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A NONLOCAL PROBLEM WITH INTEGRAL CONDITIONS FOR HYPERBOLIC EQUATION

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In this article, we consider two initial-boundary value problems with nonlocal conditions. The main goal is to show the method which allows to prove solvability of a nonlocal problem with integral conditions of the first kind. This method is based on equivalence of a nonlocal problem with integral conditions of the first kind and nonlocal problem with integral conditions of the second kind in special form. Existence and uniqueness of generalized solutions to both problems are proved.

Keywords: hyperbolic equation, nonlocal problem, integral conditions.

1. Introduction

In this paper, we consider a mixed problem with nonlocal integral conditions for a hyperbolic equation. Before considering a problem it will be convenient to recall certain features that are common to nonlocal problems.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, Q be the cylinder $\Omega \times (0,T)$, $T < \infty$, $S = \partial\Omega \times (0,T)$ be the lateral boundary of Q.

Consider an equation

$$u_{tt} - (a_{ij}(x,t)u_{x_i})_{x_j} + c(x,t) = f(x,t) \tag{1}$$

(repeated indices imply summation from 1 to n) and set a problem: find a function $u(x,t)$ that is a solution of (1) in Q, satisfies initial data

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{2}$$

and the following condition for $n > 1$:

$$\alpha \frac{\partial u}{\partial \nu}\Big|_S + \int\limits_\Omega K(x,t)u(x,t)dx = 0. \tag{3}$$

Here $\frac{\partial u}{\partial \nu} \equiv a_{ij}(x,t)u_{x_i}(x,t)\nu_i\Big|_S$, $\nu(x) = (\nu_1, ..., \nu_n)$ is outward normal to $\partial\Omega$ at the current point, $K(x,t)$ is given.

In a special case $n = 1$ the lateral boundary of $Q = (0,l) \times (0,T)$ separates into two parts: $x = 0$ and $x = l$. As a consequence the condition (3) separates into two conditions:

$$\begin{aligned} \gamma_1 u_x(0,t) + \rho_1 \int\limits_0^l K_1(x,t)u(x,t)dx = 0, \\ \gamma_2 u_x(0,t) + \rho_2 \int\limits_0^l K_2(x,t)u(x,t)dx = 0, \end{aligned} \tag{4}$$

where $\rho_1^2 + \rho_2^2 > 0$.

Note that (3) and (4) are nonlocal conditions. By nonlocal condition we mean any relation between values of required solution at boundary and interior points of the domain Q. If $\alpha = 0$

then condition (3) is called the nonlocal condition of the first kind, if $\alpha \neq 0$ — of the second kind. Likewise, each of (4) is the nonlocal condition of the first kind if $\gamma_i = 0$.

Let us remark here that nonlocal integral conditions of the form

$$\begin{gathered} u(0,t) + \rho_1 \int\limits_0^l K_1(x,t)u(x,t)dx = 0, \\ u(l,t) + \rho_2 \int\limits_0^l K_2(x,t)u(x,t)dx = 0 \end{gathered} \tag{5}$$

for $n = 1$, and

$$u(x,t) + \int\limits_\Omega K(x,y,t)u(y,t)dy = 0, \ (x,t) \in S$$

for $n > 1$ are also of interest and are considered in [14], [15] (see also references in these papers). Nonlocal problems with time-dependent Steklov's conditions — in [16,17] for hyperbolic and parabolic equations respectively.

Recently, nonlocal boundary value problems for parabolic and hyperbolic equations with integral conditions have been actively studied. Such problems arise in mathematical physics in the study of heat-transfer processes (see [1–3]), plasma phenomena [4], certain technological processes [5], vibration of a media [6]. Note that inverse problems with integral overdetermination are closely related to nonlocal problems [7, 8]. Studies have shown that classical methods often do not work when we deal with nonlocal problems [3, 9, 13]. To date, several methods have been proposed for overcoming the difficulties arising from nonlocal conditions. The choice of method depends on a kind of nonlocal conditions. If $\alpha \neq 0$ in (3) then we can use compactness method. The major advantage of this approach is possibility to derive an identity that is a base of a definition of a solution to the problem. Using Sobolev's embedding theorems we obtain a priori estimates and prove solvability [10]. It is easy to see that this approach fails for $\alpha = 0$ in (3) or $\gamma_1 = \gamma_2 = 0$ in (4)

This difficulty can be overcome easily and with elegance by suggested in this paper approach when $n = 1$.

