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Original article

Method of reference problems for obtaining approximate analytical solution of multi-parametric Sturm–Liouville problems

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ABSTRACT Approximate analytical formulas are obtained for the eigenfrequencies of longitudinal oscillations of an elastic rod with different mechanical fixings of the ends. The eigenfrequencies are found by solving Sturm–Liouville problems with the third kind boundary conditions as roots of transcendental equations. Homogeneous boundary conditions contain one or more parameters whose values are calculated through the indices of mechanical system. Approximate expression for analytical dependencies of the eigenfrequencies on the single parameter are obtained for one-parametric problems, which are called reference ones. We propose a method for obtaining approximate analytical expression for dependencies of the eigenfrequencies on several parameters in boundary conditions by sequentially solving the reference problems. The two-parametric Sturm–Liouville problem is solved by the proposed method.

KEYWORDS Sturm–Liouville problem, elastic rod, longitudinal oscillations, eigenfrequencies, approximation, least-squares method.

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1. Introduction

For the time being high-precision measuring instruments which use resonant microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS) are of theoretical and practical interests [1–11]. Simple elastic constructions with resonance properties are used as primary converters of physical quantities in MEMS and NEMS [1, 6, 11]. The main characteristic parameters of such systems are their resonant frequencies. Mathematical modeling of time-harmonic oscillations of one-dimensional distributed elastic constructions such as strings, rods, beams, supported by elastic elements leads to the Sturm–Liouville problems. The Sturm–Liouville problem is the boundary value problem for an ordinary linear homogeneous differential equation with homogeneous boundary conditions at the ends of the interval [12–19].

The Sturm–Liouville problem for the second-order differential equation with boundary conditions of the third kind is of practical interest [20, 21]. The boundary condition at each edge of the interval has the form of annihilation of the linear combination of the function value and its derivative calculated at the end point of the interval. The coefficients of the linear combination in the boundary condition are the parameters of the problem. We call the Sturm–Liouville problem containing n parameters n-parametric. The Sturm–Liouville problem eigenvalues are roots of a transcendental equation which depend on the problem parameters.

We have proposed earlier (in [22–24]) an analytical method for obtaining approximate values for the eigenfrequencies of the one-parametric Sturm–Liouville problem. Using the proposed method, we obtain approximate analytical solutions of two one-parametric Sturm–Liouville problems, which will be called the reference ones. We extend this method to solving of the $n-$parametric Sturm–Liouville problem. As an example, we solve the two-parametric Sturm–Liouville problem.

2. Reference one-parametric Sturm–Liouville problems

Let the elastic homogeneous rod of length l be located on the interval $[0, l]$ along the axis OX. The cross-sectional area of the rectangular rod is F. The Young's modulus and the linear rod density are E and ρ, respectively.

Small longitudinal displacements $U = U(X, t)$ of the cross section of the rod with coordinate X from the equilibrium position at time moment t satisfy to the following equation

$$\rho F \frac{\partial^2 U(X,t)}{\partial t^2} = EF \frac{\partial^2 U(X,t)}{\partial X^2}, \quad 0 < X < l. \tag{1}$$

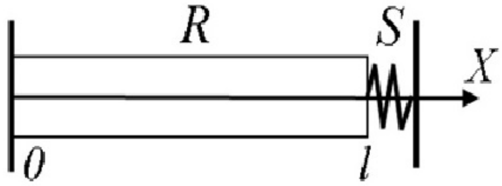

FIG. 1. Choice of the coordinate system for the first reference one-parametric problem. R is an elastic rod, the left end of the rod at $X = 0$ is rigidly embedded into the wall, the right end at $X = l$ is supported by a spring S

The left end of the rod at $X = 0$ is rigidly embedded into the wall. It is described by the boundary condition

$$U(0, t) = 0. \tag{2}$$

The right end of the rod at $X = l$ is supported by an elastic spring with the stiffness K. Correspondingly, one has the following boundary condition [13]:

$$-EF\frac{\partial U(l,t)}{\partial X} = K\,U(l,t). \tag{3}$$

To find the eigenfrequencies of the rod oscillations, we assume that the dependence of longitudinal displacement on time is harmonic, $U(X,t) = Y(X)e^{-i\omega t}$, where ω is the circular frequency, and $Y(X)$ is the amplitude of the longitudinal displacement of the cross section at the point with coordinate X.

