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Original article

Translation-invariant *p*-adic quasi Gibbs measures for the Potts model with an external field on the Cayley tree

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ABSTRACT The study is focused on investigation of p-adic Gibbs measures for the q-state Potts model with an external field and determination of the conditions for the existence of a phase transition. In this work, we derive a functional equation that satisfies the compatibility condition for p-adic quasi-Gibbs measures on a Cayley tree of order $k \geq 2$. Furthermore, we prove that if $|q|_p = 1$ there exists a unique p-adic Gibbs measure for this model. Additionally, for the Potts model on a binary tree, we identify three p-adic quasi-Gibbs measures under specific circumstances: one bounded and two unbounded, which implies a phase transition.

KEYWORDS p-adic numbers, the Potts model with external field, p-adic quasi Gibbs measure, translation-invariant, Cayley tree

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1. Introduction

A comprehensive understanding of the interactions between individual atoms and molecules within nanosystems, along with their statistical mechanical modeling, is crucial for the development in nanotechnology [1]. To formulate the thermodynamics of small systems, one has to start evaluating thermodynamics from the first principles reviewing the concepts, laws, definitions, and formulations, and to draw a set of guidelines for their applications to small systems [2–5]. Such questions as property relations and phase transitions in small (nano) systems are subjects to be investigated and formulated provided the formulation of working equations of thermodynamics and statistical mechanics of small systems. It is worth mentioning that the molecular self-assembly (bottom-up technology) that was originally proposed by Feynman [6] has its roots in phase transitions.

p-adic probabilities, a novel concept in theoretical physics, have spontaneously appeared in physical models based on p-adic numbers, similar to the p-adic string, first proposed by I. Volovich [7]. In [8], a theory of stochastic processes was developed for values in p-adic and more general non-Archimedean fields. These processes have probability distributions with non-Archimedean values. A non-Archimedean analog of the Kolmogorov theorem was established, enabling the construction of a wide range of stochastic processes using finite-dimensional probability distributions. This foundation has opened the door for investigating and developing certain problems in statistical mechanics within the framework of p-adic probability theory.

The Potts model is a statistical mechanics model that generalizes the Ising model to allow for more than two components [9]. It has been extensively studied in recent years due to its rich mathematical structure and its applications to various physical systems [10, 11]. The studies in [12–16] for the Ising, in [17–19] for the Potts have contributed to our understanding of these models. Note that papers [20–23] are focused on translation-invariant p-adic Gibbs measures In [24–29], different aspects or specific cases of non-periodic, constructive p-adic quasi-Gibbs measures for the Ising and Potts models are explored.

In this paper, we investigate translation-invariant p-adic quasi Gibbs measures for the Potts model with an external field. The theory immediately shows the effect of an external force. For example, in [30], translation-invariant p-adic Gibbs measures were investigated in the Ising model with an external field, and a phase transition was identified for $p \equiv 1 (\mathrm{mod}\, 4)$. In [31], weakly periodic Gibbs measures were investigated for the same model, and the existence of a phase transition was shown for any odd prime number. Moreover, in [22], it was proved: if $|q|_p = 1$, then there is no translation-invariant p-adic Gibbs measure for the Potts model corresponding to $\mathbf{h}_x$ on the set $\mathcal{E}_p \setminus \{1\}$. However, we prove that, if $|q|_p = 1$, then there is a unique p-adic Gibbs measure for the Potts model with an external field. Therefore, we apply those ideas to a more complicated situation.

The purpose of this research is to examine p-adic Gibbs measures for the q-state Potts model with an external field and to provide sufficient conditions for a phase transition. In contrast to a real case, such measures for the model are not explained in a p-adic setting. In this work, we have derived a functional equation satisfying the compatible condition for p-adic quasi-Gibbs measures on a Cayley tree of order k for the given model. Moreover, we have proved the existence of a unique p-adic Gibbs measure for this model. Additionally, for the Potts model on a binary tree, we have determined under the some specific cases three p-adic quasi Gibbs measures which one of them is bounded, and others are unbounded and derived a new conditions for the existence of a phase transition.

1.1. p-adic numbers

Let $\mathbb{Q}$ be a field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \dfrac{n}{m}$ where, $r, n \in \mathbb{Z}$, m is a positive integer, and (n, p) and (m, p), where number r is called a p-order of x and it is denoted by $ord_p(x) = r$. The p-adic norm of x is given by

$$| x|_p = \begin{cases} p^{-r}, \ x \neq 0, \\ 0, \quad x = 0. \end{cases}$$

The norm of $| \, . \, |_p$ is non-Archimedean, i.e., it satisfies the strong triangle inequality:

$$| x + y |_p \leq \max\{| x |_p, | y |_p\}, \quad \forall x, y \in \mathbb{Q}.$$

We note that the following essential properties are relevant to the non-Archimedeanity of the norm:
i) if $| x|_p \neq | y|_p$, then $| x \pm y |_p = \max\{| x|_p, | y|_p\}$;
ii) if $| x|_p = | y|_p$, then $| x - y|_p \leq | x|_p$.
The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form $x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$, where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0, \ 0 \leq x_j \leq p - 1$. In this case $| x|_p \ = p^{-\gamma(x)}$.

An integer $b \in \mathbb{Z}$ is called *quadratic residue modulo p* if the congruent equation $x^2 \equiv b (\bmod\, p)$ has a solution $x \in \mathbb{Z}$.

Let p be odd prime and a be an integer not divisible by p. The *Legendre symbol* (see [32]) is defined by

$$\left(\frac{b}{p}\right) = \begin{cases} \ \ 1, & \text{if } b \text{ is quadratic residue of } p, \\ -1, & \text{if } b \text{ is quadratic nonresidue of } p. \end{cases} \tag{1}$$

Let $a \in \mathbb{Q}_p$, $a \neq 0, \ a = p^{\gamma(a)}(a_0 + a_1 p + a_2 p^2 + ...), \ 0 \leq a_j \leq p - 1, j \in \mathbb{N}, \ a_0 > 0.$

Lemma 1. [33] *The equation $x^2 = a$ has a solution in $x \in \mathbb{Q}_p$ iff the followings hold:*
i) $\gamma(a)$ is even;
ii) a_0 is a quadratic residue modulo p if $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

