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Original article

The point spectrum of the three-particle Schrödinger operator for a system comprising two identical bosons and one fermion on $\mathbb{Z}$

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ABSTRACT We consider the Hamiltonian of a system of three quantum particles (two identical bosons and a fermion) on the one-dimensional lattice interacting by means of zero-range attractive or repulsive potentials. We investigate the point spectrum of the three-particle discrete Schrödinger operator $H(K)$, $K \in \mathbb{T}$ which possesses infinitely many eigenvalues depending on repulsive or attractive interactions, under the assumption that the bosons in the system have infinite mass.

KEYWORDS Schrödinger operator, dispersion functions, zero-range pair potentials, discrete spectrum, essential spectrum.

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1. Introduction

In physics, stable composite objects often result from attractive forces, enabling the constituents to lower their energy by binding together. In contrast, repulsive forces tend to scatter particles when they are in free space. However, within structured environments such as a periodic potential, and in the absence of dissipation, stable composite objects can exist even with repulsive interactions arising from the lattice band structure [1].

The Bose-Hubbard model, which describes repulsive pairs, serves as the theoretical basis for various applications.

The work referenced by [1] exemplifies the crucial link between the Bose-Hubbard model [2, 3] and atoms in optical lattices, and helps pave the way for many more interesting developments and applications, as discussed in [4]. In cold atom lattice physics, stable repulsively bound objects are expected to be common, which leading to potential composites with fermions [5] or Bose-Fermi mixtures [6], and can even be formed with more than two particles in a similar manner.

The Efimov effect, first discovered by V. Efimov in 1970 [7], is one of the most intriguing phenomena in physics. It occurs in three-body systems in three-dimensional space that interact through short-range pairwise potentials. It is always possible to ensure the couplings of the interactions in such a way that none of the particle pairs has a negative energy bound state, but at least two pairs exhibit a resonance at zero energy. The existence of this effect of a three-particle Schrödinger operator was discovered by Yafaev [8] and in the discrete case by others [9–14].

Recently, the authors [15–17] considered perturbations of the system of three arbitrary quantum particles on lattices $\mathbb{Z}^d$, $d = 1, 2$, interacting through attractive zero-range pairwise potentials. They established that the three-particle Schrödinger operators posses infinitely many negative eigenvalues for all positive non-zero point interactions, under the assumption that two particles in the system have infinite mass. Also, note that the author of [17] obtained asymptotics for these eigenvalues.

The main goal of the paper is to investigate the existence of infinite number of bound states of the three-particle discrete Schrödinger operator associated with a system of two identical bosons and a fermion, where the bosons have infinite mass and the fermion has a finite mass. This investigation is performed on the one-dimensional lattice $\mathbb{Z}$ and involves repulsive or attractive zero-range pairwise interactions. The problem is reduced to the study the Fredholm determinant of a diagonal operator.

It should be noted that, unlike the last three articles [15–17], we study eigenvalues below and above the essential spectrum of the unperturbed operator for all repulsive or attractive zero-range pairwise interactions.

It is worth mentioning that in the continuous case, the authors of [18] studied a very similar system, specifically a system consisting of two infinitely heavy quantum particles and one light particle in three-dimensional space $\mathbb{R}^3$, interacting via long-range Coulomb pair potentials. Although the authors of this book are not interested in the number of eigenvalues, they briefly explained the scheme for solving problems of this type using the Coulomb spheroidal function decomposition. In the present paper, the problem is reduced to the study of eigenvalues of convolution type compact integral operators.

The paper is organized as follows. In Section 2, we introduce the three-particle discrete Schrödinger operator $H(K)$ and the two-particle discrete Schrödinger operators associated with subsystems of the system of two identical bosons and a fermion. In Section 3, we study the essential spectrum of the three-particle discrete Schrödinger operator $H(K)$. The eigenvalues of $H(K)$ below and above the spectrum of the non-perturbed operator $H_0(K)$ are investigated in Section 4. Section 5 is devoted to showing main result, Theorem 5.1.

2. Three-particle discrete Schrödinger operator on the lattice $\mathbb{Z}$

Let $L^2_s(\mathbb{T}^2)$ be the linear subspace of the symmetric functions of the Hilbert space $L^2((\mathbb{T})^2)$.

Let us consider the discrete Schrödinger operator $H(K)$, where $K \in \mathbb{T}$, associated with a system consisting of two identical bosons and a fermion moving on the one-dimensional lattice $\mathbb{Z}$ (see [15, 17]).

$$H(K) = H_0(K) - V$$

with zero-range attractive potentials

$$V = V_1 + V_2 + V_3,$$

where

$$(V_1 f)(p,q) = \frac{\lambda}{2\pi}\int\limits_{\mathbb{T}} f(p,t)dt, \quad (V_2 f)(p,q) = \frac{\lambda}{2\pi}\int\limits_{\mathbb{T}}, f(t,q)dt$$

$$(V_3 f)(p,q) = \frac{\mu}{2\pi}\int\limits_{\mathbb{T}} f(t,p+q-t)dt, \quad f \in L^2_s(\mathbb{T}^2),\ p,q \in \mathbb{T},$$

and numbers λ and μ serve as the parameters of boson-fermion interaction and boson-boson interaction, respectively.

