dist}(x, S_1)dx, \qquad F(S_2) = \int_\Omega \mathrm{dist}(x, S_2)dx$$

and $S_1 \subseteq S_2$ implies $\mathrm{dist}(x, S_1) \geqslant \mathrm{dist}(x, S_2)$ thus integrating on Ω we have

$$\int_\Omega \mathrm{dist}(x, S_1)dx \geqslant \int_\Omega \mathrm{dist}(x, S_2)dx.$$

□

A consequence of this is that prescribing the maximum length is the same as prescribing the length: this helps when we have to pass to the limit, as for any fixed $l > 0$, A_l is sequentially compact, but $A_l \setminus \bigcup_{0 \leqslant j < l} A_j$ is not.

This paper will be structured as follows:

- in Section 2 we will present preliminaries, in particular results concerning regularity of optimal sets;
- in Section 3 we will analyze conditions sufficient to force a bifurcation;
- in Section 4 we will exhibit an explicit example, and use results from Section 3 in this particular case.

This paper is an extended version of the talk given during the conference "Operator theory and boundary value problems" in Orsay in May 2011, and some proofs are somewhat synthetic; we refer to [9] for more details.

Notations

In order to simplify notations, unless explicitly specified, if a notation is used in two different Definitions/ Propositions/ Lemma/ Theorems, there is no connection between them.

The only notable exceptions are:

- A_l (with $l \geqslant 0$), and A: if there is a given domain Ω, they always denote the sets defined after (1.1),
- F which always stands for the average distance functional
- $V(\cdot)$ which stands for the Voronoi cell of the point.

We will work only with compact connected domain in $\mathbb{R}^2$ with positive Lebesgue measure, and *"domain"* will always refer to a similar domain.

2. Geometry of optimal sets in the static case

In this section we present some results about the geometry of optimal sets in the static case, as they will be useful later in discussing the evolution case. All these results can be found on [4], [5] and [9].

The following are results concerning prohibited subsets of minimizers of the average distance functional:

Proposition 2.1. *Let be Ω a given domain, $l > 0$ a fixed quantity, and $\Sigma_{opt} \in argmin_{A_l} F$. Then Σ_{opt} cannot contain*

(1) *a loop (a subset homeomorphic to S^1);*
(2) *a cross (a subset homeomorphic to $\{x^2 + y^2 \leqslant 1 : xy = 0\}$);*
(3) *a triple point P with an angle among the three angles here that does not measure $\frac{2}{3}\pi$.*

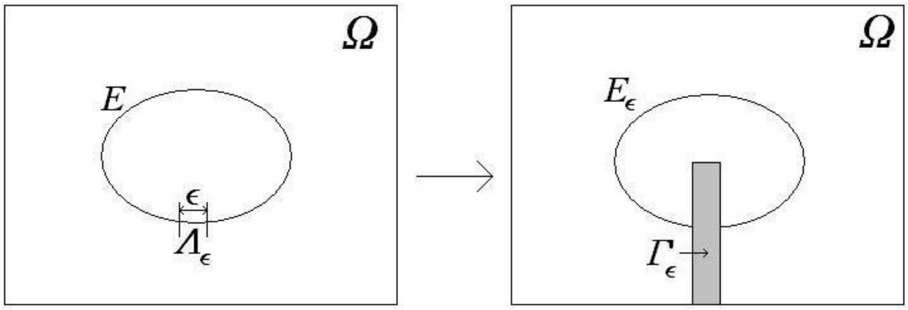

Fig. 1. This is a schematic representation of what happens if we remove the portion Λ_ε