Lemma 2.(Hensel's lemma [34]) *Let $f(x) = c_0 + c_1 x + ... + c_n x^n$ be a polynomial whose coefficients are p-adic integers. Let $f'(x) = c_1 + 2c_2 x + ... + nc_n x^{n-1}$ be the derivative of $f(x)$. Let x^* be a p-adic integer such that $f(x^*) \equiv 0 (\bmod\, p)$ and $f'(x^*) \not\equiv 0 (\bmod\, p)$. Then there exists a unique p-adic integer root x_* such that*

$$f(x_*) = 0 \ \ and \ \ x_* \equiv x^* (\bmod\, p).$$

In [35], the authors introduced new symbols, "O" and "o", which simplify certain calculations. Essentially, these symbols help us to write down the calculations in our work more concisely. To understand their meanings, one can note: for a given p-adic number x, $O[x]$ refers to a p-adic number whose norm satisfies $| \ x \ |_p = | \ O[x] \ |_p$. On the other hand, $o[x]$ refers to a p-adic number such that $| \ o[x] \ |_p < | \ x \ |_p$. For example, if $x = 1 + p + p^3$, we write $O[1] = x$, $o[1] = x - 1$ or $o[p^2] = x - 1 - p$.

For any $a \in \mathbb{Q}_p$ and $r > 0$, we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : | \ x - a|_p < r\},$$

and the set

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p :| \ x \ |_p \leq 1\}, \ \mathbb{Z}_p^* = \mathbb{Z}_p \backslash p\mathbb{Z}_p.$$

$\mathbb{Z}_p$ is called the set of p- adic integers, $\mathbb{Z}_p^*$ is called the set of p- adic units. Note that the p-adic exponential is defined by the series

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, \frac{1}{2})$ if $p = 2$ and $x \in B(0, 1)$ if $p \neq 2$. For simplicity of notation, we write $\exp(x)$ instead of $\exp_p(x)$.
Put

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : \mid x - 1 \mid_p < p^{-1/(p-1)} \right\}.$$

A more thorough explanation of p-adic calculus and p-adic mathematical physics is provided in [36, 37].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, A_2, ..., A_n \in \mathcal{B}$ such that
$A_i \cap A_j = \emptyset,\ i \neq j$, the following holds:

$$\mu \left(\bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu(A_j).$$

If $\mu(X) = 1$, then a p-adic measure is called *probability*. One of the important conditions is boundedness, namely, a p-adic measure μ is called *bounded* if $\sup\{\mid \mu(A)|_p : A \in \mathcal{B}\} < \infty$. For more detail information about p-adic measures we refer to [36, 38].

1.2. Cayley Tree

Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree [39] of order $k \geq 1$ with the root $x^0 \in V$. Here V is the set of vertices and L is the set of edges. The vertices x and y are referred to as *nearest neighbors* when there is an edge l connecting them and this is shown by the notation $l = \langle x, y \rangle$. Note that each vertex of Γ^k_+ has exactly $k+1$ nearest neighbors, except for the root x^0, which has k nearest neighbors. A collection of the pairs $\langle x, x_1 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y. The distance $d(x, y)$ on the Cayley tree is the length (number of edges) of the shortest path from x to y.

Let us set

$$W_n = \{x \in V : \ d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m,$$

$$L_n = \{\langle x, y \rangle \in L : \ x, y \in V_n\}.$$

We introduce a coordinate structure in Γ^k_+: every vertex x (except for x^0) of Γ^k_+ has coordinates $(i_1, \ldots, i_n)$, here $i_m \in \{1, \ldots, k\},\ 1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \ldots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice. Let us define on Γ^k_+ binary operation $\circ : \Gamma^k_+ \times \Gamma^k_+ \to \Gamma^k_+$ as follows: for any two elements $x = (i_1, \ldots, i_n)$ and $y = (j_1, \ldots, j_m)$ put

$$x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m) \tag{2}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \ldots, i_n) \circ (0) = (i_1, \ldots, i_n). \tag{3}$$

By means of the defined operation Γ^k_+ becomes a noncommutative semigroup with a unit. Let us denote this group $(G^k, \circ)$. Using this semigroup structure one defines translations $\tau_g : G^k \to G^k,\ g \in G_k$ by

$$\tau_g(x) = g \circ x.$$

It is clear that $\tau_{(0)} = id$.

Let $G \subset G^k$ be a sub-semigroup of G^k and $h : G^k \to Y$ be a Y-valued function defined on G^k. We say that h is G-*periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in G^k$. We say that any G^k-periodic function is *translation-invariant*.

Now, for each $m \geq 2$ we put

$$G_m = \{x \in G^k : d(x, x^0) \equiv 0 (\operatorname{mod} m)\}. \tag{4}$$

It is easy to verify that G^k is a sub-semigroup of G_m.

2. p-adic quasi Gibbs measure for the Potts model

Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\Phi = \{1, 2, ..., q\}$ be a finite set. A configuration σ on V is defined as $x \in V \mapsto \sigma(x) \in \Phi$. The set of all configurations coincides with the set $\Omega = \Phi^V$ For given configurations $\sigma \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$, we define their concatenation by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), \text{ if } \ x \in V_{n-1}, \\ \omega(x), \ \text{ if } x \in W_n. \end{cases}$$

It is clear that $\sigma \vee \omega \in \Omega_{V_n}$.

We consider *p-adic q-state Potts model* on a Cayley tree with an external field.

The (formal) Hamiltonian of p-adic Potts model is

$$H(\sigma) = J \sum_{\langle x,\, y\rangle \in L} \delta_{\sigma(x)\sigma(y)} + \alpha \sum_{x\in V} \delta_{q\sigma(x)}, \tag{5}$$

where $J, \alpha \in B(0, p^{-1/(p-1)})$ are constant, $\langle x,\, y\rangle$ stands for nearest neighbor vertices and δ_{ij} is the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Assume that $h : V \to \mathbb{Q}_p^{|\Phi|}$ is a mapping, i.e. $h_x = (h_{1,x}, h_{2,x}, ..., h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ $(i \in \Phi)$ and $x \in V$. Given $n \in \mathbb{N}$, we consider a p-adic probability measure $\mu_{h,\sigma}^{(n)}$ on Ω_{V_n} defined by

$$\mu_h^{(n)}(\sigma) = \frac{1}{Z_n^{(h)}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}, \tag{6}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(h)}$ is the corresponding normalizing factor or a partition function given by

$$Z_n^{(h)} = \sum_{\sigma\in\Omega_{V_n}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{7}$$

We say that p-adic probability distributions (6) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega\in\Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \omega) = \mu_h^{(n-1)}(\sigma_{n-1}) \tag{8}$$

We notice that a non-Archimedean analogue of the Kolmogorov extension theorem was proved in [40, 41]. According to this theorem, there exists a unique p-adic quasi measure μ_h on $\Omega = \Phi^V$ such that for all $n \geq 1$ and $\sigma \in \Phi^{V_n}$,

$$\mu(\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n)$$

Such measure is called a *p-adic quasi Gibbs measure* corresponding to the Hamiltonian (5) and vector-valued function $\mathbf{h}_x$, $x \in V$. By $QG(H)$ we denote the set of all p-adic quasi Gibbs measure associated with function $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$. If all coordinates of $\mathbf{h}_x$ belong to the set $\mathcal{E}_p$ then it is called p-adic Gibbs measure. If there are at least two distinct p-adic quasi Gibbs measure $\mu, \nu \in QG(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition* occurs.