Here the numbers λ and μ indicate repulsive pair-wise interaction when $\lambda < 0$ and $\mu < 0$, and attractive pair-wise interaction when $\lambda > 0$ and $\mu > 0$. The operator $H_0(K)$ is defined on the Hilbert space $L^2_s(\mathbb{T}^2)$ by

$$\big(H_0(K)f\big)(p,q) = E(K;p,q)f(p,q), \quad f \in L^2_s(\mathbb{T}^2),$$

and

$$E(K;p,q) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{b}}(q) + \varepsilon_{\mathfrak{f}}(K-p-q), \quad p,q \in \mathbb{T}.$$

Here, the real-valued continuous function $\varepsilon_{\mathfrak{b}}(\cdot)$ and $\varepsilon_{\mathfrak{f}}(\cdot)$, referred to as *the dispersion relation* associated with the free boson and fermion, is defined as

$$\varepsilon_{\mathfrak{b}}(p) = \frac{1}{m}\varepsilon(p), \quad \varepsilon_{\mathfrak{f}}(p) = \frac{1}{\mathfrak{m}}\varepsilon(p), \quad \varepsilon(p) = 1 - \cos(p), \quad p \in \mathbb{T}, \tag{1}$$

respectively, and m and $\mathfrak{m}$ represents the mass of the boson and fermion, respectively.

Let $k \in \mathbb{T}$ and $L^2_k(\mathbb{T})$ be a linear subspace of the Hilbert space $L^2(\mathbb{T})$ defined by

$$L^2_k(\mathbb{T}) = \{f \in L^2(\mathbb{T}) | f(p) = f(k-p)\}.$$

A two-particle discrete Schrödinger operator corresponding to the subsystem {bozon,fermion} and {bozon,bozon}, of the three-particle system acts on the Hilbert space $L^2(\mathbb{T})$ and $L^2_k(\mathbb{T})$ as

$$h_1(k) = h^0_1(k) - v_1, \quad \text{and} \quad h_2(k) = h^0_2(k) - v_2, \quad k \in \mathbb{T}, \tag{2}$$

respectively.

Here, the operators $h^0_\alpha(k)$

$$\big(h^0_1(k)f\big)(p) = E^{(1)}_k(p)f(p), \quad f \in L^2(\mathbb{T}),$$

and

$$\big(h^0_2(k)f\big)(p) = E^{(2)}_k(p)f(p), \quad f \in L^2_k(\mathbb{T}),$$

where

$$E^{(1)}_k(p) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{f}}(k-p), \quad E^{(2)}_k(p) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{b}}(k-p), \quad p \in \mathbb{T}. \tag{3}$$

The operators v_1 and v_2 are defined as

$$(v_1 f)(p) = \frac{\mu}{2\pi}\int\limits_{\mathbb{T}} f(q)dq, \quad f \in L^2(\mathbb{T}),\ p \in \mathbb{T}.$$

and

$$(v_2 f)(p) = \frac{\lambda}{2\pi}\int\limits_{\mathbb{T}} f(q)dq, \quad f \in L^2_k(\mathbb{T}),\ p \in \mathbb{T},$$

respectively.

2.1. Spectral properties of the two-particle discrete Schrödinger operators when $m = \infty$ and $0 < \mathfrak{m} < \infty$

With $m = \infty$ and $0 < \mathfrak{m} < \infty$ and the equality (1), the functions (3) can be written as

$$E_k^{(1)}(p) = \varepsilon_{\mathfrak{f}}(k-p) = \varepsilon(k-p)/\mathfrak{m},\ E_k^{(2)}(p) = 0,\ p \in \mathbb{T}.$$

Consequently, since the potentials v_α, $\alpha = 1, 2$ have a convolution-type property, all two-particle Schrödinger operators do not depend on the quasi-momentum $k \in \mathbb{T}$,

$$h_1 := h_1(k) \quad \text{and} \quad h_2 := h_2(k).$$

Then, the operators $h_1(k)$ and $h_2(k)$ act as

$$h_1(k)f(p) = \varepsilon_{\mathfrak{f}}(p)f(p) - (v_1 f)(p), \quad f \in L^2(\mathbb{T}) \quad \text{and} \qquad h_2(k)f(p) = -(v_2 f)(p), \quad f \in L_k^2(\mathbb{T}).$$

As v_1 is a finite rank operator, according to the Weyl theorem, the essential spectrum $\sigma_{ess}\big(h_1(k)\big)$ of the operator $h_1(k)$ in (2) coincides with the spectrum $\sigma\big(h_1^0(k)\big)$ of the non-perturbed operator $h_1^0(k)$. More specifically,

$$\sigma_{ess}\big(h_1(k)\big) = \big[E_{\min}^{(1)}(k),\, E_{\max}^{(1)}(k)\big],$$

where

$$E_{\min}^{(1)}(k) \equiv \min_{p\in\mathbb{T}} E_k^{(1)}(p), \quad E_{\max}^{(1)}(k) \equiv \max_{p\in\mathbb{T}} E_k^{(1)}(p).$$