Let's transform the boundary value problem (1)–(3) to the dimensionless form. We introduce the dimensionless coordinate x, $x = X/l$, $0 \le x \le 1$ and the dimensionless amplitude of the longitudinal displacement of the cross-section $y(x) = Y(X)/l$. We also introduce the following dimensionless quantities: eigenfrequency $\lambda = \omega l\sqrt{\rho/E}$ and stiffness of the spring $k = K/(EFl)$.

Taking into account the introduced dimensionless quantities, we have the first reference one-parametric Sturm–Liouville problem for finding the set of eigenfunctions $y(x)$ and dimensionless eigenfrequencies λ, $\lambda > 0$:

$$y''(x) = -\lambda^2 y(x), \quad 0 < x < 1 \tag{4}$$

With the boundary conditions:

$$y(0) = 0, \quad y'(1) + k\,y(1) = 0. \tag{5}$$

The general solution of equation (4) is as follows

$$y = y(x) = C_1 \sin\lambda x + C_2 \cos\lambda x, \tag{6}$$

where C_1, C_2 are arbitrary constants.

Let's find a particular solution of the differential equation (4) satisfying boundary conditions (5). It follows from the boundary condition $y(0) = 0$ that $C_2 = 0$ in the representation (6). Correspondingly, the general solution transforms to the form $y = C_1 \sin\lambda x$. The second boundary condition at $x = 1$ gives one $\Delta(\lambda, k) = 0$, where $\Delta(\lambda, k) = \lambda\cos\lambda + k\sin\lambda$. Keeping in mind that $\lambda \ne \pi n$, $n \in \mathbb{Z}$, we transform this equation to the form

$$\lambda\mathrm{ctg}\lambda + k = 0. \tag{7}$$

The spectral equation (7) gives the dependence of eigenfrequencies on the parameter k

$$\lambda = \Lambda(k), \tag{8}$$

where Λ is a multivalued function implicitly specified by equation (7). The equation is transcendental with respect to the frequency λ and does not allow one to find an analytical solution of the form (8) in elementary functions.

Graphical, numerical and analytical solutions of the spectral equations similar to (7) are considered in [25, 27]. To obtain an approximate analytical solution of the spectral equation (7), we apply the method proposed in [24]. For this purpose, we note that this equation is linear in respect to the parameter k. Therefore, at the first stage of solving the spectral equation we find the dependence of the parameter k on the eigenfrequency λ:

$$k = \Lambda^{-1}(\lambda) = -\lambda\mathrm{ctg}\lambda.$$

This dependence is an elementary function, and is the inverse of the required dependence (8).

The function $k = \Lambda^{-1}(\lambda)$ is a meromorphic function which is defined for those values of $\lambda > 0$ for which the values of function are nonnegative. The behavior of the function $k = -\lambda\mathrm{ctg}\lambda$ is determined by its zeros $w_n = \pi(n - 0.5)$ and by its poles $v_n = \pi n$. We assume here and elsewhere below that $n \in N$. The function takes nonnegative values for $w_n \le \lambda < v_n$. On each interval of such type, the function has the positive derivative and is monotonically increasing.

We denote the function $\Lambda^{-1}(\lambda)$ for $\lambda \in [w_n, v_n)$ as $\Lambda_n^{-1}(\lambda)$. Note that each function $\Lambda_n^{-1}(\lambda)$ is continuous and monotonically increasing in the domain of $\lambda \in [w_n, v_n)$. As for the values of these functions at the ends of the intervals, one has $\Lambda_n^{-1}(w_n) = 0$ and $\Lambda_n^{-1}(\lambda) \to +\infty$ as $\lambda \to v_n - 0$. The functions $\Lambda_n^{-1}(\lambda)$ have inverse functions with the range of values $\lambda \in [w_n, v_n)$. Let's denote these inverse functions by Λ_n so that $\lambda = \Lambda_n(k)$. Graphs of the functions $\lambda = \Lambda_n(k)$ are shown in Fig. 2. In constructing the graphs, we used the fact that the functions $k = \Lambda^{-1}(\lambda)$ and $\lambda = \Lambda_n(k)$ are inverse functions. The points on the graphs have coordinates $(\Lambda^{-1}(\lambda), \lambda)$, $\lambda \in [w_n, v_n)$.