We are now equipped to state a main problem and prove solvability.

2. The Main Result

Let $Q = (0,l) \times (0,T), \quad l, T < \infty$.

Problem 1. Find a function $u(x,t)$ that is a solution of an equation

$$u_{tt} - u_{xx} + c(x,t)u = f(x,t) \tag{6}$$

in Q, satisfies the initial data (2) and nonlocal conditions

$$\int\limits_0^l K_i(x)u(x,t)dx = 0, \quad i = 1, 2. \tag{7}$$

The main objective is to show that under certain conditions on data there exists the unique solution to Problem 1.

Theorem 1. *Let*

$$\begin{gathered} c(x,t) \in C(\bar{Q}), c_t(x,t) \in C(Q), \quad K_i(x) \in C^1[0,l] \cap C^2(0,l), \\ K_1(0)K_2(l) - K_1(l)K_2(0) \neq 0, \\ K_{1x}(0)K_2(0) - K_{2x}(0)K_1(0) = K_{1x}(l)K_2(l) - K_{2x}(l)K_1(l), \end{gathered}$$

$$\Delta\Big\{[K_{1x}(0)K_2(l)-K_{2x}(0)K_1(l)]\zeta_1^2+2[K_{2x}(l)K_1(l)-K_{1x}(l)K_2(l)]\zeta_1\zeta_2-$$
$$-[K_{2x}(l)K_1(0)-K_{1x}(l)K_2(0)]\zeta_2^2\Big\}\geqslant 0\ \forall z=(\zeta_1,\zeta_2),$$
$$f(x,t)\in L_2(Q), f_t(x,t)\in L_2(Q),\ \varphi(x)\in W_2^1(0,l),\ \psi(x)\in L_2(0,l).$$

Then there exists a unique generalized solution to Problem 1.

(A definition of a solution to Problem 1 will be given later.)

We shall divide the proof of this statement into two steps:

1. Proof of equivalence of Problem 1 and the problem (name it problem 2) with integral conditions of the second kind.

2. Proof of solvability of Problem 2.

Now we begin to carry out this scheme.

Step 1. Equivalence.

Lemma. *Let $u(x,t)$ satisfies equation* (6)*, initial data* (2)*,*

$$c(x,t)\in C(\bar{Q}), f(x,t)\in L_2(Q), K_i(x)\in C^2[0,l],$$
$$\Delta\equiv K_1(0)K_2(l)-K_1(l)K_2(0)\neq 0$$

and consistency conditions hold:

$$\int\limits_0^l K_i(x)\varphi(x)dx=0,\qquad \int\limits_0^l K_i(x)\psi(x)dx=0. \tag{8}$$

Then (7) *are equivalent to the conditions of the second kind:*

$$\begin{aligned} u_x(0,t)&=\alpha_1u(0,t)+\beta_1u(l,t)+\int\limits_0^l M_1(x,t)u(x,t)dx+\int\limits_0^l P_1(x)fdx,\\ u_x(l,t)&=\alpha_2u(0,t)+\beta_2u(l,t)+\int\limits_0^l M_2(x,t)u(x,t)dx+\int\limits_0^l P_2(x)fdx,\end{aligned} \tag{9}$$

