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THRESHOLD EIGENFUNCTIONS AND THRESHOLD RESONANCES OF SOME RELATIVISTIC OPERATORS

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We give a review of recent developments on the study of threshold eigenfunctions and threshold resonances of magnetic Dirac operators and Pauli operators. Emphasis is placed on a proof of the non-existence of threshold resonances of the magnetic Dirac operators in a concise manner.

Keywords: Dirac operators, magnetic potentials, threshold energies, threshold resonances, threshold eigenfunctions, zero modes.

1. Threshold eigenfunctions

The relativistic operators we consider are magnetic Dirac operators

$$H_A = \alpha \cdot \left(\frac{1}{i} \nabla_x - A(x) \right) + m\beta, \quad x \in \mathbb{R}^3, \quad (1.1)$$

and Pauli operators

$$P_A = \sum_{j=1}^3 \left(\frac{1}{i} \frac{\partial}{\partial x_j} - A_j(x) \right)^2 - \sigma \cdot B, \quad x \in \mathbb{R}^3. \quad (1.2)$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \quad (j = 1, 2, 3)$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}.$$

The constant m is assumed to be positive. $A(x) = (A_1(x), A_2(x), A_3(x))$ is a vector potential, and $B = \nabla \times A$ is the magnetic field. By $\alpha \cdot \left(\frac{1}{i} \nabla_x - A(x) \right)$ in (1.1), we mean

$$\sum_{j=1}^3 \alpha_j \left(\frac{1}{i} \frac{\partial}{\partial x_j} - A_j(x) \right),$$

and similarly by $\sigma \cdot B$ in (1.2) we mean $\sum_{j=1}^3 \sigma_j B_j$.

We need to make various assumptions on the vector potential in the present section. It is notable that all these assumptions share one common feature that each component of A is a

real-valued function decaying at infinity in a certain sense. Therefore, any of these assumptions assures that H_A is essentially self-adjoint on $[C_0^\infty(\mathbb{R}^3)]^4$. The unique self-adjoint realization in the Hilbert space $[L^2(\mathbb{R}^3)]^4$ will be denoted by H_A again. Note that the domain of the self-adjoint operator H_A is given by the Sobolev space of order 1, *i.e.* $\text{Dom}(H_A) = [H^1(\mathbb{R}^3)]^4$. It is straightforward that the spectrum of the self-adjoint operator H_A is given as follows:

$$\sigma(H_A) = (-\infty, -m] \cup [m, \infty).$$

By the threshold energies of H_A , we mean the values m and $-m$.

It is a natural question whether these threshold energies become eigenvalues of H_A . It is well-known that $\pm m$ are generically not the eigenvalues of H_A . Precise description of this fact is given as follows.

Theorem 1.1 (Balinsky-Evans-Saitō-Umeda). *The set*

$$\{ A \in [L^3(\mathbb{R}^3)]^3 \mid \text{Ker}(H_A \mp m) = \{0\} \}$$

contains an open and dense subset of $[L^3(\mathbb{R}^3)]^3$.

For the proof of Theorem 1.1, see Balinsky-Evans [3, Theorem 2], together with Saitō-Umeda [13, Corollary 2.1 and Theorem 4.2]. Theorem 1.1 says that the set of vector potentials which give rise to threshold eigenfunctions is sparse. A similar result in a different class of vector potentials also holds true. Actually, Elton [8] analyzed the structure of the set of vector potentials which produce threshold eigenfunctions.

Theorem 1.2 (Elton-Saitō-Umeda). *Let \mathcal{A} be the Banach space defined by*

$$\mathcal{A} := \{ A \in [C^0(\mathbb{R}^3)]^3 \mid |A(x)| = o(|x|^{-1}) \text{ as } |x| \rightarrow \infty \}$$

equipped with the norm

$$\|A\|_{\mathcal{A}} = \sup_x \{ \langle x \rangle |A(x)| \}.$$

Define

$$\mathcal{Z}_k^\pm = \{ A \in \mathcal{A} \mid \dim(\text{Ker}(H \mp m)) = k \}$$