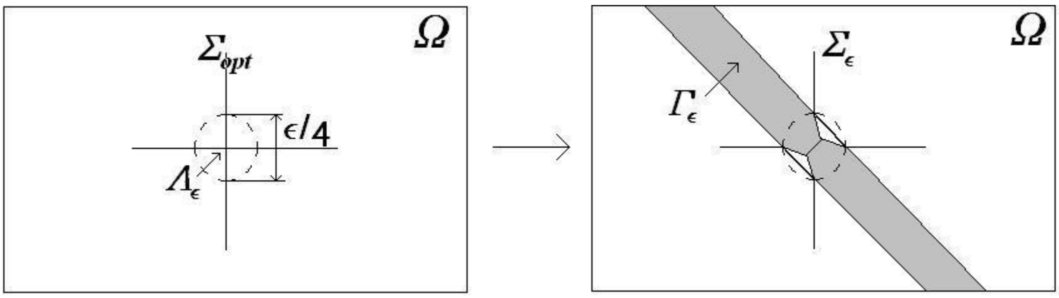

Fig. 2. Σ_ε is obtained from Σ_{opt} by replacing the infinitesimal cross Λ_ε with a slightly shorter Steiner graph

The proof rely on the "cut and paste" technique, in which we first remove a subset, estimate the variation in energy, and then add it elsewhere. These figures show what happens for cases (1) and (2) (case (3) is similar), and in both cases the loss in energy is comparable with ε^3. The next result shows that the gain in energy by adding similar sets elsewhere, is larger:

Proposition 2.2. *Given a domain Ω, let be $S \subset \Omega$ be a connected set, if we add a segment λ_ε to a non endpoint of S (with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$), then the "gain" $F(S) - F(S_\varepsilon)$ is comparable with $\varepsilon^{3/2}$, where $S_\varepsilon := S \cup \lambda_\varepsilon$.*

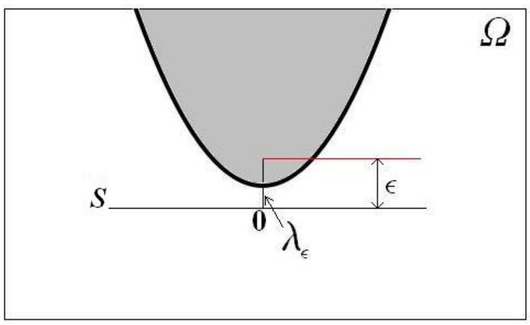

Fig. 3. All the shaded area, whose area is comparable with $\varepsilon^{1/2}$, gains something in path

The detailed proof can be found in [9]. Here we limit to present a sketch:
Step 1: scale the configuration (see Fig. 4).

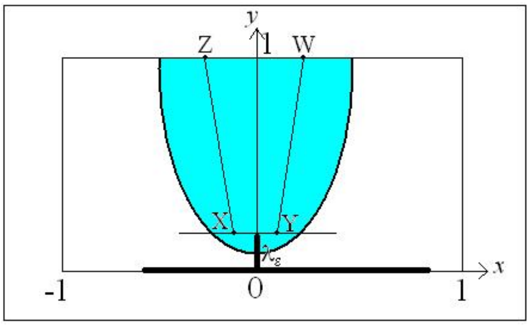

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Fig. 4. After scaling, we can do all the computation in this configuration

Step 2: points (x, y) which project on $(0, \varepsilon)$ satisfy

$$|y| \geqslant \text{dist}((x, y), (0, \varepsilon)) = \sqrt{x^2 + (y - \varepsilon)^2},$$

which corresponds to the parabola, of area $O(\varepsilon^{1/2})$;

Step 3: consider the trapezium, its area is $O(\varepsilon^{1/2})$ while points on it gain $O(\varepsilon)$ in path, thus the total gain for the energy if $O(\varepsilon^{3/2})$.

Minimizing movements

Now we present the minimizing movement problem in our case:

let be Ω a given domain, we work on the space is A, endowed with the Hausdorff distance metric, and our kinetic term is

$$\mathcal{F}(t, \mathcal{X}_1, \mathcal{X}_2) := \begin{cases} F(\mathcal{X}_1) & \text{if } \mathcal{X}_2 \subseteq \mathcal{X}_1 \text{ and } \mathcal{X}_1 \in A_{t+\mathcal{H}^1(\Sigma_0)} \\ \infty & \text{otherwise} \end{cases},$$

where $\Sigma_0 \in A$ is the initial datum.