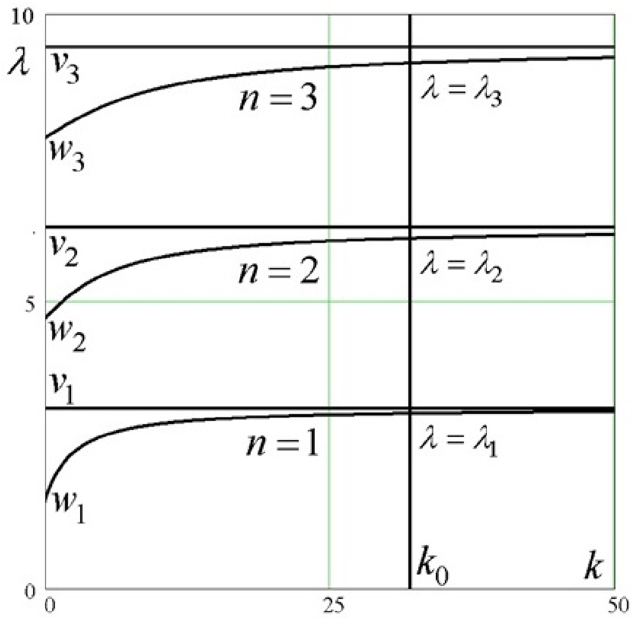

FIG. 2. Graphs of the functions $\lambda = \Lambda_n(k)$ for n=1, 2, 3. Graphical finding of eigenfrequencies λ_1, λ_2, λ_3 is presented

As the first step of the Sturm–Liouville problem solving, we find graphically (Fig. 2) the values $\lambda_n = \lambda_n(k_0)$ as ordinates of the intersection points of function graphs $\lambda = \Lambda_n(k)$ with the vertical straight line $k = k_0$. The disadvantage of the graphical solution is the impossibility of using the eigenfrequencies found graphically in the subsequent computer modeling of the physical problem.

To obtain approximate analytical expressions for the eigenfrequencies, we use the method proposed in [24]. We choose the approximation of functions $\Lambda_n^{-1}(\lambda)$ by functions $G_n(\lambda)$, following the conditions listed below:

– the functions $G_n(\lambda)$ should be elementary functions with elementary inverse functions G_n^{-1}, i.e. analytic representations of $G_n(\lambda)$ and G_n^{-1} functions should contain only basic elementary functions;

– the functions $G_n(\lambda)$ should provide sufficient approximation accuracy $G_n(\lambda) \approx \Lambda_n^{-1}(\lambda)$, $\lambda \in [w_n, v_n)$.

As in [28], we use the interpolation method and the approximation by the method of least-squares (LSM) in combination to find the $G_n(\lambda)$ functions.

To interpolate the functions $G_n(\lambda)$, $\lambda \in [w_n, v_n)$, we choose the ends of intervals $\lambda = v_n$ and $\lambda = w_n$, and the middle of the intervals $\lambda = \gamma_n = 0.5(v_n + w_n)$ as interpolation nodes. We look for the functions in the form

$$G_n(\lambda) = A_n \left(\frac{\lambda - w_n}{v_n - \lambda} \right)^{r_n}, \tag{9}$$

where A_n and r_n are some positive constants. The choice of the representation (9) ensures the equality of the functions $G_n(\lambda)$ and $\Lambda_n^{-1}(\lambda)$ at $\lambda = w_n$: $G_n(w_n) = \Lambda_n^{-1}(w_n) = 0$. The choice of the representation (9) also ensures that both functions $G_n(\lambda)$ and $\Lambda_n^{-1}(\lambda)$ tend to $+\infty$ as $\lambda \to v_n - 0$.