for $k = 0, 1, 2, \dots$. Then

- (i) $\mathcal{Z}_k^+ = \mathcal{Z}_k^-$ for all k , and $\mathcal{A} = \bigcup_{k \geq 0} \mathcal{Z}_k^\pm$.
- (ii) \mathcal{Z}_0^\pm is an open and dense subset of \mathcal{A} .
- (iii) For any k and any open subset $\Omega (\neq \emptyset)$ of \mathbb{R}^3 there exists an $A \in \mathcal{Z}_k^\pm$ such that $A \in [C_0^\infty(\Omega)]^3$.
- (iv) For $k = 1, 2$ the set \mathcal{Z}_k^\pm is a smooth sub-manifold of \mathcal{A} with co-dimension k^2 .

For the proof of Theorem 1.2, see Elton [8, Theorems 1 and 2], together with Saitō-Umeda [13, Corollary 2.1 and Theorem 5.2].

By simple computations, one can see that $P_A = \{ \sigma \cdot (\frac{1}{i} \nabla - A) \}^2$. Hence one can define the Friedrichs extension in $[L^2(\mathbb{R}^3)]^2$ of P_A on $[C_0^\infty(\mathbb{R}^3)]^2$ under appropriate assumptions on A and B . Balinsky-Evans [2, Theorem 4.2] showed the following result.

Theorem 1.3 (Balinsky-Evans). *The set*

$$\{ B \in [L^{3/2}(\mathbb{R}^3)]^3 \mid \text{Ker}(P_A) = \{0\} \text{ and } \nabla \times A = B \}$$

contains an open and dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

As was shown in [2, Lemma 2.2], there exists a unique vector potential A such that $A \in [L^3(\mathbb{R}^3)]^3$, $\nabla \times A = B$ and $\operatorname{div} A = 0$. Theorem 1.3 says that the set of magnetic fields which give rise to a *zero mode* of P_A (an eigenfunction of P_A corresponding to the eigenvalue 0) is sparse.

It is not only necessary but also important to mention examples of vector potentials $A(x)$ which yield threshold eigenfunctions of H_A as well as P_A . For this purpose, it is convenient to introduce Weyl-Dirac operator

$$W_A = \sigma \cdot \left(\frac{1}{i} \nabla_x - A(x) \right).$$

When A is sufficiently smooth, it is well-known (Thaller [16, p. 195, Theorem 7.1]) that

$$W_A \varphi = 0 \iff H_A \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = m \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \iff H_A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = -m \begin{pmatrix} 0 \\ \varphi \end{pmatrix}.$$

Since $P_A = W_A^2$ (see the paragraph before Theorem 1.3), it follows that

$$P_A \varphi = 0 \iff W_A \varphi = 0,$$

provided that A is in $[C^1(\mathbb{R}^3)]^3$ and satisfies an appropriate condition. Therefore, it is apparent that a zero mode of W_A provides a threshold eigenfunction of H_A corresponding to either one of the energies $\pm m$, as well as a threshold eigenfunction of P_A corresponding to the energy 0.

Example 1.1 (Loss-Yau). Define

$$A(x) = 3\langle x \rangle^{-4} \{ (1 - |x|^2)w_0 + 2(w_0 \cdot x)x + 2w_0 \times x \} \quad (1.3)$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$, $\phi_0 = {}^t(1, 0)$, and

$$w_0 = \phi_0 \cdot (\sigma \phi_0) := ((\phi_0, \sigma_1 \phi_0)_{\mathbb{C}^2}, (\phi_0, \sigma_2 \phi_0)_{\mathbb{C}^2}, (\phi_0, \sigma_3 \phi_0)_{\mathbb{C}^2}).$$

Note that $w_0 \cdot x$ and $w_0 \times x$ denotes the inner product and the exterior product of \mathbb{R}^3 respectively. Then

$$\varphi(x) := \langle x \rangle^{-3} (I_2 + i\sigma \cdot x) \phi_0 \quad (1.4)$$

is a zero mode of the Weyl-Dirac operator W_A .