So, given a positive time step $\eta > 0$ and an initial datum $S_0 \in A$, our Euler scheme is

$$\begin{cases} w(0) = S_0 \\ w(n+1) \in \text{argmin}_{\mathcal{H}^1(\mathcal{X}) \leqslant \mathcal{H}^1(S_0) + (n+1)\eta, w(n) \subseteq \mathcal{X}} F(\mathcal{X}) \end{cases}.$$

A minimizing movement can be thought as the limit case for $\eta \downarrow 0$ of Euler schemes:

Definition 2.3. *Given $T > 0$, the function $u : [0, T] \longrightarrow A$ is a minimizing movement associated with initial datum u_0 and kinetic term $\mathcal{F}$, and we will write $u \in MM(\mathcal{F}, A, u_0)$ if there exists a sequence $\varepsilon_n \downarrow 0$ for which*

$$\forall t \in [0, T] \quad u_{\varepsilon_n}(t) \to u(t).$$

Existence is guaranteed by [2]: it states that limit functions exist if the following conditions are verified:

- the convergence in $(A_l, d_{\mathcal{H}})$ is sequentially compact;
- the irreversibility condition is compatible with the convergence;
- every nondecreasing function $\psi : \mathbb{R} \longrightarrow (A_l, d_{\mathcal{H}})$ is continuous up to countably many points.

The first two conditions are easy to verify.

The third arises from the following argument: consider a generic nondecreasing function ψ :

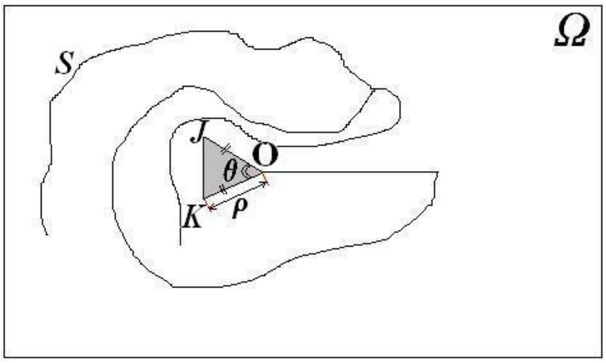

Fig. 5. The presence of the shaded triangle T' makes adding at an endpoint more convenient than at a non endpoint at least when the added portion has sufficient small length

$\mathbb{R} \longrightarrow (A_l, d_{\mathcal{H}})$, and suppose that it has discontinuity points $\{x_i\}_{i\in I}$, with $x_i < x_j$ when $i < j$. As ψ is nondecreasing, we can write

$$\psi(x_1) \subset \psi(x_2) \subset \cdots \psi(x_i) \subset \psi(x_{i+1}) \subset \cdots \subset \psi(\sup_{i\in I} x_i),$$

and passing to the $\mathcal{H}^1$ measures,

$$\mathcal{H}^1(\psi(x_1)) < \mathcal{H}^1(\psi(x_2)) < \cdots \mathcal{H}^1(\psi(x_i)) < \mathcal{H}^1(\psi(x_{i+1})) < \cdots < \mathcal{H}^1(\psi(\sup_{i\in I} x_i)) < \infty,$$

which is possible only if I is at most countable, as being $\{x_i\}_{i\in T}$ discontinuity points, the difference $\mathcal{H}^1(\psi(x_{j+1})) - \mathcal{H}^1(\psi(x_j))$ are positive for any j.

3. Bifurcation condition

In this section we will try to find a condition sufficient to force a branching behavior. Several tools are needed.

Definition 3.1. *Given a domain Ω, $S \in A$ a generic element, a non endpoint $P \in S$ is "smooth" if there exists $r > 0$ such that:*

(1) *there exists an homeomorphism $f : B(P, r) \cap S \longrightarrow (0, 1)$;*
(2) *there exists an unique direction θ such that for any sequence $P_n \longrightarrow P$ in $B(P, r)$ the directions of the line $L(P_n, P)$ converge to θ.*

A subset of S is smooth is all its non endpoints are smooth.

For these points the estimate of Proposition 2.2 applies. The next results compares the gain for F when adding at smooth points with when adding at other points.