The following statement describe conditions $\mathbf{h}_x$ providing compatibility of $\mu_{\mathbf{h}}^{(n)}(\sigma)$.

Theorem 1. The measures $\mu_{\mathbf{h}}^{(n)}(\sigma), n = 1, 2, ...$ (6) associated with the Potts model (5) satisfy the compatibility condition (8) if and only if for any $n \in \mathbb{N}$ the equation that follows holds:

$$\widehat{\mathbf{h}}_x = \prod_{y\in S(x)} F(\widehat{\mathbf{h}}_y, \theta, \eta), \tag{9}$$

here $\theta = \exp\{J\}$, $\eta = \exp\{\alpha\}$ and below a vector $\widehat{\mathbf{h}}_x = \left(\widehat{h}_{1,x}, \widehat{h}_{2,x}, ..., \widehat{h}_{q-1,x}\right) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\mathbf{h}_x = (h_{1,x}, h_{2,x}, ..., h_{q,x}) \in \mathbb{Q}_p^q$ as follows

$$\widehat{h}_{i,x} = \frac{h_{i,x}}{h_{q,x}},\ i = 1, 2, ..., q-1$$

and mapping

$F : \mathbb{Q}_p^{q-1} \to \mathbb{Q}_p^{q-1}$ is defined by $F(x; \theta, \eta) = (F_1(x; \theta, \eta), ..., F_{q-1}(x; \theta, \eta))$ with

$$F_i(x; \theta, \eta) = \frac{(\theta-1)x_i + \sum\limits_{j=1}^{q-1} x_j + \eta}{\sum\limits_{j=1}^{q-1} x_j + \theta\eta},\ \ x = \{x_i\} \in \mathbb{Q}_p^{q-1},\ i = 1, 2, ..., q-1.$$

Proof *Necessity*. Assume that (8) holds. We must demonstrate (9). Substituting (6) into (8), we have

$$\sum_{\omega\in\Phi^{W_n}} \frac{1}{Z_n^{(h)}} \exp\left\{H_{n-1}(\sigma) + \left(\sum_{x\in W_{n-1}} \sum_{y\in S(x)} (J\delta_{\sigma_{n-1}(x)\omega_n(y)} + \alpha\delta_{q\omega_n(y)})\right)\right\} \prod_{x\in W_n} h_{\sigma(x),x}$$

$$= \frac{1}{Z_{n-1}^{(h)}} \exp\{H_{n-1}(\sigma)\} \prod_{x\in W_{n-1}} h_{\sigma(x),x}.$$

By eliminating the expressions on the left side of the equality outside the sign of the sum that do not depend on the sum, we obtain the following equality:

$$\frac{Z_{n-1}}{Z_n}\sum_{\omega\in\Phi^{W_n}}\exp\left(\sum_{x\in W_{n-1}}\sum_{y\in S(x)}(J\delta_{\sigma_{n-1}(x)\omega_n(y)}+\alpha\delta_{q\omega_n(y)})\right)\prod_{x\in W_n}h_{\omega_n(x),x}$$
$$=\prod_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}.$$

It yields that

$$\frac{Z_{n-1}}{Z_n}\sum_{\omega\in\Phi^{W_n}}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\exp(J\delta_{\sigma_{n-1}(x)\omega_n(y)}+\alpha\delta_{q\omega_n(y)})h_{\omega_n(y),y}=\prod_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}. \tag{10}$$

Fix $x\in W_{n-1}$ and consider two configurations $\sigma_{n-1}=\overline{\sigma}_{n-1}$ and $\sigma_{n-1}=\widetilde{\sigma}_{n-1}$ on W_{n-1} which coincide on $W_{n-1}\backslash\{x\}$, and the equality (10) for $\overline{\sigma}_{n-1}$ is divided by (10) for $\widetilde{\sigma}_{n-1}$. Then we obtain

$$\prod_{y\in S(x)}\frac{\sum\limits_{j\in\Phi}\exp(J\delta_{ij}+\alpha\delta_{qj})h_{j,y}}{\sum\limits_{j\in\Phi}\exp(J\delta_{qj}+\alpha\delta_{qj})h_{j,y}}=\frac{h_{i,x}}{h_{q,x}}.$$

It follows that

$$\prod_{y\in S(x)}\frac{\sum\limits_{j=1}^{q-1}\widehat{h}_{j,y}+(\theta-1)\widehat{h}_{i,y}+\eta}{\sum\limits_{j=1}^{q-1}\widehat{h}_{j,y}+\theta\eta}=\widehat{h}_{i,x},$$

where $\widehat{h}_{i,x}=\dfrac{h_{i,x}}{h_{q,x}}$ which implies (9).