where

$$\alpha_1=\frac{K_{1x}(0)K_2(l)-K_{2x}(0)K_1(l)}{\Delta},\ \alpha_2=\frac{K_{1x}(0)K_2(0)-K_{2x}(0)K_1(0)}{\Delta},$$
$$\beta_1=\frac{K_{2x}(l)K_1(l)-K_{1x}(l)K_2(l)}{\Delta},\ \beta_2=\frac{K_{2x}(l)K_1(0)-K_{1x}(l)K_2(0)}{\Delta},$$
$$M_1(x,t)=\frac{[K_{1xx}(x)-c(x,t)K_1(x)]K_2(l)-[K_{2xx}(x)-c(x,t)K_2(x)]K_1(l)}{\Delta},$$
$$M_2(x,t)=\frac{[K_{1xx}(x)-c(x,t)K_1(x)]K_2(0)-[K_{2xx}(x)-c(x,t)K_2(x)]K_1(0)}{\Delta},$$
$$P_1(x)=\frac{K_1(x)K_2(l)-K_2(x)K_1(l)}{\Delta},\ P_2(x)=\frac{K_1(x)K_2(0)-K_2(x)K_1(0)}{\Delta}.$$

Proof. Let $u(x,t)$ satisfies equation (6), initial data (2) and conditions (7). Multiplying (6) by $K_i(x)$, integrating over $(0,l)$ and using (7) we get:

$$\begin{aligned} &K_1(0)u_x(0,t)-K_1(l)u_x(l,t)=K_{1x}(0)u(0,t)-K_{1x}(l)u(l,t)+\\ &+\int\limits_0^l(K_{1xx}(x)-c(x,t)K_1(x))u(x,t)dx+\int\limits_0^l K_1(x)f(x,t)dx,\\ &K_2(0)u_x(0,t)-K_2(l)u_x(l,t)=K_{2x}(0)u(0,t)-K_{2x}(l)u(l,t)+\\ &+\int\limits_0^l(K_{2xx}(x)-c(x,t)K_2(x))u(x,t)dx+\int\limits_0^l K_2(x)f(x,t)dx.\end{aligned} \tag{10}$$

As $\Delta \equiv K_1(0)K_2(l) - K_1(l)K_2(0) \neq 0$, we can solve this system with respect to $u_x(0,t)$ and $u_x(l,t)$. Then we immediately get (9).

Let now (9) holds for the solution $u(x,t)$ of (6). Obviously (10) also holds. Multiplying (6) by $K_i(x)$ and integrating over $(0,l)$ we get a system of ODE:

$$\frac{\partial^2}{\partial t^2}\int_0^l K_i(x)u(x,t)dx = 0. \tag{11}$$

Consistency conditions (8) give initial data:

$$\int_0^l K_i(x)u(x,0)dx = 0, \ \int_0^l K_i(x)u_t(x,0)dx = 0.$$

By virtue of uniqueness of a solution to the Cauchy problem

$$\int_0^l K_i(x)u(x,t)dx = 0.$$

This means that conditions (7) hold.

Problem 2. Find a function $u(x,t)$ that is a solution of (6) and satisfies (2) and (9).

Step 2. Solvability of Problem 2

Denote

$$\hat{W}_2^1(Q) = \{v(x,t) : v \in W_2^1(Q), \ v(x,T) = 0\},$$

where $W_2^1(Q)$ is Sobolev space.

Definition. Function $u(x,t) \in W_2^1(Q)$ is said to be a generalized solution to Problem 2 (as well as to Problem 1) if $u(x,0) = \varphi(x)$ and identity

$$\int_0^T\int_0^l(-u_tv_t + u_xv_x + cuv)dxdt + \int_0^T v(0,t)[\alpha_1u(0,t) + \beta_1u(l,t)]dt+$$
$$+\int_0^T v(0,t)\int_0^l M_1(x,t)u(x,t)dxdt - \int_0^T v(l,t)[\alpha_2u(0,t) + \beta_2u(l,t)]dt-$$
$$-\int_0^T v(l,t)\int_0^l M_2(x,t)u(x,t)dxdt = \int_0^l v(x,0)\psi(x)dx+$$
$$+\int_0^T\int_0^l fvdxdt + \int_0^T[v(0,t)\int_0^l P_1(x)fdx - v(l,t)\int_0^l P_2(x)fdx]dt \tag{12}$$

holds for every $v(x,t) \in \hat{W}_2^1(Q)$.