Proposition 3.2. *Given a domain Ω, let $S \in A$ be a smooth set, and let it have an endpoint O which satisfies:*

($*$) *there exist $\rho, \theta > 0$ and a triangle $T' \subset V(O)$ with a vertex in O and sides ρ, $\rho, \rho\sqrt{2(1 - 2\cos\theta)}$ (the order is not relevant) that does not intersect S.*

Then there exists ε_0 such for any $\varepsilon < \varepsilon_0$ adding a segment λ_ε at O, with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$ in O is more convenient that adding any connected set with same length at any non endpoint.

We present here a sketch of the proof. for more details we refer to [9]:

Step 1: adding a straight segment along the bisector of the angle in O, the gain is positive on at least half of the triangle T' (which has finite area), and it is comparable with $O(\varepsilon)$;

Step 2: from Proposition 2.2 we know that adding sets with length ε to smooth non endpoints the gain is comparable with $O(\varepsilon^{3/2})$, thus the choice in Step 1 is better.

3.1. Bifurcation

We investigate now the situations that may appear during the evolution. Given an initial datum $S_0 \in A$, $\Sigma : [0, T] \longrightarrow A$ a minimizing movement function, a time $T_0 \in (0, T]$, we are interested in the following behaviors:

(1) any point $X \in S_0$ has the same multiplicity as $i_t(X) \in \Sigma(t)$, where $i_t : S_0 \longrightarrow \Sigma(t)$ denotes the identical inclusion, except for endpoints which have multiplicity 1 or 2 in $\Sigma(t)$;
(2) there exists a non endpoint $X_0 \in S_0$, $t_0 > 0$ such that $i_{t_0}(X_0)$ has different multiplicity from X_0, or some endpoint $X_1 \in S_0$ has $i_{t_0}(X_1) \in \Sigma(t)$ with multiplicity at least 3, with $i_{t_0} : S_0 \longrightarrow \Sigma(t_0)$ denoting the identical inclusion.

In order to provide an upper bound to the branching (which falls into case (2), where a point increases its multiplicity) time, we need to establish when choice (2) becomes necessary preferable to choice (1).

Proposition 3.2 shows that under those conditions a branching is not optimal, so to obtain a contradiction, we must let the hypothesis of Proposition 3.2 fail. The only ways to reach a contradiction is admit the existence of non smooth points, or negate the existence of endpoints, or negate condition $(*)$, i.e. all its endpoints do not satisfy $(*)$. The last reads:
"for any endpoint O', for any $\rho, \theta > 0$, for all triangles with a vertex in O' and sides $\rho, \rho, \rho\sqrt{2 - 4\cos\theta}$ the set $\Sigma(t)$ intersects that triangle".

Let us try to negate condition $(*)$.

These following tools will be used:

Definition 3.3. *Let S be a compact connected set in a given domain Ω, $P \in S$ a point, and a positive value $\mathcal{R} > 0$. The "the inner radial projection" is the function*

$$\pi_{P,\mathcal{R}} : B(P, \mathcal{R}) \longrightarrow \partial B(P, \mathcal{R}), \qquad \pi_{P,\mathcal{R}}(x) := \partial\overline{B(P, \mathcal{R})} \cap Px$$

where Px denotes the halfline starting from P and passing through x.

In other words the inner radial projection maps a point to the the point on the border having its same direction (when putting P as the center).

Now we can define the equivalent of a loop:

Definition 3.4. *Given a domain Ω, let be Γ a curve, a subset $\gamma \subseteq \Gamma$ is "general loop" around a point $Q \in \Omega$ if it is a closed connected set satisfying:*

(1) *There exists a $\mathcal{R}'$ for which $\gamma \subseteq B(Q, \mathcal{R}')$ and $\pi_{Q,\mathcal{R}'}(\gamma \cap B(Q, \mathcal{R}')) = \partial B(Q, \mathcal{R}')$;*
(2) *No connected proper subsets of γ satisfies the first condition.*

Graphically a general loop may be thought as a minimal set that "wraps around" a point.