Therefore, in our case, we have

$$\sigma_{ess}\big(h_1(k)\big) = \big[0, 2/\mathfrak{m}\big] \quad \text{and} \quad \sigma_{ess}\big(h_2(k)\big) = \{0\}.$$

The Fredholm determinants associated with the operators $h_1(k)$ are defined as

$$\Delta(\lambda; z) = 1 - \lambda d_0(z), \quad d_0(z) = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \frac{ds}{\varepsilon_{\mathfrak{f}}(s) - z}, \quad z \in \mathbb{C} \setminus \big[0, 2/\mathfrak{m}\big].$$

Lemma 2.1. *(a) The number $z \in \mathbb{C} \setminus \big[0, 2/\mathfrak{m}\big]$ is an eigenvalue of $h_1(k)$ if and only if $\Delta(\lambda; z) = 0$.*

(b) If $\lambda < 0$ and $\lambda > 0$, then there exists a unique simple eigenvalue $z = z_1^0$ of $h_1(k)$ in the interval $\big(-2/\mathfrak{m} - \mu, \infty\big)$ and $\big(-\infty, 0\big)$, respectively. Moreover, z_1^0 does not depend on $k \in \mathbb{T}$.

Proof. (a) The equation

$$h_1(k)f = zf \quad \text{i.e.,} \quad f = \big(h_1^0(k) - z\big)^{-1} v_1 f$$

has a non-trivial solution if and only if

$$\Delta(\lambda; z)C = 0, \quad C \in \mathbb{C},$$

has a non-trivial solution.

Therein, the solutions $C \in \mathbb{C}$ and $f \in L^2(\mathbb{T})$ are related as follows

$$C = v_1 f \quad \text{and} \quad f = \big(h_1^0(k) - z\big)^{-1} C.$$

(b) Let $\lambda > 0$. The function $\Delta(\lambda; z)$ is monotonic decreasing in $(-\infty, 0)$ and $\Delta(\lambda; z) > 1$ in $\big(-2/\mathfrak{m} - \mu, \infty\big)$. Since

$$\lim_{z\to-\infty} \Delta(\lambda; z) = 1 \quad \text{and} \quad \lim_{z\to 0-} \Delta(\lambda; z) = -\infty,$$

the intermediate-value theorem implies the existence of a unique simple zero $z = z_1^0$, $z_1^0 \in (-\infty, 0) \cup \big(-2/\mathfrak{m} - \mu, \infty\big)$ of the function $\Delta(\lambda; \cdot)$, and furthermore $z_1^0 \in (-\infty, 0)$.

The lemma can be proven in a similar way when $\lambda < 0$. □

Now, we can summarize the results of this section in the following lemma.

Lemma 2.2. *We have*

$$\sigma_{\rm disc}\big(h_1(k)\big) = \{z_1^0\}, \quad \textit{if} \quad \lambda \neq 0,$$
$$\sigma\big(h_1(k)\big) = \{z_1^0\} \cup \big[0, 2/\mathfrak{m}\big], \quad \textit{if} \quad \lambda \neq 0$$

and

$$\sigma_{\rm disc}\big(h_2(k)\big) = \{-\mu\}, \quad \sigma\big(h_2(k)\big) = \{-\mu\} \cup \{0\}.$$

3. Essential spectrum of $H(K)$

One of the notable outcomes in the spectral theory of multi-particle continuous Schrödinger operators involves characterizing the essential spectrum of the Schrödinger operators in terms of cluster operators (the HVZ-theorem. See Refs. [19–23] for the discrete case and [24] for a pseudo-relativistic operator).

Lemma 3.1. *The essential spectrum of $H(K)$ satisfies the relation*

$$\sigma_{ess}\big(H(K)\big)=\bigcup_{k\in\mathbb{T}}\Big\{\sigma\big(h_1(K-k)\big)+\varepsilon_{\mathfrak{b}}(k)\Big\}\cup\bigcup_{k\in\mathbb{T}}\Big\{\sigma\big(h_2(K-k)\big)+\varepsilon_{\mathfrak{f}}(k)\Big\}.$$