Sufficiency. Suppose that (9) holds. It yields

$$\prod_{y\in S(x)}\frac{\sum\limits_{j\in\Phi}\exp(J\delta_{ij}+\alpha\delta_{qj})h_{j,y}}{\sum\limits_{j\in\Phi}\exp(J\delta_{qj}+\alpha\delta_{qj})h_{j,y}}=\frac{h_{i,x}}{h_{q,x}},$$

then for some function $a(x)\in\mathbb{Q}_p$, $x\in V$, we have

$$\prod_{y\in S(x)}\sum_{j\in\Phi}\exp(J\delta_{ij}+\alpha\delta_{qj})h_{j,y}=a(x)\exp(h_{i,x}),\ i\in\Phi. \tag{11}$$

We rewrite (6) as

$$\sum_{\omega\in\Omega_{W_n}}\mu_h^{(n)}(\sigma_{n-1}\vee\omega)=\frac{1}{Z_n}\exp\{H(\sigma_{n-1})\}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{j\in\Phi}\exp(J\delta_{\sigma_{n-1}(x)j}+\alpha\delta_{qj})h_{j,y}. \tag{12}$$

Substituting (11) into (12) and denoting $A_{n-1}=\prod\limits_{x\in W_{n-1}}a(x)$, we obtain

$$\sum_{\omega\in\Omega_{W_n}}\mu_h^{(n)}(\sigma_{n-1}\vee\omega)=\frac{A_{n-1}}{Z_n}\exp\{H(\sigma_{n-1})\}\prod_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}. \tag{13}$$

Since $\mu^{(n)}$ is a probability measure, we have

$$\sum_{\sigma\in\Omega_{V_{(n-1)}}}\sum_{\omega\in\Omega_{W_n}}\mu_h^{(n)}(\sigma_{n-1}\vee\omega)=1.$$

(13) yields

$$\sum_{\omega\in\Omega_{W_n}}\mu_h^{(n)}(\sigma_{n-1}\vee\omega)=\frac{A_{n-1}}{Z_n}\mu_h^{(n-1)}(\sigma_{n-1})Z_{n-1} \tag{14}$$

or

$$1=\sum_{\sigma\in\Omega_{V_{(n-1)}}}\sum_{\omega\in\Omega_{W_n}}\mu_h^{(n)}(\sigma_{n-1}\vee\omega)=\frac{A_{n-1}}{Z_n}Z_{n-1}\sum_{\sigma\in\Omega_{V_{(n-1)}}}\mu_h^{(n-1)}(\sigma_{n-1})=\frac{A_{n-1}}{Z_n}Z_{n-1}.$$

It follows that

$$Z_n=A_{n-1}Z_{n-1}. \tag{15}$$

Substituting (15) into (14), we have

$$\sum_{\omega\in\Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1}\vee\omega)=\mu_h^{(n-1)}(\sigma_{n-1}).$$

Theorem was proven.

Remark 1. If $\eta=1$, then Theorem 1 coincides with Theorem 3.1 in [39].

3. Translation-invariant p-adic quasi Gibbs measure for the Potts model with external field

We try to find the translation-invariant solutions of the system of equations (9). It requires to solve the following system of equations

$$\widehat{h}_i=\left(\frac{(\theta-1)\widehat{h}_i+\sum\limits_{j=1}^{q-1}\widehat{h}_j+\eta}{\sum\limits_{j=1}^{q-1}\widehat{h}_j+\theta\eta}\right)^k,\ i=1,2,...,q-1. \tag{16}$$

We assume that $\widehat{h}:\equiv\widehat{h}_1=\widehat{h}_2=...=\widehat{h}_{q-1}$. Then equation (16) reduces to the following one

$$\widehat{h}=\left(\frac{(\theta+q-2)\widehat{h}+\eta}{(q-1)\widehat{h}+\theta\eta}\right)^k. \tag{17}$$

Lemma 3. For equation (17), the following statements hold:
1) Equation (17) has no solution on $p\mathbb{Z}_p$;
2) If $q\notin\mathcal{E}_p$ then the solutions of (17) belong to $\mathbb{Z}_p^*$.

Proof At first, we show that equation (17) has no solution on $p\mathbb{Z}_p$. Assume that $\widehat{h}\in p\mathbb{Z}_p$, i.e. $|\ \widehat{h}\ |_p<1$. Since $\eta,\theta\in\mathcal{E}_p$ and $q\in\mathbb{Z}_p$, we obtain

$$|\ \widehat{h}\ |_p=\left|\left(\frac{(\theta+q-2)\widehat{h}+\eta}{(q-1)\widehat{h}+\theta\eta}\right)^k\right|_p=\left|\frac{\eta}{\theta\eta}\right|_p^k=1.$$

However, it contradicts to our assumption. Therefore, equation (17) has no solution on $p\mathbb{Z}_p$.

Now, we proof the second part of the theorem. We assume that $q\notin\mathcal{E}_p$, $|\ \widehat{h}\ |_p>1$. From (17), we have

$$\left|\left(\frac{(\theta+q-2)\widehat{h}+\eta}{(q-1)\widehat{h}+\theta\eta}\right)^k\right|_p=\left|\left(\frac{(\theta-1+q-1)\widehat{h}}{(q-1)\widehat{h}}\right)^k\right|_p=1\neq|\ \widehat{h}\ |_p.$$

However, it contradicts to our assumption. Thus, equation (17) has no solution on $p\mathbb{Z}_p$, if $q\notin\mathcal{E}_p$, (17) has no solution on $\mathbb{Q}_p\setminus\mathbb{Z}_p$. To conclude, if equation (17) has a solution, it must belong to $\mathbb{Z}_p^*$. *Lemma was proven.*

Lemma 4. Let $|q|_p=1$, $p\geq 3$. Then there is a unique solution of (17) in the form of $h^*\in\mathcal{E}_p$.

Proof We rewrite (17) as

$$\widehat{h}((q-1)\widehat{h}+\theta\eta)^k-((\theta+q-2)\widehat{h}+\eta)^k=0$$

Set the notation

$$F(\widehat{h},\theta,\eta,q)=\widehat{h}((q-1)\widehat{h}+\theta\eta)^k-((\theta+q-2)\widehat{h}+\eta)^k.$$

It can be seen that $F(\widehat{h},\theta,\eta,q)$ is a polynomial with p-adic integer coefficients. For $h\equiv 1(\operatorname{mod}p)$, we verify that $F(\widehat{h},\theta,\eta,q)$ satisfies the conditions of Lemma 2. Then we obtain that

$$\begin{aligned}F(1,\theta,\eta,q)&\equiv((q-1+1+o[1])^k-(q-1+o[1]+1+o[1])^k)\\&\equiv((q+o[1])^k-(q+o[1])^k)\equiv 0(\operatorname{mod}p)\end{aligned}$$

and

$$F'(h,\theta,\eta,q)=((q-1)\widehat{h}+\theta\eta)^k+k(q-1)h((q-1)\widehat{h}+\theta\eta)^{k-1}-k(\theta+q-2)((\theta+q-2)\widehat{h}+\eta)^{k-1}.$$

We consider $F'(1,\theta,\eta,q)\equiv 0(\operatorname{mod}p)$, i.e.,
$F'(1,\theta,\eta,q)\equiv(q-1+1+o[1])^k+k(q-1)(q-1+1+o[1])^{k-1}-k(1+o[1]+q-2)(1+o[1]+q-2+1+o[1])^{k-1}\equiv(q+o[1])^k\not\equiv 0(\operatorname{mod}p)$.
Thus, the polynomial fulfills the requirements of Lemma 2. It implies that there is a unique integer root h^* such that

$$F(h^*,\theta,\eta,q)=0,\ h^*\equiv 1(\operatorname{mod}p).$$

It yields $h^*\in\mathcal{E}_p$.