Using the above notations:

Definition 3.5. *Given a domain Ω, let be Γ a curve, $P \in \Gamma$ an endpoint, and suppose that there exist a sequence $\{\rho_n\}_{n=0}^{\infty}$ with $\rho_n \downarrow 0$ such that for any n $\pi_{P,\rho_n}(\Gamma \cap B(P, \rho_n)) = \partial B(P, \rho_n)$.*

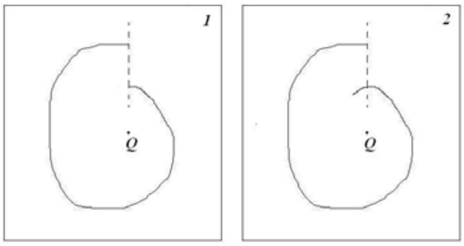

Fig. 6. (1) is what a general loop may look like, while (2) is not a general loop

Then if there exists a partition of Γ *in general loops, namely* $\Gamma = \bigcup_{n=0}^{\infty} L_n$ *in which every* L_n *is a general loop and* $P \notin L_0$ *and for every* n $\partial B(P, \rho_n)$ *contains the farthest point of* L_n *from* P *and* $L_n \subset \overline{B(P, \rho_n) \backslash B(P, \rho_{n+1})}$*, then* $\{\rho_n\}_{n=0}^{\infty}$ *is a "distance sequence" for* P.

Not every endpoint has a distance sequence, and even if it had one, this is not unique, as a distance sequence can be arbitrary truncated at its beginning (i.e. if $\{r_n\}_{n\geqslant k}$ is a distance, then $\{r_n\}_{n\geqslant h}$ with $h \geqslant k$ is a distance sequence for the same point).

Theorem 3.6. *Given a domain* Ω*, let be* $S_0 \in A$ *(A defined just after* (1.1)*) be the initial datum of e rate-independent evolution* $\Sigma : [0, T] \longrightarrow A$*. Moreover, suppose the evolution does not stop.*

Then, a change in topology occurs, i.e. $\Sigma(t)$ *is not homeomorphic to* $\Sigma(0)$ *for any* $t > 0$*, if the following condition is satisfied:*

$(**)$ *any endpoint* $P' \in S_0$ *has a distance sequence* $\{\rho_n^{(P')}\}_{n=0}^{\infty}$ *and a constant* $Wr(P')$ *which verifies*

$$\limsup_{n\to\infty} \log_{\rho_n^{(P')}} \rho_{n+1}^{(P')} \leqslant Wr(P') < 2.$$

We need to prove it first for Euler schemes:

Proposition 3.7. *Given a domain* Ω*, let be* $S_0 \in A$ *(A defined just after* (1.1)*) the initial datum, and consider the Euler scheme*

$$\begin{cases} w(0) = S_0 \in A \\ w(k-1) \subseteq w(k) \\ w(k) \in \operatorname{argmin}_{\mathcal{H}^1(\mathcal{X}) = k\varepsilon + \mathcal{H}^1(S_0)} F(\mathcal{X}) \end{cases}.$$

Then if condition $(**)$ *of Theorem 3.6 is verified, there exists* ε_0 *such that for* $\varepsilon < \varepsilon_0$*,* $w(1)$ *presents a bifurcation.*

Proof. We assume first that $\Sigma(0) = S_0$ has an unique endpoint P. Let us analyze what happens if we add some set $J_{\varepsilon'}$ (with length $\varepsilon' > 0$ small) at P: we have to estimate the gain for the energy. As $J_{\varepsilon'} \subset B(P, \varepsilon')$, the gain is upper bounded by the quantity

$$\varepsilon' |B(P, \rho_{m(n)-1}^{(P)})|$$

where $\rho_{m(n)-1}$ will be explained in the following.