Let's choose the constants A_n so that the values of the functions $G_n(\lambda)$ and $\Lambda_n^{-1}(\lambda)$ are equal in the centers of the intervals (w_n, v_n) i.e. at $\lambda = \gamma_n = 0.5\,(v_n + w_n)$:

$$G_n(\gamma_n) = \Lambda_n^{-1}(\gamma_n). \tag{10}$$

Substitution the values of $\lambda = \gamma_n$ into formula (9) gives one the equalities $G_n(\gamma_n) = A_n$. Given equality (10), we obtain

$$A_n = \Lambda_n^{-1}(\gamma_n) = -\gamma_n\, \mathrm{ctg}\gamma_n. \tag{11}$$

To find the constants r_n, we use the LSM. Taking logarithm of the left and the right hand sides of equation (9), we obtain

$$\psi_n(\lambda) = r_n \varphi_n(\lambda), \quad w_n < \lambda < v_n. \tag{12}$$

TABLE 1. Values of approximation constants r_n for the selected intervals

j	1	2	3	4	5	6	7	8	9	$J = 10$
n_j	1	2	3	4	5	7	10	15	30	$N = n_J = 50$
r_{n_j}	0.961	0.878	0.864	0.854	0.849	0.843	0.838	0.834	0.832	$r_J = r_{50} = 0.831$

Here $\psi_n(\lambda) = \ln(G_n(\lambda)/A_n)$ and $\varphi_n(\lambda) = \ln((\lambda - w_n)/(v_n - \lambda))$. According to the LSM, on each interval (w_n, v_n), we choose M points $\lambda_{nm} \in (w_n, v_n)$, where m is the ordinal number of the chosen argument λ_{nm} and pick the coefficients r_n to minimize the residual sum of squares

$$\varepsilon_n(r_n) = \frac{1}{M}\sum_{m=1}^{M}\left(\psi_n(\lambda_{nm}) - r_n\varphi_n(\lambda_{nm})\right)^2. \tag{13}$$

Differentiating with respect to r_n, one obtains

$$\frac{d\varepsilon_n(r_n)}{dr_n} = \frac{2}{M}\sum_{m=1}^{M}\psi_n(\lambda_{nm})\varphi_n(\lambda_{nm}) - \frac{2r_n}{M}\sum_{m=1}^{M}\left(\varphi_n(\lambda_{nm})\right)^2 = 0. \tag{14}$$

The solution of equation (14) gives one the values of the constants r_n

$$r_n = \sum_{m=1}^{M}\psi_n(\lambda_{nm})\varphi_n(\lambda_{nm}) \Bigg/ \sum_{m=1}^{M}\left(\varphi_n(\lambda_{nm})\right)^2. \tag{15}$$

Let's find an *approximate analytical* dependence of the constant r_n on the interval number n. We calculate r_n for J intervals with ordinal numbers $n_1, n_2, ..., n_j, ..., n_J = N$. On each interval n_j, we choose M points $\lambda_{n_j m} = w_{n_j} + m\,\Delta_{n_j}$, where $\Delta_{n_j} = (\nu_{n_j} - w_{n_j})/(M+1)$, and calculate the values of r_n by formula (15). Table 1 gives one the values of r_{n_j} for $J = 10$, $M = 20$, $N = 50$.

Keeping in mind the obtained values r_{n_j}, we can suggest the following approximate analytical dependence of the constant r_n on the number n:

$$r_n \approx \hat{r}_n = \psi(n, \alpha) = r_N + \frac{r_1 - r_N}{n^\alpha}, \tag{16}$$

provided that the approximation parameter α is positive: $\alpha > 0$. The choice of formula (16) ensures that equality $r_1 = \hat{r}_1$ and approximation equality $r_N \approx \hat{r}_N$ are satisfied, since $N = 50 \gg 1$ and the second summand in the right side of formula (16) is small.

To find the value of the parameter α, we use the LSM. Dependence (16) will be reduced to a linear model on the parameter α after elementary transformations:

$$\ln\left(\frac{r_1 - r_N}{\hat{r}_n - r_N}\right) = \alpha\ln n, 2 \le n \le N-1.$$

According to the LSM, by formulas similar to (12)–(15), we obtain the value

$$\alpha = \sum_{j=2}^{J-1}\ln\left(\frac{r_1 - r_N}{\hat{r}_{n_j} - r_N}\right)\ln n_j \Bigg/ \sum_{j=2}^{J-1}(\ln n_j)^2.$$

We find the inverse functions G_n^{-1} for the functions G_n by solving the equations $k = G_n(\lambda)$ with respect to λ. Taking into account that the functions G_n^{-1} are approximations of the functions $\Lambda_n^{-1}(\lambda)$, we find approximate values $\lambda_n \approx \hat{\lambda}_n$ for the eigenfrequencies of the Sturm–Liouville problem

$$\hat{\lambda}_n = \frac{k^{q_n}w_n + (A_n)^{q_n}v_n}{k^{q_n} + (A_n)^{q_n}},$$

where $q_n = 1/\hat{r}_n$.