Proof. The proof can be found in [17, 21]. □

3.1. The essential spectrum of $H(K)$ with $m=\infty$, and $\mathfrak{m}<\infty$

Due to Lemma 2.2 and the relations $\varepsilon_{\mathfrak{b}}(p)=0$ and $\varepsilon_{\mathfrak{f}}(p)=\varepsilon(p)/\mathfrak{m}$, we obtain

$$\bigcup_{k\in\mathbb{T}}\Big\{\sigma\big(h_1(K-k)\big)+\varepsilon_{\mathfrak{b}}(k)\Big\}=\sigma\big(h_1(k)\big)=\{z_1^0\}\cup\big[0,2/\mathfrak{m}\big],$$

$$\bigcup_{k\in\mathbb{T}}\Big\{\sigma\big(h_2(K-k)\big)+\varepsilon_{\mathfrak{f}}(k)\Big\}=\bigcup_{k\in\mathbb{T}}\Big\{\{-\mu\}\cup\{0\}+\varepsilon_{\mathfrak{f}}(k)\Big\}=\big[-\mu,2/\mathfrak{m}-\mu\big]\cup\big[0,2/\mathfrak{m}\big].$$

According to the last two relations and Lemma 3.1, we have

Lemma 3.2.

$$\sigma_{ess}\big(H(K)\big)=\{z_1^0\}\cup\big([-\mu,2/\mathfrak{m}-\mu]\cup[0,2/\mathfrak{m}]\big).$$

4. The point spectrum of $H(K)$ for $m=\infty$ and $\mathfrak{m}<\infty$

One can show easily that the subspace

$$\mathcal{A}_0=\{f\in L_s^2(\mathbb{T}\times\mathbb{T})|f(p,q)=g(p+q),\ g\in L^2(\mathbb{T})\}$$

is invariant under the operator $H(K)$, and so it is $\mathcal{A}_0^{\perp}=L_s^2(\mathbb{T}\times\mathbb{T})\ominus\mathcal{A}_0$.

Therefore, we have

$$\sigma_{pp}\big(H(K)\big)=\sigma_{pp}\big(A_0(K)\big)\cup\sigma_{pp}\big(A_1(K)\big),$$

where $A_0(K) and\ A_1(K)$ are restrictions of $H(K)$ on the linear subspaces $\mathcal{A}_0$ and $\mathcal{A}_0^{\perp}$, respectively.

Since $\mathcal{A}_0$ and $L^2(\mathbb{T})$ are isomorphic, the operator $A_0(K)$ is unitarily equivalent to the operator B_0 on $L^2(\mathbb{T})$, where

$$B_0=E_0(K)-\mu I-2\lambda v, \tag{4}$$

$E_0(K)$ denotes the multiplication by the function $\varepsilon_{\mathfrak{f}}(K-p)$, I is the identity operator, and v is an integral operator defined by

$$(vf)(p)=\frac{1}{2\pi}\int_{\mathbb{T}}f(q)dq,\quad f\in L^2(\mathbb{T}),\ p\in\mathbb{T}.$$

The operator $A_1(K)$ takes the form

$$A_1(K)=H_0(K)-\lambda V_1-\lambda V_2.$$

Let $U_K:L_s^2(\mathbb{T}\times\mathbb{T})\to L_s^2(\mathbb{T}\times\mathbb{T})$ be a unitary operator defined as

$$(U_Kf)(p,q)=f\big(-K/2+p,-K/2+q\big). \tag{5}$$

It establishes a unitary equivalence between $H(K)$ and $H(0)$, and so we can prove the coming statements for $H(0)$.

4.1. Spectrum of $A_0(K)$

The equivalence (4) and the Weyl theorem imply that

$$\sigma_{ess}\big(A_0(K)\big)=\big[-\mu,2/\mathfrak{m}-\mu\big].$$

The following lemma describes the behaviour of the eigenvector of $H(K)$ in the linear space $\mathcal{A}_0$.

Lemma 4.1. *(a) If $\lambda<0$, then $A_0(K)$ has an unique eigenfunction with the corresponding eigenvalue η, $\eta\in\big(2/\mathfrak{m}-\mu,\infty\big)$ for any $\mu\in\mathbb{R}$.*

(b) If $\lambda=0$, then $A_0(K)$ has no eigenvalues for any $\mu\in\mathbb{R}$.

(c) If $\lambda>0$, then $A_0(K)$ has an unique eigenfunction with the corresponding eigenvalue η, $\eta\in(-\infty,-\mu)$ for any $\mu\in\mathbb{R}$.

Proof. The equation

$$H(0)f = zf, \quad f \in \mathcal{A}_0,$$

has a solution if and only if the equation

$$C\big(1 - 2\lambda d_0(\mu + z)\big) = 0, \tag{6}$$

has one, and their solutions are related by

$$f(p,q) := g(p+q) = \frac{2\lambda}{\varepsilon_{\mathfrak{f}}(-p-q) - \mu - z} C,$$

where

$$C = \frac{1}{2\pi}\int_{\mathbb{T}} g(t)dt.$$

Equation (6) has a nontrivial solution C if and only if the equation

$$\Delta(2\lambda; z) = 1 - 2\lambda d_0(\mu + z) = 0$$

has a root. The function $\Delta(2\lambda; z)$ is defined on $(-\infty, -\mu) \cap (2/m - \mu, \infty)$ and according to Lemma 2.1: (a) If $\lambda > 0$, then $\Delta(2\lambda; z) = 1 - 2\lambda d_0(\mu + z)$ has a unique zero in $(-\infty, -\mu)$, but does not have one in $\big(2/\mathfrak{m} - \mu, \infty\big)$; (c) If $\lambda < 0$, then $\Delta(2\lambda; z)$ has a unique zero in $\big(2/\mathfrak{m} - \mu, \infty\big)$, but does not have one in $\big(-\infty, -\mu\big)$.