Theorem 2. *Let*

$$c(x,t) \in C(\bar{Q}), \ \ M_i(x,t) \in C(\bar{Q}), M_{it}(x,t) \in C(\bar{Q}), \ P_i(x) \in C[0,l],$$
$$f(x,t) \in L_2(Q), f_t(x,t) \in L_2(Q), \ \varphi(x) \in W_2^1(0,l), \ \psi(x) \in L_2(0,l),$$
$$\alpha_2 + \beta_1 = 0, \ \alpha_1\zeta_1^2 + (\beta_1 - \alpha_2)\zeta_1\zeta_2 - \beta_2\zeta_2^2 \geqslant 0 \ \ \forall z = (\zeta_1, \zeta_2).$$

Then there exists a unique generalized solution to Problem 2.

Proof. We start by choosing

$$v(x,t)=\begin{cases}\int\limits_{\tau}^{t}u(x,\eta)d\eta, & 0\leqslant t\leqslant\tau,\\ 0, & \tau\leqslant t\leqslant T\end{cases}\tag{13}$$

in the inequality (12) with $f(x,t)=0,\ \psi(x)=0$. After some simple manipulation we obtain:

$$\frac{1}{2}\int\limits_0^l[u^2(x,\tau)+v_x^2(x,0)]dx+\frac{1}{2}[\alpha_1v^2(0,0)+(\beta_1-\alpha_2)v(0,0)v(l,0)-\beta_2v^2(l,0)]=$$

$$=\int\limits_0^\tau\int\limits_0^l cvv_t dxdt-$$

$$-\int\limits_0^\tau v(0,t)\int\limits_0^l M_1(x,t)u(x,t)dxdt+\int\limits_0^\tau v(0,t)\int\limits_0^l M_2(x,t)u(x,t)dxdt.\tag{14}$$

In order to derive a priori estimate note that

$$v^2(x,t)\leqslant\tau\int\limits_0^\tau u^2(x,t)dt,\tag{15}$$

$$v^2(\xi_i,t)\leqslant 2l\int\limits_0^l v_x^2(x,t)dx+\frac{2}{l}\int\limits_0^l v^2(x,t)dx\tag{16}$$

where $\xi_1=0,\ \xi_2=l$. These inequalities follow easily from (13) and a relation

$$v(\xi_i,t)=\int\limits_0^{\xi_i}v_\xi(\xi,t)d\xi+v(x,t),\quad i=1,2.$$

Now by using Cauchy, Cauchy-Bunyakovskii inequalities as well as (15) and (16) under the conditions of Theorem 2 we get from (14):

$$\int\limits_0^l[u^2(x,\tau)+v_x^2(x,0)]dx\leqslant C_1\int\limits_0^\tau\int\limits_0^l u^2(x,t)dxdt+4l\int\limits_0^\tau\int\limits_0^l v_x^2(x,t)dxdt,\tag{17}$$

where $C_1>0$ depends only on $c(x,t),K_i(x)$ and T.

Introduce a function $w(x,t)=\int\limits_0^t u_x(x,\eta)d\eta$. It is easy to see that $v_x(x,t)=w(x,t)-w(x,\tau),\ v_x(x,0)=-w(x,\tau)$ and we get from (17) an inequality

$$\int\limits_0^l[u^2(x,\tau)+w^2(x,\tau)]dx\leqslant C_2\int\limits_0^\tau\int\limits_0^l[u^2(x,t)+w^2(x,t)dxdt\tag{18}$$

that is true for $\tau:\ 1-8l\tau>0$. Taking into account that τ is arbitrary set $\tau\in[0,\frac{1}{16l}]$. Then, by using the Gronwall lemma, we obtain: $u(x,t)=0$ for $t\in[0,\frac{1}{16l}]$. At a subsequent step we get $u(x,t)=0$ for $t\in[\frac{1}{16l},\frac{1}{8l}]$. By repeating the above argument several times, it follows that (see [11], p.212)

$$u(x,t)=0\ \forall(x,t)\in\bar{Q}.$$

This means that there exists at most one solution to Problem 2.