As the point P satisfies condition $(**)$, there exists a maximum $m(n)$ for which $\rho_{m(n)}^{(P)} \leqslant \varepsilon' < \rho_{m(n)-1}^{(P)}$ so the total gain can be estimated by $\pi(\rho_{m(n)-2}^{(P)})^2 \varepsilon'$, and as

$$\rho_{m(n)}^{(P)} \leqslant \varepsilon' < \rho_{m(n)-1}^{(P)}$$

the logarithmic condition in $(**)$ gives

$$\varepsilon'^2 < (\rho^{(P)}_{m(n)-2})^2 \leqslant (\rho^{(P)}_{m(n)})^{2/Wr(P)^2} \leqslant \varepsilon'^{2/Wr(P)^2} = o(\varepsilon^{1/2})$$

and the total gain is an $O(\varepsilon'^{1+2/Wr(P)}) = o(\varepsilon'^{3/2})$.

So, considering the gain obtained in Proposition 2.2, adding $J_{\varepsilon'}$ in this way (when ε' becomes small enough) is not optimal. This argument can be generalized to S_0 having more endpoints (by applying it to all endpoints of S_0), so the proof is complete.

□

Now we can prove the result for the rate-independent case:

Proof. (of Theorem 3.6)
Step 1:
By hypothesis $\Sigma : [0,T] \longrightarrow A$ is the minimizing movement with initial datum $\Sigma(0) = S_0$, i.e there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ with $\varepsilon_n \downarrow 0$ such that, put

$$\begin{cases} w(0,n) = S_0 \in A \\ w(k-1,n) \subseteq w(k,n) \\ w(k,n) \in \text{argmin}_{\mathcal{H}^1(\mathcal{X})=k\varepsilon_n+\mathcal{H}^1(S_0)} F(\mathcal{X}) \end{cases},$$

$$\Sigma_{\varepsilon_n}(t) := w(\left[\frac{t}{\varepsilon_n}\right], n),$$

and for any $t \in [0,T]$ $\Sigma(t) = \lim_{n\to\infty} \Sigma_{\varepsilon_n}(t)$; then by hypothesis $t = 0$ is the time at which condition $(**)$ is satisfied, and let us analyze the topology of $\Sigma(t)$ for $t > 0$.

We assume first that $\Sigma(0) = S_0$ has an unique endpoint P.

Step 2:
Suppose that there exists $\delta' > 0$ such that $\Sigma(t)$ has the same topology for any $t \in [0, \delta')$ (obviously, if this is true for δ', it holds for any positive $0 < \delta'' < \delta'$ too).
The proof of Proposition 3.7 shows that adding length in P is not optimal for Euler schemes with small enough time step. To pass to the limit, we need uniformity for the estimate, i.e. there is a positive ξ such that adding length in $B(P,\xi)$ is not optimal. This is done by considering that there is always a non point Z such that adding length ε here the gain is at least $B_Z\varepsilon^{3/2}$, with $B_Z > 0$ depending only on the geometry of $\Sigma(t)$ near Z, while adding near P the gain is $o(\varepsilon^{3/2})$ for sufficiently small ε, as P verifies condition $(**)$.

Step 3:
This argument can be generalized to S_0 having more endpoints (by applying it to all endpoints of S_0). As the evolution does not stop, the new part added must be connected to the original set by a connected path, thus a bifurcation arises.

□

The entire proof relies on the fact that evolving too close to P becomes not optimal for the presence of the distance sequence.

4. Examples

In the previous section, we have found a condition (given by Theorem 3.6) sufficient to force a branching behavior, so now we look for an example in which Theorem 3.6 is applicable.

Let the logarithmic spiral S_0^* with proportion 3 (i.e. naming $Y_1, Y_2, \cdots$ the points as in Figure 8, $\frac{\mathcal{H}^1(OY_k)}{\mathcal{H}^1(OY_{k+1})} = 3$ for any k) be our initial datum (see Fig. 7).