(b) If $\lambda = 0$, then $B_0 = E_0(K) - \mu I$ is a multiplication operator, and so $A_0(K)$ has no eigenvalues for any $\mu \in \mathbb{R}$. □

4.2. Spectrum of $A_1(K)$

As the operator $A_1(K)$ does not contain the parameter μ, Lemma 3.2 implies that

$$\sigma_{ess}\big(A_1(K)\big) = \{z_1^0\} \cup [0, 2/\mathfrak{m}].$$

In order to study need, we define the following integral depending on $n \in \mathbb{Z}$:

$$d_n(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{int}}{\varepsilon_{\mathfrak{f}}(t) - z} dt, \quad z \in \mathbb{C} \setminus [0, 2/\mathfrak{m}].$$

Lemma 4.2. *For any fixed $n \in \mathbb{Z}$, the function $d_n(z)$ satisfies the equality*

$$d_n(z) = \mathfrak{m}\frac{\big(1 - \mathfrak{m}z - \sqrt{\mathfrak{m}^2 z^2 - 2z/\mathfrak{m}}\big)^{|n|}}{\sqrt{\mathfrak{m}^2 z^2 - 2z/\mathfrak{m}}} \quad z \in \mathbb{C} \setminus [0, 2/\mathfrak{m}].$$

Proof. The theory of residues provides the proof (see [17, Lemma 6]). □

The operators V_1 and V_2 are represented as $V_1 = \lambda\varphi_1^*\varphi_1$ and $V_2 = \lambda\varphi_2^*\varphi_2$, respectively, where the operators $\varphi_1, \varphi_2 : L_s^2(\mathbb{T}\times\mathbb{T}) \to L^2(\mathbb{T})$ are of the form

$$(\varphi_1 f)(p) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{T}} f(p,t)dt \quad \text{and} \quad (\varphi_2 f)(q) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{T}} f(t,q)dt.$$

Lemma 4.3. *The number $z \in \mathbb{C} \setminus \sigma_{ess}\big(A_1(K)\big)$ is an eigenvalue of the operator $A_1(K)$ if and only if $D(z) = 0$, where*

$$D(z) = \frac{1}{\Delta^2(\lambda; z)}\prod_{n\in\mathbb{Z}} \Delta_n^+(\lambda; z), \tag{7}$$

with

$$\Delta_n^+(\lambda; z) = 1 - \lambda\big(d_0(z) + d_n(z)\big).$$

If $z_n \in \mathbb{C} \setminus \sigma_{ess}\big(A_1(K)\big)$ is an eigenvalue of $A_1(K)$, then $D_n(z_n) = 0$, $n \in \mathbb{Z}$ and the corresponding eigenfunction is of the form

$$f_n(p,q) = \frac{\lambda}{\varepsilon_{\mathfrak{f}}(K - p - q) - z_n}\Big(\cos\big(n(-K/2 + p)\big) + \cos\big(n(-K/2 + q)\big)\Big) \tag{8}$$

Proof. Given the unitary equivalence of $H(K)$ and $A_1(K)$ to $H(0)$ and $A_1(0)$, respectively, we first establish the claim for the latter operators.

Let $z \in \mathbb{C} \setminus \sigma_{ess}\big(A_1(0)\big)$ be an eigenvalue of $A_1(0)$, and let f be the corresponding eigenfunction, then

$$f = R_0(z)\big[\lambda V_1 + \lambda V_2\big]f, \tag{9}$$

where $R_0(z) = (H_0(0) - zI)^{-1}$ is a resolvent of $H_0(0)$.

This equation has a non-trivial solution if and only if the system of two linear equations

$$\tilde{\varphi}_\alpha = \varphi_\alpha\big(R_0(z)[\lambda\varphi_1^*\tilde{\varphi}_1 + \lambda\varphi_2^*\tilde{\varphi}_2]\big), \ \alpha = 1, 2, \ \tilde{\varphi}_1, \tilde{\varphi}_2 \in L^2(\mathbb{T}) \tag{10}$$

on the space $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ has a non-zero solution.