Let $w_k(x) \in C^2[0,l]$ be arbitrary system of linearly independent functions that is complete in $W_2^1(0,l)$. Without loss of generality we assume $(w_k, w_l)_{L_2(0,l)} = \delta_{kl}$.

We seek an approximate solution of Problem 2 in the form

$$u^m(x,t) = \sum_{k=1}^{m} c_k(t) w_k(x) \tag{19}$$

from relations

$$\int_0^l (u_{tt}^m w_j + u_x^m w_j' + c(x,t)u^m w_j)dx + w_j(0)\int_0^l K_1(x,t)u^m dx -$$

$$-w_j(l)\int_0^l K_2(x,t)u^m dx = \int_0^l f(x,t)w_j dx, \tag{20}$$

in addition,

$$c_k(0) = \alpha_k, \quad c_k'(0) = \beta_k \tag{21}$$

where α_k, β_k are coefficients of the sums

$$\varphi^m(x) = \sum_{k=1}^{m} \alpha_k w_k(x),\ \psi^m(x) = \sum_{k=1}^{m} \beta_k w_k(x),$$

approximating as $m \to \infty$ the functions $\varphi(x),\ \psi(x)$ in the norms $W_2^1(0,l)$ and $L_2(0,l)$ respectively. Under the conditions of Theorem 2 the Cauchy problem (20)—(21) has a unique solution such that $c_k''(t) \in L_1(0,T)$. It follows that a sequence $u^m(x,t)$ is constructed.

Let us now prove that this sequence converges and its limit is a required solution to Problem 2. To this end we need to derive an estimate. Multiplying (20) by $c_l'(t)$, summing with respect to l from 1 to m and integrating over $(0,\tau)$ we obtain:

$$\int_0^\tau\int_0^l (u_{tt}^m u_t^m + u_x^m u_{xt}^m + c(x,t)u^m u_t^m)dxdt + \int_0^\tau u_t^m(0,t)\int_0^l M_1(x,t)u^m(x,t)dxdt -$$

$$-\int_0^\tau u_t^m(l,t)\int_0^l M_2(x,t)u^m(x,t)dxdt + \int_0^\tau u_t^m(0,t)[\alpha_1 u^m(0,t) + \beta_1 u^m(l,t)]dt -$$

$$-\int_0^\tau u_t^m(l,t)[\alpha_2 u^m(0,t) + \beta_2 u^m(l,t)]dt =$$

$$= \int_0^\tau\int_0^l f(x,t)[u^m(x,t) + u^m(0,t)P_1(x) - u^m(l,t)P_2(x)]dxdt. \tag{22}$$

Consider first term in the left part of (22). Integrating by parts we obtain

$$\int_0^\tau\int_0^l (u_{tt}^m u_t^m + u_x^m u_{xt}^m + c(x,t)u^m u_t^m)dxdt = \int_0^\tau\int_0^l cu^m u_t^m dxdt +$$

$$+\frac{1}{2}\int\limits_0^l [(u_t^m(x,\tau))^2+(u_x^m(x,\tau))^2]dx-\frac{1}{2}\int\limits_0^l [(u_t^m(x,0))^2+(u_x^m(x,0))^2]dx.$$