Following and developing the ideas in [9], Adam-Muratori-Nash [1] constructed a series of vector potentials which give rise to zero modes of the corresponding Weyl-Dirac operators. All of their vector potentials share the property that $|A(x)| \leq C\langle x \rangle^{-2}$ with the one given by (1.3).

Recently, Saitō-Umeda [14] found an interesting connection between the series of the zero modes constructed in [1] and a series of solvable polynomials.

It follows that the zero mode defined by (1.4) has the asymptotic limits $\varphi(x) \asymp |x|^{-2}$ as $|x| \rightarrow \infty$. According to Theorem 1.4 below, every zero mode exhibits the same asymptotic limit.

Theorem 1.4 (Saitō-Umeda). *Suppose that $|A(x)| \leq C\langle x \rangle^{-\rho}$, $\rho > 1$. Let φ be a zero mode of the Weyl-Dirac operator W_A . Then for any $\omega \in \mathbb{S}^2$, the unit sphere of \mathbb{R}^3 ,*

$$\begin{aligned} & \lim_{r \rightarrow +\infty} r^2 \varphi(r\omega) \\ &= \frac{i}{4\pi} \int_{\mathbb{R}^3} \{ (\omega \cdot A(y)) I_2 + i\sigma \cdot (\omega \times A(y)) \} \varphi(y) dy, \end{aligned}$$

where the convergence being uniform with respect to $\omega \in \mathbb{S}^2$.

See Saitō-Umeda [11] for the proof of Theorem 1.4.

Theorem 1.4 excludes the case $\rho = 1$. In Balinsky-Evans-Saitō [6], they were successful to derive estimates for zero modes of the Dirac operator of the form $H_Q = \alpha \cdot \frac{1}{i}\nabla + Q(x)$ with $Q(x) = O(|x|^{-1})$ as $|x| \rightarrow \infty$, where $Q(x)$ is a 4×4 -matrix-valued function. Their estimates are as follows:

$$\int_{|x| \geq 1} \{|x|^{-2}|\varphi(x)|\}^k |x|^{-6} dx < \infty$$

for any $k \in [1, 10/3)$.

2. Threshold resonances

To define threshold resonances, we introduce a weighted Hilbert space

$$L^{2,-s}(\mathbb{R}^3) = \{u \mid \|\langle x \rangle^{-s}u\|_{L^2} < \infty\}.$$

A \mathbb{C}^4 -valued function f is said to be an m -resonance (resp. a $-m$ -resonance) if and only if f belongs to $[L^{2,-s}(\mathbb{R}^3)]^4 \setminus [L^2(\mathbb{R}^3)]^4$ for some $s \in (0, 3/2]$ and satisfies $H_A f = mf$ (resp. $H_A f = -mf$) in the distributional sense. By a threshold resonance of H_A , we mean an m -resonance or a $-m$ -resonance.

Theorem 2.1 (Saitō-Umeda). *Suppose that $|A(x)| \leq C\langle x \rangle^{-\rho}$, $\rho > 3/2$. Let $f = {}^t(\varphi^+, \varphi^-) \in [L^{2,-s}(\mathbb{R}^3)]^4 = [L^{2,-s}(\mathbb{R}^3)]^2 \oplus [L^{2,-s}(\mathbb{R}^3)]^2$ for some s with $0 < s < \min(1, \rho - 1)$.*

- (i) *If f satisfies $H_A f = mf$ in the distributional sense, then $f \in [H^1(\mathbb{R}^3)]^4$ and $\varphi^- = 0$.*
- (ii) *If f satisfies $H_A f = -mf$ in the distributional sense, then $f \in [H^1(\mathbb{R}^3)]^4$ and $\varphi^+ = 0$.*

Theorem 2.1 implies the non-existence of the threshold resonance of H_A as far as $\rho > 3/2$ and $0 < s < \min(1, \rho - 1)$. As was mentioned in Section 1, the vector potentials by Loss-Yau [9] and by Adam-Muratori-Nash [1] satisfy the inequality $|A(x)| \leq C\langle x \rangle^{-2}$. Therefore, these vector potentials do not yield $\pm m$ -resonances.

As an easy corollary to Theorem 2.1, one can get the following result, which seems a more natural formulation of the non-existence of threshold resonances from the physics point of view.