Solutions of equations (9) and (10) are linked by

$$f(p,q) = R_0(z)\big[\lambda\varphi_1^*\tilde{\varphi}_1 + \lambda\varphi_2^*\tilde{\varphi}_2\big], \tag{11}$$

and

$$\tilde{\varphi}_\alpha = \varphi_\alpha f, \quad \alpha = 1,2.$$

Since f is a symmetric function, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the same function, and only their arguments are different, that is, the argument of $\tilde{\varphi}_1$ is the variable p, while q is an independent variable of $\tilde{\varphi}_1$:

$$\tilde{\varphi}_2(p) = \tilde{\varphi}_1(p). \tag{12}$$

We note that the functions

$$\Delta_\alpha := I - \lambda\varphi_\alpha R_0(z)\varphi_\alpha^*, \quad z \in \mathbb{C}\setminus\sigma_{ess}\big(H_0(0)\big), \quad \alpha = 1,2,$$

are nonzero for any $z \in \mathbb{C}\setminus\sigma_{ess}\big(A_1(0)\big)$, therefore, their inverses exist,

$$\Delta_\alpha^{-1} = \big(I - \lambda\varphi_\alpha R_0(z)\varphi_\alpha^*\big)^{-1}.$$

Then, the solutions $\tilde{\varphi}_\alpha, \alpha = 1,2$, of the equation (10) satisfy the following system of integral equations

$$\begin{cases} \tilde{\varphi}_1 = \lambda\Delta_1^{-1}Q\tilde{\varphi}_2, \\ \tilde{\varphi}_2 = \lambda\Delta_2^{-1}Q^*\tilde{\varphi}_1, \end{cases} \tag{13}$$

where

$$Q = \varphi_1 R_0(z)\varphi_2^*$$

is the integral operator on $L^2(\mathbb{T})$ defined as

$$(Q\varphi)(p) = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \frac{\varphi(t)dt}{\varepsilon_{\mathfrak{f}}(-p-t) - z}$$

and Q^* is the adjoint of Q.

Using the substitution method, system (13) can be reduced into the form

$$\tilde{\varphi}_1 = Q(z)\tilde{\varphi}_1, \quad \text{i.e.,} \quad Q(z) = \lambda^2\Delta_1^{-1}\Delta_2^{-1}QQ^*. \tag{14}$$

Moreover, if $\Phi = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ is a solution to (13), then $\tilde{\varphi}_1$ is an eigenfunction of $Q(z)$ corresponding to the eigenvalue 1. Conversely, suppose that $\tilde{\varphi}_1$ is an eigenfunction corresponding to the eigenvalue 1 of the operator $Q(z)$. Then $\Phi = (\tilde{\varphi}_1, \tilde{\varphi}_2)$, with $\tilde{\varphi}_2 = \dfrac{\lambda}{\Delta(\lambda;z)}Q\tilde{\varphi}_1$ is a solution to (13) (i.e. (10)). Notice that the multiplicities of the linearly independent eigenvectors $\tilde{\varphi}_1$ and Φ coincide.

We also note that the function f defined in (11) is an eigenfunction of $A_1(0)$ corresponding to an eigenvalue $z \in \mathbb{C}\setminus\sigma_{ess}\big(A_1(0)\big)$. Moreover, the multiplicity of the eigenvalues z of $A_1(0)$ is the same as the multiplicity of the eigenvalue $\mu = 1$ of $Q(z)$.

The operator $Q(z)$ is a convolution-type trace-class integral operator. The standard Fourier transform $\mathcal{F}_1 : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$,

$$(\mathcal{F}_1 g)(n) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{T}} e^{inp}g(p)dp, \quad g \in L^2(\mathbb{T}),\ n \in \mathbb{Z},$$

establishes that $\widehat{Q}(z) := \mathcal{F}_1 Q(z)\mathcal{F}_1^*$ acts as a multiplication operator on the space $\ell^2(\mathbb{Z})$ by the function

$$\kappa_n(z) = \frac{\lambda^2}{\big(\Delta(\lambda;z)\big)^2}d_n(z)d_{-n}(z),\ n \in \mathbb{Z}. \tag{15}$$

Thus, the spectrum of $\widehat{Q}(z)$ consists of the following union

$$\sigma\big(\widehat{Q}(z)\big) = \{0\} \cup \bigcup_{n\in\mathbb{Z}} \{\kappa_n(z)\},$$

with the space of eigenfunctions

$$\hat{\varphi}_n(m) = \delta_{n,m},\ n,m \in \mathbb{Z},$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta function on $\mathbb{Z}$.

Note that the compact operator $Q(z)$ has eigenvalues $\kappa_n(z)$, $n \in \mathbb{Z}$ with the corresponding eigenfunctions

$$\psi_n(p) = e^{inp},\ n \in \mathbb{Z},\ p \in \mathbb{T}.$$

Therefore, the determinant of the operator $I - Q(z)$ can be written as the following product

$$\det(I - Q(z)) = \prod_{n\in\mathbb{Z}}(1 - \kappa_n(z)), \tag{16}$$

which takes the form (7), since $d_{-n}(z) = d_n(z)$ and (15).

Let $\kappa_n(z_n) = 1$ be an eigenvalue of $Q(z_n)$, then ψ_n is the first component of the solution $\varphi_n = (\psi_n, \tilde{\psi}_n)$ of equation (10), and the second component is defined as

$$\tilde{\psi}_n(q) := \tilde{\varphi}_2(q) = \lambda\Delta_2^{-1}Q^*\psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)}\tilde{\psi}_{-n}(q). \tag{17}$$

Using the equality (12) in the last relation, we get

$$e^{i(n,q)} = \lambda\Delta_2^{-1}Q^*\psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)}e^{-i(n,q)},$$

which is a contradiction.