Consider in more detail the terms generated by nonlocal conditions. At first we integrate by parts:

$$\int\limits_0^\tau u_t^m(\xi_i,t)\int\limits_0^l M_i(x,t)u^m(x,t)dxdt=-\int\limits_0^\tau u^m(\xi_i,t)\int\limits_0^l M_i(x,t)u_t(x,t)dxdt-$$

$$+\int\limits_0^\tau u^m(\xi_i,t)\int\limits_0^l M_{it}u^m(x,t)dxdt+u^m(\xi_i,t)\int\limits_0^l M_i(x,t)u^m(x,t)dx\Big|_0^\tau. \tag{23}$$

Now we can derive following inequalities:

$$\Big|\int\limits_0^\tau u^m(\xi_i,t)\int\limits_0^l M_i(x,t)u_t^m(x,t)dxdt\Big|\leqslant$$

$$\leqslant\frac{1}{2}\int\limits_0^\tau (u^m(\xi_i,t))^2dt+\frac{m_i^0}{2}\int\limits_0^\tau\int\limits_0^l (u_t^m(x,t))^2dxdt;$$

$$\Big|\int\limits_0^\tau u^m(\xi_i,t)\int\limits_0^l M_{it}(x,t)u_t^m(x,t)dxdt\Big|\leqslant$$

$$\leqslant\frac{1}{2}\int\limits_0^\tau (u^m(\xi_i,t))^2dt+\frac{m_i^1}{2}\int\limits_0^\tau\int\limits_0^l (u_t^m(x,t))^2dxdt;$$

$$\Big|u^m(\xi_i,t)\int\limits_0^l M_{it}(x,t)u_t^m(x,t)dx\Big|_0^\tau\Big|\leqslant\frac{1}{2}(u^m(\xi_i,\tau))^2+\frac{1}{2}(u^m(\xi_i,0))^2+$$

$$+\frac{m_i^0}{2}\int\limits_0^l (u^m(x,\tau))^2dx+\frac{m_i^0}{2}\int\limits_0^l (u^m(x,0))^2dx,$$

where $m_i^0=\max\limits_{[0,T]}\int\limits_0^l M_i^2(x,t)dx,\quad m_i^1=\max\limits_{[0,T]}\int\limits_0^l M_{it}^2(x,t)dx.$

As (see [11])

$$\begin{gathered}(u^m(\xi_i,\tau))^2\leqslant\varepsilon\int\limits_0^l (u_x^m(x,\tau))^2dx+c(\varepsilon)\int\limits_0^l (u^m(x,\tau))^2dx,\\ \int\limits_0^l (u^m(x,\tau))^2dx\leqslant 2\tau\int\limits_0^\tau\int\limits_0^l (u_t^m(x,t))^2dxdt+2\int\limits_0^l (u^m(x,0))^2dx\end{gathered} \tag{24}$$

then

$$\Big|\int\limits_0^\tau u_t^m(0,t)\int\limits_0^l M_1(x,t)u^m(x,t)dxdt\Big|+\Big|\int\limits_0^\tau u_t^m(l,t)\int\limits_0^l M_2(x,t)u^m(x,t)dxdt\Big|\leqslant$$

$$\leqslant \varepsilon\int\limits_0^l (u_x^m(x,\tau))^2dx + C_3\int\limits_0^\tau\int\limits_0^l [(u_x^m(x,t))^2+(u_t^m(x,t))^2+(u^m(x,t))^2]dxdt+$$
$$+C_4\int\limits_0^l [u_x^m(x,0)]^2+(u^m(x,0))^2]dx. \tag{25}$$

Consider now next two terms and after some manipulation we obtain:

$$\int\limits_0^\tau u_t^m(0,t)[\alpha_1u^m(0,t)+\beta_1u^m(l,t)]dt-\int\limits_0^\tau u_t^m(l,t)[\alpha_2u^m(0,t)+\beta_2u^m(l,t)]dt=$$
$$=\frac{1}{2}[\alpha_1(u^m(0,\tau))^2+(\beta_1-\alpha_2)u^m(0,\tau)u^m(l,\tau)-\beta_2(u^m(l,\tau))^2]-$$
$$-\frac{1}{2}[\alpha_1(u^m(0,0))^2+(\beta_1-\alpha_2)u^m(0,0)u^m(l,0)-\beta_2(u^m(l,0))^2]. \tag{26}$$