Table 2 shows the results of solving two reference problems. The first problem ($s = 1$) is solved above; the second problem ($s = 2$) has different boundary condition and was solved similarly to the first problem.

In Table 2, we use the following notations: s is the number of the problem $s = 1, 2$, $v_n = \pi n$, $w_n = \pi(n - 0,5)$, $n \in N$,

$$\hat{r}_n^{(s)} = r_N^{(s)} + \frac{r_1^{(s)} - r_N^{(s)}}{n^{\alpha_s}}, \tag{17}$$

$$\varphi_n^{(s)}(k, a, b) = \frac{a\,k^{q_n^{(s)}} + b(A_n^{(s)})^{q_n^{(s)}}}{k^{q_n^{(s)}} + (A_n^{(s)})^{q_n^{(s)}}}, \tag{18}$$

TABLE 2. One-parametric reference Sturm–Liouville problems: boundary conditions, intermediate solution results and approximate formulas for eigenfrequencies

s	Boundary conditions	Spectral equation $\Delta^{(s)}(\lambda, k) = 0$	Function $k = (\Lambda^{(s)})^{-1}(\lambda)$, domain of function	Approximate function $k = \left(\Lambda_n^{(s)}\right)^{-1}(\lambda)$	Eigenfrequencies $\lambda_n^{(s)} = \varphi_n^{(s)}(k, a, b)$	α_s $r_1^{(s)}$ $r_{50}^{(s)}$
1	$y(0) = 0$ $y'(1) + ky(1) = 0$	$\Delta^{(1)}(\lambda, k) =$ $= \lambda \cos \lambda + k \sin \lambda = 0$	$k = -\lambda \text{ctg}\lambda$, $\lambda \in [w_n, v_n)$	$k =$ $= A_n^{(1)} \left(\frac{\lambda - w_n}{v_n - \lambda} \right)^{\hat{r}_n^{(1)}}$	$\lambda_n^{(1)} =$ $= \varphi_n^{(1)}(k, w_n, v_n)$	0.9 0.961 0.831
2	$y'(0) = 0$ $y'(1) + ky(1) = 0$	$\Delta^{(2)}(\lambda, k) =$ $= \lambda \sin \lambda - k \cos \lambda = 0$	$k = \lambda \text{tg}\lambda$, $\lambda \in [v_{n-1}, w_n)$	$k =$ $= A_n^{(2)} \left(\frac{\lambda - v_{n-1}}{w_n - \lambda} \right)^{\hat{r}_n^{(2)}}$	$\lambda_n^{(2)} =$ $\varphi_n^{(2)}(k, v_{n-1}, w_n)$	0.8 1.100 0.840

where

$$A_n = \left(\Lambda_n^{(s)}\right)^{-1}(\gamma_n) \tag{19}$$

$$q_n^{(s)} = \frac{1}{\hat{r}_n^{(s)}}. \tag{20}$$

The quality of the problem solution is characterized by the relative errors $\delta_n = \delta_n(k) = \left|\lambda_n - \hat{\lambda}_n\right| / \lambda_n$ of the determined values $\hat{\lambda}_n$. The functions $\delta_n = \delta_n(k)$ are determined parametrically using the parameter λ:

$$\begin{cases} k = \Lambda^{-1}(\lambda), \\ \delta_n = \left| 1 - \dfrac{\hat{\lambda}_n(\Lambda^{-1}(\lambda))}{\lambda_n} \right|. \end{cases}$$

Numerical calculations have shown that the relative errors of the approximate calculation of $\lambda_n \approx \hat{\lambda}_n$ values do not exceed the magnitude of 0.002.

3. The two-parametric Sturm–Liouville problem

In this section, we show how to use the obtained solutions of one-parametric reference problems to solve the *two-parametric* Sturm–Liouville problem.

We consider the problem on longitudinal oscillations of an elastic rod, whose ends are supported by elastic springs S_1 and S_2 with stiffnesses K_1 and K_2 (Fig. 3).