Remark 2. In [22] authors studied all translation-invariant p-adic Gibbs measures for the Potts model without external fields. It was shown that if $\mid q \mid_p = 1$, $\eta = 1$ then the system of equations (16) on $\mathcal{E}_p \setminus \{1\}$ does not have any solution. However, we proved that if $\mid q \mid_p = 1$, $\eta \neq 1$ then the system of equations (16) has a unique solution on $\mathcal{E}_p \setminus \{1\}$.

Remark 3. Further calculations are needed in order to study equation (17) for the case $q \in \mathcal{E}_p$. Hence, this problem will be studied in our upcoming work.

From Lemma 3, if $q \notin \mathcal{E}_p$, the solutions of equation (17) belonging to $\mathbb{Z}_p^*$. We obtain the following congruence from (17) after slight modification:

$$\widehat{h}((q-1)\widehat{h}+\theta\eta)^k - ((\theta+q-2)\widehat{h}+\eta)^k \equiv 0(\mathrm{mod}\, p). \tag{18}$$

Theorem 2. For congruence (18), the following statements hold:

i) If $\mid q \mid_p < 1$ then (18) has a solution with $\widehat{h} \equiv 1(\mathrm{mod}\, p)$;

ii) If $q \in \mathbb{Z}_p^* \setminus \mathcal{E}_p$ then (18) has the solutions with $\widehat{h}^{(1)} \equiv 1(\mathrm{mod}\, p)$ and $h^{(2)} \equiv -(q-1)^{-1}(\mathrm{mod}\, p)$, here $(q-1)^{-1}$ is inverse of $q-1$ modulo p.

Proof Let $\mid q \mid_p < 1$. Then it can be seen that

$$\widehat{h}((q-1)\widehat{h}+\theta\eta)^k - ((\theta+q-2)\widehat{h}+\eta)^k \equiv (-1)^k(\widehat{h}-1)^{k+1}(\mathrm{mod}\, p).$$

It follows that the solution of the congruence (17) is $\widehat{h}_1 \equiv 1(\mathrm{mod}\, p)$.

Let $\mid q \mid_p = 1$ and $q \notin \mathcal{E}_p$. Then we get

$$\widehat{h}((q-1)\widehat{h}+\theta\eta)^k - ((\theta+q-2)\widehat{h}+\eta)^k \equiv (\widehat{h}-1)((q-1)\widehat{h}+1)^k(\mathrm{mod}\, p).$$

From this, we have two solutions $\widehat{h}_1 \equiv 1(\mathrm{mod}\, p)$ and $\widehat{h}_2 \equiv -(q-1)^{-1}(\mathrm{mod}\, p)$ which implies (17).

Remark 5. We note that it is essential to find the first coefficient of the canonical form of the solution of (17). It gives a possibility to check the boundedness of the Gibbs measure.

If $q \in \mathbb{Z}^*$ then according to Lemma 4, equation (17) has a unique solution in $\mathcal{E}_p$. Now, we show that there is a solution of (16) such that $h \notin \mathcal{E}_p$. It is difficult to solve this problem in general case. We concentrate on the simplest case $k = 2$. In this case, we have

$$\widehat{h} = \left(\frac{(\theta+q-2)\widehat{h}+\eta}{(q-1)\widehat{h}+\theta\eta}\right)^2. \tag{19}$$

Let us consider the following depressed cubic equation

$$x^3 + ax = b.$$

In [42], the criteria for solvability of the depressed cubic equation over $\mathbb{Z}_p^*$ are given.

Let $D = -4(a \mid a \mid_p)^3 - 27(b \mid b \mid_p)^2 \neq 0$, $D = \dfrac{D^*}{\mid D \mid_p}$, $D^* \in \mathbb{Z}_p^*$, $D^* = d_0 + d_1 p + \dots$, $D_0 = -4a_0^3 - 27b_0^2$ and $u_1 = 0$, $u_2 = -a_0$, $u_3 = b_0$ and $u_{n+3} = b_0 u_n - a_0 u_{n+1}$.

Theorem 3. [42] Let $p>3$ be a prime number and $\mathcal{N}$ be the cardinality of the set of solution to $x^3+ax-b=0$ in $\mathbb{Z}_p$. Then the following statements hold:

$$\mathcal{N}=\begin{cases} 3, & \mid a\mid_p^3<\mid b\mid_p^2\leq 1,\ 3\mid \log_p\mid b\mid_p \ \ p\equiv 1(\bmod 3),\ b_0^{\frac{p-1}{3}}\equiv 1(\bmod p);\\ 3, & \mid a\mid_p^3=\mid b\mid_p^2\leq 1,\ D=0;\\ 3, & \mid a\mid_p^3=\mid b\mid_p^2\leq 1,\ 0<\mid D\mid_p<1,\ \ 2\mid \log_p\mid D\mid_p,\ d_0^{\frac{p-1}{2}}\equiv 1(\bmod p);\\ 3, & \mid a\mid_p^3=\mid b\mid_p^2\leq 1,\ \mid D\mid_p=1 \text{ and } u_{p-2}\equiv 0(\bmod p);\\ 3, & \mid b\mid_p^2<\mid a\mid_p^3\leq 1,\ ,\ 2\mid \log_p\mid a\mid_p,\ (-a_0)^{\frac{p-1}{2}}\equiv 1(\bmod p);\\ 1, & \mid a\mid_p^3<\mid b\mid_p^2\leq 1\ \ 3\mid \log_p\mid b\mid_p,\ p\equiv 2(\bmod 3);\\ 1, & \mid a\mid_p^3=\mid b\mid_p^2\leq 1,\ 0<\mid D\mid_p<1,\ \ 2\mid \log_p\mid D\mid_p,\ d_0^{\frac{p-1}{2}}\not\equiv 1(\bmod p);\\ 1, & \mid a\mid_p^3=\mid b\mid_p^2\leq 1,\ 0<\mid D\mid_p<1,\ \ 2\nmid \log_p\mid D\mid_p;\\ 1, & \mid a\mid_p^3=\mid b\mid_p^2\leq 1,\ D_0u_{p-2}^2\not\equiv 0(modp),\ D_0u_{p-2}^2\not\equiv 9a_0^2(\bmod p);\\ 1, & \mid b\mid_p^2<\mid a\mid_p^3\leq 1,\ \ 2\mid \log_p\mid a\mid_p,\ (-a_0)^{\frac{p-1}{2}}\not\equiv 1(\bmod p);\\ 1, & \mid b\mid_p^2<\mid a\mid_p^3\leq 1,\ \ 2\nmid \log_p\mid a\mid_p;\\ 1, & \mid b\mid_p^2<\mid a\mid_p^3,\ \mid b\mid_p\leq\mid a\mid_p,\ \mid a\mid_p>1;\\ 0, & otherwise, \end{cases}$$

where $a\mid b$ means a divides b.