Corollary 2.1. *Suppose that $f \in [L^2_{loc}(\mathbb{R}^3)]^4$ and that*

$$f(x) = C_1|x|^{-1} + C_2|x|^{-2} + o(|x|^{-2}) \text{ as } |x| \rightarrow \infty.$$

If f satisfies either of $H_A f = \pm mf$ in the distributional sense, then $C_1 = 0$.

We shall give an outline of the proof of Theorem 2.1. Although the reader can find the proof in Saitō-Umeda [13], it heavily relies on Saitō-Umeda [12], hence the whole story of the proof is separated into two different papers. For this reason, we believe that it is beneficial to illustrate the whole story in the present article in a concise manner.

Before giving the outline, we prepare two lemmas.

Lemma 2.1. *Let K be an integral operator define by*

$$K\varphi(x) = \int_{\mathbb{R}^3} i \frac{\sigma \cdot (x - y)}{4\pi|x - y|^3} \varphi(y) dy.$$

Then $K(\sigma \cdot \frac{1}{i}\nabla)\varphi = \varphi$ if $\varphi \in [L^{2,-3/2}(\mathbb{R}^3)]^2$ and $(\sigma \cdot \frac{1}{i}\nabla)\varphi \in [L^{2,t}(\mathbb{R}^3)]^2$ for some $t > 1/2$.

Proof. We give a formal proof. A rigorous proof can be found in [12, Section 4].

Note that $K = (\sigma \cdot \frac{1}{i}\nabla)I_2 = I_2(\sigma \cdot \frac{1}{i}\nabla)$, where I_2 denotes the Riesz potential (cf. Stein [15, Chapter V]). It follows that

$$K(\sigma \cdot \frac{1}{i}\nabla)\varphi = I_2(\sigma \cdot \frac{1}{i}\nabla)^2\varphi = I_2(-\Delta)\varphi = \varphi,$$

since $(\sigma \cdot \frac{1}{i}\nabla)^2 = -\Delta$. Here the anti-commutation relations $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I_2$ were used. \square

Lemma 2.2. *If $t \geq 1$, then the Riesz potential I_1 is a bounded operator from $L^{2,t}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$.*

Proof. Let $\hat{u} \in C_0^\infty(\mathbb{R}^3)$, where \hat{u} denotes the Fourier transform of u . Then

$$\|I_1 u\|_{L^2}^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{1}{|\xi|^2} |\hat{u}(\xi)|^2 d\xi \leq \frac{1}{\pi^2} \int_{\mathbb{R}^3} |\nabla_\xi \hat{u}(\xi)|^2 d\xi \leq \frac{1}{\pi^2} \|u\|_{L^{2,1}}^2,$$

where we have used the Hardy inequality with respect to ξ variable. See, e.g., [5, p. 19] for the Hardy inequality. \square

Outline of the proof of Theorem 2.1. We shall give the proof only for an m -resonance. The proof for a $-m$ -resonance is similar and shall be omitted.

Let $f = {}^t(\varphi^+, \varphi^-)$ be in $[L^{2,-s}(\mathbb{R}^3)]^4$ and satisfy $H_A f = m f$ in the distributional sense. We then have

$$\begin{cases} m\varphi^+ + \sigma \cdot (\frac{1}{i}\nabla - A(x))\varphi^- = m\varphi^+ \\ \sigma \cdot (\frac{1}{i}\nabla - A(x))\varphi^+ - m\varphi^- = m\varphi^- \end{cases}$$

in the distributional sense, which is equivalent to

$$\begin{cases} \sigma \cdot (\frac{1}{i}\nabla - A(x))\varphi^- = 0 \\ \sigma \cdot (\frac{1}{i}\nabla - A(x))\varphi^+ = 2m\varphi^- \end{cases} \quad (2.1)$$

in the distributional sense. Since $\varphi^- \in [L^{2,-s}(\mathbb{R}^3)]^2$, it follows from the first equation in (2.1) that

$$(\sigma \cdot \frac{1}{i}\nabla)\varphi^- = (\sigma \cdot A)\varphi^- \in [L^{2,\rho-s}(\mathbb{R}^3)]^2, \quad (2.2)$$

where $\rho - s > 1$. Hence Lemma 2.1 is applicable to φ^- , and we have

$$\varphi^- = K(\sigma \cdot \frac{1}{i}\nabla)\varphi^- = K(\sigma \cdot A)\varphi^-. \quad (2.3)$$