Then using Cauchy inequality and (24) we get:

$$\Big|\int\limits_0^\tau\int\limits_0^l f(x,t)[u^m(x,t)+u^m(0,t)P_1(x)-u^m(l,t)P_2(x)]dxdt\Big|\leqslant$$
$$\leqslant\frac{1}{2}\int\limits_0^\tau\int\limits_0^l (u^m(x,t))^2dxdt+C_5\int\limits_0^\tau\int\limits_0^l (u_x^m(x,t))^2dxdt+$$
$$+\frac{3}{2}\int\limits_0^\tau\int\limits_0^l f^2(x,t)dxdt \tag{27}$$

where $C_5=\max\limits_i\{\int\limits_0^l P_i^2(x)dx\}$.

Letting now $\varepsilon=\frac{1}{4}$ in (25) we get from (22), (25), (26), (27) required estimate:

$$||u^m(x,t)||_{W_2^1(Q)}\leqslant L, \tag{28}$$

where L does not depend on m.

The above-proved estimate implies that we can extract a subsequence $\{u^{m_k}(x,t)\}$ from $\{u^m(x,t)\}$ such that $u^{m_k}(x,t)$ converges weakly to $u(x,t)\in W_2^1(Q)$. It remains to show that this limit function is the required solution to Problem 2. For this purpose multiply (20) by $d_l(t)\in C^2[0,T]$ with $d_l(T)=0$. After summing over l from 1 to m_k and integrating over $[0,T]$ we get an equality

$$\int\limits_0^T\int\limits_0^l(-u_t^{m_k}\eta_t^k+u_x^{m_k}\eta_x^k+cu^{m_k}\eta^k)dxdt+\int\limits_0^T\eta^k(0,t)[\alpha_1u^{m_k}(0,t)+\beta_1u^{m_k}(l,t)]dt+$$
$$+\int\limits_0^T\eta^k(0,t)\int\limits_0^l M_1(x,t)u^{m_k}(x,t)dxdt-\int\limits_0^T\eta^k(l,t)[\alpha_2u^{m_k}(0,t)+\beta_2u^{m_k}(l,t)]dt-$$
$$-\int\limits_0^T\eta^k(l,t)\int\limits_0^l M_2(x,t)u^{m_k}(x,t)dxdt=\int\limits_0^l\eta^k(x,0)u^{m_k}(x,0)dx+$$

$$+\int\limits_0^T\int\limits_0^l f(x,t)\eta^k(x,t)dxdt+$$

$$+\int\limits_0^T[\eta^k(0,t)\int\limits_0^l P_1(x)fdx-\eta^k(l,t)\int\limits_0^l P_2(x)f(x,t)dx]dt \tag{29}$$

that is true for every $\eta^k(x,t)=\sum\limits_{l=1}^{m_k}d_l(t)w_l(x)$.

By taking into account the above-proved weak convergence of the subsequence $\{u^{m_k}(x,t)\}$ in $W_2^1(Q)$, one can pass in (29) to the limit as $m_k\Rightarrow\infty$ and certain $\eta^k(x,t)$ is fixed. Denote the set of all $\eta^k(x,t)$ by $\mathcal{M}_k$. Since $\bigcup\limits_{k=1}^{\infty}\mathcal{M}_k$ is dense in $\hat{W}_2^1(Q)$ ([11], p.215) it follows that (29) holds for any $v(x,t)\in\hat{W}_2^1(Q)$ which implies that the required solution to Problem 2 exists.

The solvability of the main problem 1 follows from Lemma and Theorem 2. The proof of Theorem 1 is completed.

Remark. All results are true for more general hyperbolic equation $u_{tt}-(a(x,t)u_x)_x+c(x,t)u=f(x,t)$ if $a(x,t),a_t(x,t)\in L_2(Q)$.

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