Lemma 5. Let $p>3$, $q\in\mathbb{Z}_p^*\setminus\mathcal{E}_p$, $\mathcal{N}$ be the cardinality of the set of the solutions of (19). Then we have

$$\mathcal{N}=\begin{cases} 3, & \text{if } (1-q)^{\frac{p-1}{2}}\equiv 1(\bmod p);\\ 1, & otherwise. \end{cases}$$

Proof We rewrite equation (17) as follows

$$\widehat{h}^3+\frac{2\theta\eta(q-1)-(\theta+q-2)^2}{(q-1)^2}\widehat{h}^2+\frac{(\theta^2\eta^2-2\eta(\theta+q-2))}{(q-1)^2}\widehat{h}-\frac{\eta^2}{(q-1)^2}=0. \tag{20}$$

We denote

$$z:=\widehat{h}-\frac{2\theta\eta(q-1)-(\theta+q-2)^2}{3(q-1)^2}. \tag{21}$$

From (20) and (21), we obtain

$$z^3+az-b=0, \tag{22}$$

where

$$\begin{gathered} a=-\frac{1}{3}\frac{(2\theta\eta(q-1)-(\theta+q-2)^2)^2}{(q-1)^4}+\frac{\theta^2\eta^2-2\eta(\theta+q-2)}{(q-1)^2},\\ b=\frac{1}{3}\frac{(\theta^2\eta^2-2\eta(\theta+q-2))(2\theta\eta(q-1)-(\theta+q-2)^2)}{(q-1)^4}+\frac{\eta^2}{(q-1)^2}-\\ \frac{2}{27}\frac{(2\theta\eta(q-1)-(\theta+q-2)^2)^3}{(q-1)^6}. \end{gathered} \tag{23}$$

It should be noted that due to Lemma 4, equation (17) has a unique solution $\widehat{h}^*\equiv 1(\bmod p)$ and this statement also holds for (19). Therefore, we check the conditions of Theorem 3 for $\mathcal{N}=3$. Since $q\notin\mathcal{E}_p,\ \mid q\mid_p=1$, we obtain that $\mid a\mid_p=\mid b\mid_p=1$.

One can see that

$$\begin{gathered} D=-4(a\mid a\mid_p)^3-27(b\mid b\mid_p)^2=\\ \frac{1}{(q-1)^8}m^2(s+1)^3(m+q)^2(-4m^3qs^2+m^4s-6m^3qs+4m^3s^2+m^2q^2s-\\ 12m^2qs^2+m^4-2m^3q+8m^3s+m^2q^2-4m^2qs+12m^2s^2+20mq^2s-12mqs^2-4m^2q+4m^2s+\\ 8mq^2-44mqs+12ms^2-8q^2s-4qs^2+4m^2-8mq+24ms+4q^2+8qs+4s^2+4q^2-4q^3), \end{gathered}$$

where $m=\theta-1,\ s=\eta-1$.

It can be checked that $\mid D\mid_p<1$. According to Theorem 3, if $2\mid\log_p\mid D\mid_p,\ \ d_0^{\frac{p-1}{2}}\equiv 1(\bmod p)$, then equation (22) has

three solutions. We show that $\operatorname{ord}_p D$ is even, $\sqrt{d_0} \in \mathbb{Q}_p$.
At first, we check that $\sqrt{d_0} \in \mathbb{Q}_p$. For the sake of simplicity, we denote

$$D_1 = -4m^3qs^2 + m^4s - 6m^3qs + 4m^3s^2 + m^2q^2s - 12m^2qs^2 + m^4 - 2m^3q + 8m^3s + m^2q^2 -$$
$$4m^2qs + 12m^2s^2 + 20mq^2s - 12mqs^2 - 4m^2q + 4m^2s + 8mq^2 - 44mqs + 12ms^2 - 8q^2s -$$
$$4qs^2 + 4m^2 - 8mq + 24ms + 8qs + 4s^2 + 4q^2 - 4q^3,$$
$$q = q_0 + o[1], q_0 \in \overline{2, p-1},\ m = p^{\beta}(m_0 + o[1]).$$

Using $|\ m\ |_p < 1,\ |\ s\ |_p < 1,\ |\ q\ |_p = 1$ and $q \notin \mathcal{E}_p$, we obtain

$$\begin{aligned} q - 1 &= q_0 - 1 + o[1]; \\ s + 1 &= 1 + o[1]; \\ m + q &= q_0 + o[1]; \\ D_1 &= 4q_0^2(1 - q_0) + o[1]. \end{aligned} \tag{24}$$

It yields that $d_0 \equiv \dfrac{4q_0^4(1-q_0)m_0^2}{(q_0-1)^8} (\operatorname{mod} p)$.

We deduce that if the Legendre symbol of $1 - q_0$ is equal to 1, then $\sqrt{d_0} \in \mathbb{Q}_p$.

Now, we define $|\ D\ |_p$ Using (24), we have

$$\begin{aligned} |\ q - 1\ |_p &= 1; \\ |\ s + 1\ |_p &= 1; \\ |\ m + q\ |_p &= 1; \\ |\ D_1\ |_p &= 1. \end{aligned}$$

It follows that $|\ D\ |_p = (|\ m\ |_p)^2$. So, $\operatorname{ord}_p D$ is even. The proof is completed.

In [43], the cubic equation (22) is examined for the case $p = 3$. If $|\ a\ |_p^3 > |\ b\ |_p^2$, $2 \mid \log_3 |\ a\ |_3$, $\dfrac{a}{|\ a\ |_3} \equiv 2(\operatorname{mod} 3)$, then equation (22) has three solutions over $\mathbb{Q}_3$. Using this criteria, we get the following lemma.