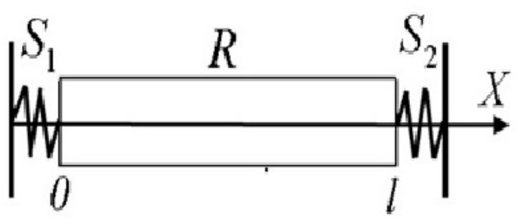

FIG. 3. Coordinate system for the two-parametric problem. The ends of the rod *R* are supported by springs S_1 and S_2 at $X = 0$ and $X = l$

Introducing the dimensionless quantities, as was done earlier in Section 2, we obtain the Sturm–Liouville problem for finding the set of eigenfunctions $y(x)$ and the set of eigenfrequencies λ: $y''(x) = -\lambda^2 y(x)$ for $0 < x < 1$ with boundary conditions

$$y'(0) - k_1 \quad y(0) = 0, \quad y'(1) + k_2 \quad y(1) = 0, \tag{21}$$

where $k_p = K_p/EFl$, $p = 1, 2$ are dimensionless spring stiffnesses.

We show that the solution of the two-parametric problem is reduced to sequential solving of the one-parametric reference problems. The general solution of the differential equation (4) has the form (6). Substituting it into the boundary conditions (21), we obtain a system of linear algebraic equations for constants C_1 and C_2.

$$\begin{cases} \lambda\, C_1 - k_1\, C_2 = 0, \\ (\lambda \cos\lambda + k_2\, \sin\lambda)\, C_1 + (k_2 \cos\lambda - \lambda \sin\lambda)\, C_2 = 0. \end{cases} \tag{22}$$

To obtain the eigenfunctions $y(x)$ we find a nonzero solution of the homogeneous system (22) from the condition that the principal determinant $\Delta(\lambda, k_1, k_2)$ is zero:

$$\Delta(\lambda,\, k_1,\, k_2) = k_1 k_2\, \sin\lambda + \lambda(k_1 + k_2) \cos\lambda - \lambda^2 \sin\lambda = 0. \tag{23}$$

Equation (23) is the spectral equation for calculating the eigenfrequencies of the two-parametric problem for given values of parameters k_1 and k_2. Let a function Γ represent the dependence of λ on parameters k_1 and k_2

$$\lambda = \Gamma(k_1, k_2). \tag{24}$$

The function $\Gamma(k_1, k_2)$ is a non-elementary multivalued function of two variables k_1 and k_2, implicitly given by equation (23). Equation (23) is linear with respect to parameters k_1, k_2, and their permutation does not change the equation. Let a function Γ^{-1} represent the dependence of k_1 on λ and k_2.

$$k_1 = \Gamma^{-1}(\lambda, k_2). \tag{25}$$

It follows from equation (23) that the function $\Gamma^{-1}(\lambda, k_2)$ has the following form

$$k_1 = \Gamma^{-1}(\lambda, k_2) = \frac{\lambda(\lambda \sin\lambda - k_2 \cos\lambda)}{(\lambda \cos\lambda + k_2 \sin\lambda)}. \tag{26}$$

The function $\Gamma^{-1}(\lambda, k_2)$ is a meromorphic function of the variable λ. It also depends on a fixed value of the parameter k_2. The values of roots W_n and poles V_n of the function $\Gamma^{-1}(\lambda, k_2)$ are found from the solutions of the reference problems given in Table 2. We find the positive poles $V_n = V_n(k_2)$ of the function $\Gamma^{-1}(\lambda, k_2)$ from the spectral equation $\lambda \cos\lambda + k_2 \sin\lambda = 0$ of the first reference problem (Table 2, $s = 1$): $V_n(k_2) = \lambda_n^{(1)}$. We find the positive roots $W_n = W_n(k_2)$ of the function $\Gamma^{-1}(\lambda, k_2)$ from the spectral equation $\lambda \sin\lambda - k_2 \cos\lambda = 0$ of the second reference problem (Table 2, $s = 2$): $W_n(k_2) = \lambda_n^{(2)}$. Using Table 2, we obtain formulas for the positive poles and roots of the function $\Gamma^{-1}(\lambda, k_2)$: $\lambda_n^{(1)} = \varphi_n^{(1)}(k_2, w_n, v_n)$ and $\lambda_n^{(2)} = \varphi_n^{(2)}(k_2, v_n, w_n)$. Note that the inequalities $W_n(k_2) < V_n(k_2)$ are satisfied. Graph of the function $\Gamma^{-1}(\lambda, k_2)$ is similar to graph of the function $k = \Lambda^{-1}(\lambda)$ in Fig. 2 if we replace quantities v_n and w_n with quantities V_n and W_n, respectively.