However, if $\vartheta_n(p) = (e^{i(n,p)} + e^{-i(n,p)})/2 = \cos((n,p))$, it satisfies (12) and (17) if and only if

$$\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad \text{i.e.} \quad \Delta_n^+(\lambda; z) = 1 - \lambda\big(d_0(z_n) + d_n(z_n)\big) = 0.$$

However, for the functions $\theta_n(p) = (e^{i(n,p)} - e^{-i(n,p)})/2i = \sin((n,p))$ the relation

$$\theta_n = \lambda\Delta_2^{-1}Q^*\theta_n = -\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)}\theta_n,$$

holds, which contradicts with (12). It implies

$$-\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad \text{i.e.} \quad \Delta_n^-(\lambda; z) = 1 - \lambda\big(d_0(z_n) - d_n(z_n)\big) = 0.$$

Since the system $\{\vartheta_n, \theta_n\}$, $n \in \mathbb{Z}$ is complete in $\ell^2(\mathbb{Z})$, the last three relations and (15) allow us to use the Fredholm determinant (7) instead of (16).

Subsequently, the number z_n is an eigenvalue of $A_1(0)$ and the corresponding eigenfunction can be found by (11) as

$$f_n^0 = R_0(z_n)\big[\lambda\varphi_1^*\psi_n + \lambda\varphi_2^*\tilde{\psi}_n\big],$$

i.e.,

$$f_n^0(p,q) = \frac{\lambda}{\varepsilon_{\mathfrak{f}}(-p-q) - z}\Big(\cos(np) + \cos(nq)\Big).$$

According to the relation $H(0) = U_K^* H(K) U_K$, the number z_n is also an eigenvalue of $H(K)$ with the eigenfunction $f_n = U_K f_n^0$ in (8), where the unitary operator U_K is defined in (5). □

Remark 4.4. *Since $\Delta_n^+(\lambda; z)$ is even with respect to $n \in \mathbb{Z}$, then $z_n = z_{-n}$, where z_n is a zero of $\Delta_n^+(\lambda; z)$, and eigenfunctions* (8) *are same.*

Lemma 4.5. *For any fixed $n \in \mathbb{Z}$, the function $d_0(z) + d_n(z)$ is positive and monotonically increasing in the interval $(-\infty, 0)$ as a function of z.*

Additionally,

$$\lim_{z\to 0-}\big(d_0(z) + d_n(z)\big) = \infty, \tag{18}$$

$$\lim_{z\to -\infty}\big(d_0(z) + d_n(z)\big) = 0. \tag{19}$$

and for any $z \in (-\infty, 0)$

$$\lim_{n\to\infty} d_n(z) = 0 \tag{20}$$

hold.

Proof. The equalities

$$d_n(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\cos nt}{\varepsilon_{\mathfrak{f}}(t) - z}dt$$

and

$$d_0(z) + d_n(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1 + \cos nt}{\varepsilon_{\mathfrak{f}}(t) - z}dt$$

imply the positivity and monotonicity of $d_0(z) + d_n(z)$ in $(-\infty, 0)$.

The limits (18), (19) and (20) follow from Lemma 4.2. □

4.3. Eigenvalues of $H(K)$ below $[0, 2/\mathfrak{m}]$

Let z_1^0 be a zero of $\Delta_\alpha(\cdot) = 0$, i.e. an eigenvalues of $h_1(k)$ in the interval $\mathbb{R} \setminus [0, 6/\mathfrak{m}]$ (see Lemma 2.1).

Lemma 4.6. *(a) Let $\lambda > 0$. Then $\Delta_n^+(\lambda; z)$ has a unique zero z_n in $(-\infty, 0)$ such that $z_n < z_1^0 < 0$. Moreover, $z_n < z_m$ if $|m| > |n|$.*

(b) The following limit

$$\lim_{n\to\infty} z_n = z_1^0 \tag{21}$$

holds.

Proof. (a) Since $\Delta_n^+(\lambda; z)$ is symmetric about the interval $[0, 2/\mathfrak{m}]$, we prove part (a). According to Lemma 4.5, the function $\Delta_n^+(\lambda; z)$ is monotonically increasing in $(-\infty, 0)$ and

$$\lim_{z\to-\infty} \Delta_n^+(\lambda; z) = 1,$$
$$\Delta_n^+(\lambda; z_1^0) < \Delta(\lambda; z_1^0) = 0,$$

holds.

Consequently, the function $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(-\infty, z_1^0)$ if $\lambda > 0$.

If $|n| < |m|$, then Lemma 4.2 implies that $\Delta_n^+(\lambda; z) < \Delta_m^+(z)$ and hence $\Delta_n^+(\lambda; z_m) < \Delta_m^+(z_m) = 0$. The last equality provides the inequality $z_n < z_m$.