Lemma 6. Let $p = 3$, $|q|_3 = 1$, then equation (19) has a unique solution.

Proof We note that, due to Lemma 4, equation (19) has a unique solution on $\mathcal{E}_3$. For this case, let us find the remaining solutions of (19).

Case I. $q \equiv 2(\operatorname{mod} 3)$.
From (23), we get $|\ a\ |_3 = 3$, $|\ b\ |_3 = 27$. This does not satisfy the conditions of the above criteria.

Case II. $q \equiv 1(\operatorname{mod} 3)$
Let $|2\theta\eta(q-1) - (\theta + q - 2)^2|_3 = 3^{\alpha}$, $|q-1|_3 = 3^m$. Then $|\ a\ |_3 = 3^{4m-2\alpha+1}$, $|\ b\ |_3 = 3^{6m-3\alpha+3}$. This also does not meet the required conditions. Therefore, we conclude that equation (19) has a unique solution.

Lemma 7. Let $p \geq 3$, $|\ q\ |_p < 1$, then equation (19) has no solution.

Proof We assume that $|\ q\ |_p < 1$. According to Lemma 3 and Theorem 3, the solutions of equation (19) belong to $\mathbb{Z}_p^*$ with $\widehat{h} \equiv 1(\operatorname{mod} p)$. Due to (21), we obtain that

$$\frac{2\theta\eta(q-1) - (\theta + q - 2)^2}{3(q-1)^2} = -1 + o[1],\ z \equiv 2(\operatorname{mod} p).$$

It follows that $|\ z\ |_p = 1$.

Using (23) and $|\ q\ |_p < 1$, we have $|\ a\ |_p < 1,\ |\ b\ |_p < 1$. We rewrite equation (22) as follows

$$z^3 = b - az.$$

It can be seen that

$$|\ b - az\ |_p < 1 \neq |\ z^3\ |_p\ .$$

It follows that equation(19) has no solution given the conditions in the lemma. Lemma was proved.

Using Lemmas 5,6, and 7, we come to the following result:

Theorem 4. The following statements are true for $p-$adic Potts model with external field on the Cayley tree of order two.

1) if $|\ q\ |_p = 1$, $p = 3$ or $|\ q\ |_p = 1$, $p > 3$, $(1-q)^{\frac{p-1}{2}} \not\equiv 1(\operatorname{mod} p)$ then there is one translation-invariant p-adic quasi Gibbs measure;

2) if $p > 3$, $\mid q\mid_p = 1$, $q \notin \mathcal{E}_p$, $(1-q)^{\frac{p-1}{2}} \equiv 1(\mathrm{mod}\, p)$, then there are three translation-invariant p-adic quasi Gibbs measures;
3) if $p \geq 3$, $\mid q\mid_p < 1$, then there is not any translation-invariant p-adic quasi Gibbs measure.

Corollary 1. Let $\mathcal{N}_{TP}$ be number of p-adic quasi Gibbs measures for the Potts model with an external field on the Cayley tree of order two. Then we obtain

$$\mathcal{N}_{TP} = \begin{cases} 0, & \text{if } p = q = 3; \\ 1, & \text{if } q = 3,\, p > 3,\, p \equiv 5(\mathrm{mod}\, 8) \text{ or } p \equiv 7(\mathrm{mod}\, 8); \\ 3, & \text{if } q = 3,\, p > 3,\, p \equiv 1(\mathrm{mod}\, 8) \text{ or } p \equiv 3(\mathrm{mod}\, 8). \end{cases}$$

Proof 1) Let $p = q = 3$. This case satisfies the third condition in Theorem 4, therefore, there is no translation-invariant p-adic quasi Gibbs measure, that is, $\mathcal{N}_{TP} = 0$.
2) If $q = 3, p > 3$, then $1 - q = -2$. In [44], the following results are obtained

$$\left(\frac{-2}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1(\mathrm{mod}\, 8) \text{ or } p \equiv 3(\mathrm{mod}\, 8); \\ -1, & \text{if } p \equiv 5(\mathrm{mod}\, 8) \text{ or } p \equiv 7(\mathrm{mod}\, 8). \end{cases}$$

These conditions satisfy the first and the second conditions in Theorem 4. Keeping in mind these results, we obtain the assertions of the corollary.

4. Boundedness of the translation-invariant p-adic quasi Gibbs measures and phase transitions

Lemma 8. Let $\mu_{\mathbf{h}}$ be an associated p-adic quasi Gibbs measure, and let $\mathbf{h}$ be a solution of (9). Then, the following equality is true for the appropriate partition function $Z_n^{(\mathbf{h})}$:

$$Z_n^{(\mathbf{h})} = A_{\mathbf{h},n-1} Z_{n-1}^{(\mathbf{h})}, \tag{25}$$

where $A_{\mathbf{h},n} = \prod_{x\in W_n} a_{\mathbf{h}}(x)$, $\prod_{y\in S(x)} \sum_{j=1}^{q} exp\{J\delta_{i,j} + \alpha\delta_{qj}\}h_{j,y} = a_{\mathbf{h}}(x)h_{i,x}$,

$$a_{\mathbf{h}}(x) \in \mathbb{Q}_p,\ i = 1, 2, ..., q.$$

Proof Assume that $\mathbf{h}$ is a solution of (9), then equation (10) hold. We rewrite (10) for ordinary $i \in \Phi$ as follows

$$Z_n = Z_{n-1} \prod_{x\in W_{n-1}} \frac{\prod_{y\in S(x)} \sum_{j=1}^{q} exp(J\delta_{ij} + \alpha\delta_{qj})h_{j,y}}{h_{i,x}}. \tag{26}$$

We present subsequent notations

$$A_{\mathbf{h},n} = \prod_{x\in W_n} a_{\mathbf{h}}(x) \text{ and } a_{\mathbf{h}}(x) = \frac{\prod_{y\in S(x)} \sum_{j=1}^{q} exp\{J\delta_{i,j} + \alpha\delta_{qj}\}h_{j,y}}{h_{i,x}}.$$

Then equation (26) is reduced to (25).

Using Lemma 8, we come to the following statement.