Then the two-parametric problem is solved by the method used above for solving the first reference problem:

1. We introduce the functions $\Gamma_n^{-1}(\lambda, k_2)$, $\lambda \in [W_n, V_n)$.
2. We approximate the functions $k_1 = \Gamma_n^{-1}(\lambda, k_2)$ by the functions $G_n(\lambda, k_2)$,
$$G_n(\lambda, k_2) = A_n(k_2) \left(\frac{\lambda - \lambda_n^{(2)}(k_2)}{\lambda_n^{(1)}(k_2) - \lambda} \right)^{\hat{r}_n(k_2)},$$
where $A_n(k_2)$ and $\hat{r}_n(k_2)$ are approximation constants to be calculated.
3. We calculate the constants $A_n(k_2) = \Gamma_n^{-1}(\gamma_n(k_2), k_2)$, where
$$\gamma_n(k_2) = 0.5\, (\lambda_n^{(1)}(k_2) + \lambda_n^{(2)}(k_2)),$$
$$\Gamma_n^{-1}(\gamma_n(k_2), k_2) = \frac{\gamma_n(k_2)(\gamma_n(k_2) \sin\gamma_n - k_2 \cos\gamma_n(k_2))}{(\gamma_n(k_2) \cos\gamma_n(k_2) + k_2 \sin\gamma_n(k_2))}.$$
4. We calculate the constants $\hat{r}_n(k_2)$ by formulas similar to formulas (12)–(15).
5. We find the inverse functions G_n^{-1} and calculate approximate analytical dependencies of eigenfrequencies $\hat{\lambda}_n = G_n^{-1}(k_1, k_2)$ on the parameters k_1 and k_2.

As a result, in the case of the two-parametric problem, we calculate the eigenfrequencies λ_n for the given values of the parameters k_1 and k_2 by the approximate formula

$$\lambda_n \approx \hat{\lambda}_n = \frac{\lambda_n^{(2)}(k_2) k_1^{q_n(k_2)} + (A_n(k_2))^{q_n(k_2)} \lambda_n^{(1)}(k_2)}{k_1^{q_n(k_2)} + (A_n(k_2))^{q_n(k_2)}}, \tag{27}$$

$$q_n(k_2) = 1/\hat{r}_n(k_2), \quad \hat{r}_n = \hat{r}_n(k_2) = r_{50}(k_2) + \frac{r_1(k_2) - r_{50}(k_2)}{n^{\alpha}}.$$

Numerical calculations using the above algorithm for the two-parameter problem at the values of the parameters $k_1 = 2$, $k_2 = 1$ gave us the following results: $\alpha = 1.320$, $r_1(k_2) = 0.947$, $r_{50}(k_2) = 0.837$. The relative errors δ_n of calculating $\lambda_n \approx \hat{\lambda}_n$ values do not exceed the magnitude of 0.002.

4. Conclusion

Mathematical modeling of applied problems may lead to multi-parameter boundary value problems. This is the case in the problem of longitudinal oscillations of a rod, the ends of which are supported by elastic springs and weighed by masses. Several parameters also appear in the problem of bending oscillations of an elastic beam with elastic springs and masses with given moments of inertia located at its ends [13].

The proposed approach makes it possible to find solutions of multi-parametric problems using the solutions of reference one-parametric problems. We obtain n one-parametric problems from an n-parametric Sturm–Liouville problem assuming that all but one parameter is zero. The one-parametric problems are reference problems for the n-parametric problem. By solving each one-parametric problem, we find an approximate analytical dependence of the eigenfrequencies on the single parameter of the problem. Then we sequentially reduce the n-parametric Sturm–Liouville problem to the solution of n one-parametric problems. As a result, we obtain an approximate analytical dependence of the eigenfrequencies on all parameters of the n-parametric Sturm–Liouville problem.

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