It follows from (2.3) that

$$|\varphi^-(x)| \leq \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|^2} |(\sigma \cdot A)(y)\varphi^-(y)| dy = \frac{\pi}{2} I_1(|(\sigma \cdot A)\varphi^-|)(x). \quad (2.4)$$

The inequality (2.4), together with Lemma 2.2, implies that $\varphi^- \in [L^2(\mathbb{R}^3)]^2$. Noting that $(\sigma \cdot \frac{1}{i}\nabla)\varphi^- \in [L^2(\mathbb{R}^3)]^2$ by (2.2), we can conclude that $\varphi^- \in [H^1(\mathbb{R}^3)]^2$.

On the other hand, it follows from the second equation of (2.1) that

$$(\sigma \cdot \frac{1}{i}\nabla)\varphi^+ = 2m\varphi^- + (\sigma \cdot A)\varphi^+ \in [L^2(\mathbb{R}^3)]^2. \quad (2.5)$$

To conclude that $\varphi^- = 0$, we need to show that

$$((\sigma \cdot \frac{1}{i}\nabla)\varphi^+, \varphi^-)_{[L^2]^2} = ((\sigma \cdot A)\varphi^+, \varphi^-)_{[L^2]^2}, \quad (2.6)$$

(see Remark 2.1 below). In fact, combining (2.5) with (2.6) and noting that $m > 0$, we can conclude that $\varphi^- = 0$.

The fact that $\varphi^- = 0$, together with the second equality in (2.1), implies that $\sigma \cdot (\frac{1}{i}\nabla - A)\varphi^+ = 0$. It is now evident that one can repeat the same arguments above for φ^- to conclude that $\varphi^+ \in [H^1(\mathbb{R}^3)]^2$. \square

Remark 2.1. A rigorous proof of (2.6) can be found in [13, Lemma 6.1], where the condition $s < 1$ is used. Since the proof of (2.6) is lengthy, we prove it by a formal argument as follows:

$$\begin{aligned} ((\sigma \cdot \frac{1}{i}\nabla)\varphi^+, \varphi^-)_{[L^2]^2} &= (\varphi^+, (\sigma \cdot \frac{1}{i}\nabla)\varphi^-)_{[L^2]^2} \\ &= (\varphi^+, (\sigma \cdot A)\varphi^-)_{[L^2]^2} \\ &= ((\sigma \cdot A)\varphi^+, \varphi^-)_{[L^2]^2}. \end{aligned} \tag{2.7}$$

Here we have made integration by part in a formal manner in the first equality in (2.7), and in the second equality in (2.7) we have used the first equality in (2.1).

As for the non-existence of zero-resonances of Pauli operators, Morita [10] recently obtained the following result.

Theorem 2.2 (Morita). *Suppose that $A \in [C^\infty(\mathbb{R}^3)]^3$ and that*

$$|A_j(x)| + |\nabla A_j(x)| \leq C\langle x \rangle^{-\rho}, \quad \rho \geq 2$$

for $j = 1, 2, 3$. If $\varphi \in [H^{1,-s}(\mathbb{R}^3)]^2$ for some s with $0 < s \leq 1$ and satisfies $P_A\varphi = 0$ in the distributional sense, then $\varphi \in [H^1(\mathbb{R}^3)]^2$. Here

$$H^{1,-s}(\mathbb{R}^3) = \{u \mid \|\langle x \rangle^{-s}u\|_{L^2} + \|\langle x \rangle^{-s}\nabla u\|_{L^2} < \infty\}.$$

One should note that $L^2(\mathbb{R}^3) \not\subset H^{1,-s}(\mathbb{R}^3)$, but $H^{1,-s}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3) \neq \emptyset$. Obviously, there is room for improvement in Theorem 2.2.

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