Lemma 9. Let $k = 2$, $\mathbf{h}$ be a translation-invariant solution of (9), then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following equality is appropriate:

$$Z_n^{(\mathbf{h})} = ((q-1)h + \eta\theta)^{3\cdot 2^n - 1}(h(q-1) + \eta). \tag{27}$$

Proof It is easy to check that $\mathbf{h} = (h, h, ..., h, 1)$ is a translation-invariant solution of (9), where h is a fixed point of (19). Since $\theta = \exp\{J\}$ and $\eta = \exp\{\alpha\}$, using (7) we obtain $Z_1^{(h)} = ((q-1)h + \theta\eta)^2((q-1)h + \eta)$. Then by Lemma 8, we come to the following equality:

$$a_h(x) = \frac{((\theta + q - 2)h_{1,y} + \eta)^2}{h_{1,x}} = \frac{(\theta + q - 2)h + \eta)^2}{h} = ((q-1)h + \theta\eta)^2.$$

From Lemma 8, we obtain

$$A_{\mathbf{h},n} = ((q-1)h + \eta\theta)^{3\cdot 2^{n-1}},$$

$$Z_n^{(\mathbf{h})} = ((q-1)h + \eta\theta)^{3\cdot(2^n - 1)}((q-1)h + \eta\theta)^2((q-1)h + \eta) =$$
$$= ((q-1)h + \eta\theta)^{3\cdot 2^n - 1}(h(q-1) + \eta).$$

Lemma is proved.

Theorem 5. Let $p \geq 3$, $|q|_p = 1$. The following statements hold for p-adic Potts model with an external field on a Cayley tree of order two:

1) if $p = 3$ or $p > 3$, $(1-q)^{\frac{p-1}{2}} \not\equiv 1(\mathrm{mod}\, p)$, then measure μ_{h^*} is bounded;
2) if $q \notin \mathcal{E}_p$, $p > 3$, $(1-q)^{\frac{p-1}{2}} \equiv 1(\mathrm{mod}\, p)$, then measure μ_{h^*} is bounded, measures μ_{h_1}, μ_{h_2} are unbounded.

Proof *Case 1.* If $p = 3$ or $p > 3$, $(1-q)^{\frac{p-1}{2}} \not\equiv 1(\mathrm{mod}\, p)$, then there exists measure μ_{h^*}. We note that $h^* \in \mathcal{E}_p$. From Lemma 9 and (6), we obtain

$$\lim_{n\to\infty} |\ \mu_{h^*}^{(n)}|_p = \lim_{n\to\infty} |\ \frac{1}{(q+o[1])^{3\cdot 2^n}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}\ |_p = 1.$$

Case 2. If $p > 3$, $(1-q)^{\frac{p-1}{2}} \equiv 1(\mathrm{mod}\, p)$, there exist translation-invariant measures μ_{h^*}, μ_{h_1}, μ_{h_2}. According to Lemma 4 and Theorem 3, $h^* = 1(\mathrm{mod}\, p)$, $h_{1,2} = -(q-1)^{-1}(\mathrm{mod}\, p)$. From Lemma 9 and (6), we obtain

$$\lim_{n\to\infty} |\ \mu_{h^*}^{(n)}|_p = \lim_{n\to\infty} |\ \frac{\prod_{x\in W_n} h_{\sigma(x),x}}{((q-1)h^* + \eta\theta)^{3\cdot 2^n - 1}(h^*(q-1)+\eta)} \exp\{H_n(\sigma)\}\ |_p = 1.$$

$$\lim_{n\to\infty} |\ \mu_{h_{1,2}}^{(n)}|_p = \lim_{n\to\infty} |\ \frac{\prod_{x\in W_n} h_{\sigma(x),x}}{((q-1)h_{1,2} + \eta\theta)^{3\cdot 2^n - 1}(h_{1,2}(q-1)+\eta)} \exp\{H_n(\sigma)\}\ |_p = \infty.$$

We have proved that the measure μ_{h^*} is bounded, μ_{h_1}, μ_{h_2} measures are unbounded as in the case 2.

Theorem 6. Let $p > 3$, $|\ q\ |_p = 1$, $q \notin \mathcal{E}_p$. Then there exists a phase transition for p-adic q-state Potts model with an external field on a Cayley tree of order two if $(1-q)^{\frac{p-1}{2}} \equiv 1(\mathrm{mod}\, p)$.

Proof The proof is straightforward due to Theorem 5.

Corollary 2. Let $q = 3$. If $p > 3$, $p \equiv 1(\mathrm{mod}\, 8)$ or $p \equiv 3(\mathrm{mod}\, 8)$ then there is a phase transition for p-adic Potts model with an external field on the Cayley tree of order two.

Remark 6. $a)$ In [23], the authors focused on the Potts model without an external field. The phase transition conditions determined in this study were consistent with the results of Corollary 2.

$b)$ In [24], the existence a quasi-phase transition is defined for the $q+1$ Potts model without an external field is proven if $|\ q\ |_p = 1$. However, we define a phase transition for this model if $|\ q\ |_p = 1$, $|\ q-1\ |_p = 1$, and $(1-q)^{\frac{p-1}{2}} \equiv 1(\mathrm{mod}\, p)$.

$c)$ Note that we have considered translation-invariant p-adic quasi Gibbs measures for the Potts model with an external field only for the case $\mathbf{h} = \{h, h, ..., h\}$, $\mathbf{h} \in \mathbb{Q}_p^{q-1}$. The remaining cases are left as an open problem.

5. Conclusion

It should be noted that so far, p-adic quasi-Gibbs measures have been obtained for the Potts model without an external field. Therefore, we have dedicated this work to the study of p-adic quasi-Gibbs measures for the Potts model with an external field. Analyzing functional equation which defines p-adic quasi Gibbs measure for the Potts model with the external field on a semi-infinite Cayley tree, we have derived three translation-invariant p-adic quasi Gibbs measures under some condition. We also obtained a system of functional equations that satisfy the consistency condition for p-adic quasi-Gibbs measures for the Potts model with an external field on the Cayley tree of order $k \geq 2$. This system corresponds to the functional equation in [39] when the external field is zero.

In [24], a quasi-phase transition was identified for the $q+1$ state Potts model when $|\ q\ |_p = 1$. Moreover, we identified the phase transition for the q-state Potts model with an external field when $|\ q\ |_p = 1$, $|\ q-1\ |_p = 1$, and $(1-q)^{\frac{p-1}{2}} \equiv 1(\mathrm{mod}\, p)$.

In [28], [23], p-adic quasi-Gibbs measures were determined for the 3-state Potts model, and we extend these results to the general case $q \geq 3$ and zero external field. In particular, if $q = 3$, our result coincides with the result in [23].

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