(b) The equalities

$$\Delta_n^+(\lambda; z_1^0) = \big(\Delta_n^+(\lambda; \xi_n)\big)'(z_1^0 - z_n),$$
$$\Delta_n^+(\lambda; z_1^0) = -\lambda d_n(z_1^0),$$
$$\big(\Delta_n^+(\lambda; z)\big)' = -\lambda(d_0'(z) + d_n'(z)),$$

where $z_n < \xi_n < z_1^0$ is a number obtained due to the intermediate value theorem, imply

$$z_n - z_1^0 = -\frac{d_n(z_1^0)}{d_0'(\xi_n) + d_n'(\xi_n)}.$$

By Lemma 4.6, the inequality $z_0 \le z_n < \xi_n < z_1^0$ and the monotonicity of the derivative of $\big(\Delta_n^+(\lambda; z)\big)'$,

$$\big(\Delta_n^+(\lambda; z_0)\big)' \ge \big(\Delta_n^+(\lambda; \xi_n)\big)' \ge \big(\Delta_n^+(\lambda; z_1^0)\big)'$$

Applying the limit (20) in the last inequality, we get the proof of the limit in (21). □

4.4. Eigenvalues of $H(K)$ above $[0, 2/\mathfrak{m}]$

The unitary operator $U_{\pi/2}$ in (5) is used to establish the equalities

$$U_{\pi/2} H_0(K) U_{\pi/2} = \frac{2}{\mathfrak{m}} - H_0(K) \quad \text{and} \quad U_{\pi/2} V U_{\pi/2} = V$$

which implies the relation

$$U_{\pi/2}\big(H_0(K) - V\big)U_{\pi/2} = \frac{2}{\mathfrak{m}} - (H_0(K) + V). \tag{22}$$

The final relationship enables us to shift the investigation of the eigenvalues of $H(K)$ from above the interval $[0, 2/\mathfrak{m}]$ to below it.

Note that z_n is a zero of $\Delta_n^+(\lambda; z)$, $n \in \mathbb{Z}$, if it exists.

Lemma 4.7. *Assume $\lambda < 0$.*

(a) The function $\Delta_n^+(\lambda; z)$ has a unique zero z_n in $(2/\mathfrak{m}, \infty)$ such that $2/\mathfrak{m} < z_1^0 < z_n$. Moreover, $z_n > z_m$ if $|m| > |n|$.

(b) The following limit

$$\lim_{n\to\infty} z_n = z_1^0$$

holds.

Proof. The proof is a consequence of Lemma 4.6 and the identity (22). □

5. Main theorem

Recall that z_1^0 be a zero of $\Delta(\lambda;\cdot)=0$, i.e. an eigenvalues of $h_1(k)$, and $z_1^0<0$ if $\lambda>0$ and $2/\mathfrak{m}<z_1^0$ if $\lambda<0$. Let η be an eigenvalue of $H(K)$ mentioned in Lemma 4.1.

Now, we are ready to formulate the main result of the paper.

Theorem 5.1. *Assume $\mu\in\mathbb{R}$ and $\lambda\in\mathbb{R}$. (a) Let $\lambda<0$. Then*

$$\sigma_{pp}\big(H(K)\big)=\bigcup_{n\in\mathbb{Z}}\{z_n\}\cup\{\eta\},$$

where $2/\mathfrak{m}<z_1^0<z_n$ and $2/\mathfrak{m}-\mu<\eta$.

(b) Let $\lambda=0$. Then

$$\sigma_{pp}\big(H(K)\big)=\varnothing.$$

(c) Let $0<\lambda$. Then

$$\sigma_{pp}\big(H(K)\big)=\bigcup_{n\in\mathbb{Z}}\{z_n\}\cup\{\eta\},$$

and $z_n<z_1^0<0$ and $\eta<-\mu$.

Proof. Lemmas 4.1,4.6 and 4.7 provide the proof. For example, by combining the assertions (a) in Lemma 4.1 and (a) in Lemma 4.7, we get the proof of the part (a) of the theorem. □

Remark 5.2. *According to Lemma 3.2*

$$\sigma_{ess}\big(H(K)\big)=\{z_1^0\}\cup\Big(\big[-\mu,2/\mathfrak{m}-\mu\big]\cup\big[0,2/\mathfrak{m}\big]\Big).$$

Theorem 5.1 demonstrates that z_n or η could be within $\sigma_{ess}\big(H(K)\big)$ or within a gap of the essential spectrum.

6. Conclusion

The discrete Schrödinger operator corresponding to the Hamiltonian of a system of three quantum particles (two identical bosons and a fermion) with masses $m=\infty$ and $\mathfrak{m}<\infty$, respectively, is considered on the one-dimensional lattice for all non-zero point interactions. The point spectrum of the three-particle discrete Schrödinger operator, which possesses infinitely many eigenvalues, has been studied for all non-zero point interactions.

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