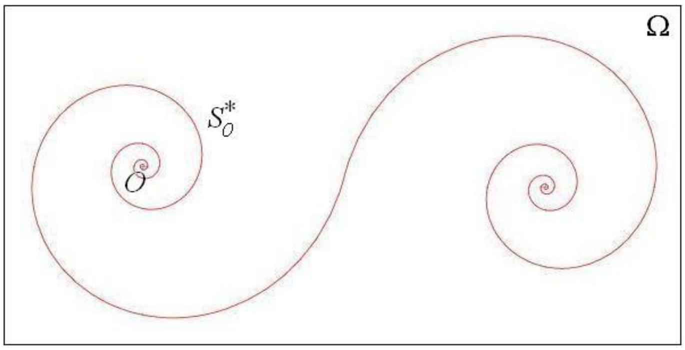

Fig. 7. This will be our datum, and it satisfies the conditions of Theorem 3.6

S_0^* has two endpoints, we do the computations on one of them. We impose the following two coordinate systems (a cartesian one and a polar one) (see Fig. 8).

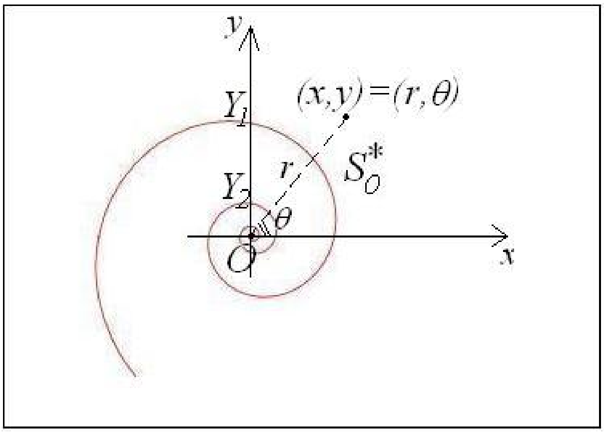

Fig. 8. Some computations are easier on cartesian coordinates, some on polar coordinates

Let us analyze the endpoint O: it verifies condition $(**)$ of Theorem 3.6, as its has a distance sequence $\{\rho_n^{(O)}\}$: if we divide the part around it into $L_1, L_2, \cdots$, with L_k the piece between Y_k and Y_{k+1}, naming $\rho_n^{(O)} := \mathcal{H}^1(OY_n)$, we have

$$\pi_{O,\rho_n^{(O)}}(S_0^* \cap B(O, \rho_n^{(O)})) = \partial B(O, \rho_n^{(O)}) \qquad \forall n$$

where $\pi_{\cdot,\cdot}$ denotes the inner radial projection. Moreover $Y_k \in L_k$ for any k, so the sequence $\{\rho_n^{(O)}\}_{n=1}^{\infty}$ is effectively a distance sequence, and it verifies

$$\log_{\rho_n^{(O)}} \rho_{n+1}^{(O)} \longrightarrow 1.$$

So hypothesis of Theorem 3.6 are all verified, and given any positive time T, any minimizing movement $\Sigma : [0,T] \longrightarrow A$ with $\Sigma(0) = S_0^*$ will exhibit a bifurcation, at the very beginning, with $\Sigma(t)$ having different topology from S_0^* for any $t > 0$.

Notice that we can alter the set in any way, it suffices to keep conditions of Theorem 3.6 verified. The next example, not as regular as the one in Figure 7, has the same bifurcation property.

Let be $\Omega := D^2$ our domain, and points $Z_n := (0, \frac{1}{2^n})$ in Ω. The we connect each Z_k with Z_{k+1} with an injective arc

$$\gamma_k : [0,1] \longrightarrow \{Z_k\} \cup \{Z_{k+1}\} \cup (conv(B((0,0),2^{-k}))\backslash conv(B((0,0),2^{-k-1})))$$

such that:

- $\gamma_k(0) = Z_k,\ \gamma_k(1) = Z_{k+1},$;
- $\pi_{(0,0),2^{-k}}(\gamma_k([0,1]) \cap B(0,2^{-k})) = \partial B(0,2^{-k})$.

Notice that these γ_k can be highly irregular.

These conditions are sufficient to force $\{\xi_n\}$, $\xi_n := 2^{-n}$ to be a distance sequence for $(0,0)$, and the logarithmic condition is satisfied, as

$$\lim_{n\to\infty} \log_{2^{-n}} 2^{-n-1} = 1 < 2.$$

So the same bifurcation result follows.

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