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Original article

Fast forward problem for adiabatic quantum dynamics: Estimation of the energy cost

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ABSTRACT We consider the problem of energy cost needed for acceleration (deceleration) of the evolution of a quantum system using the Masuda–Nakamura's fast forward protocol. In particular, we focus on dynamics by considering models for a quantum box with a moving wall and harmonic oscillator with time-dependent frequency. For both models we computed the energy needed for acceleration (deceleration) as a function of time. The results obtained are compared with those of other acceleration (deceleration) protocols.

KEYWORDS energetic cost, shortcuts to adiabaticity, statistical physics and thermodynamics

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1. Introduction

Controlling of evolution and manipulation of physical processes in quantum systems is of practical importance in emerging quantum technologies. Achieving such a goal allows one to improve device miniaturization and operational speed at nanoscale and quantum functional materials. Solving such a problem requires developing of cost-efficient methods for acceleration and deceleration of quantum processes. One of the methods allowing to accelerate (decelerate) the quantum evolution, including that in adiabatic regime has been proposed in [1], which was later modified for different cases and situations (see [2–10]). According to the protocol of this approach, evolution of a given quantum system can be accelerated (decelerated) by acting some external electromagnetic potential on it. The method was later called Masuda–Nakamura's fast forward protocol in quantum mechanics [1]. It is important to note that there are different protocols for acceleration (deceleration) of quantum evolution. Therefore, an important problem that arises in the context of their practical applications is energy-efficiency of a fast-forward protocol. Such efficiency can be estimated in terms of so called fast forward energy cost which was introduced first in the [11] and applied later to different physical systems in [12, 13]. Here we estimate such a cost for Masuda–Nakamura's fast forward protocol in case of dynamical quantum confinement by considering harmonic oscillator with time-dependent frequency and a quantum box with a moving wall.

Remarkable feature of the Masuda–Nakamura's fast forward protocol developed in [1] is the fact that it allows one to accelerate the time evolution of a quantum system by tuning the external potential that can be reduced to regulation of an additional phase in the wave function. The improved version of the prescription was proposed later in [4] which is used for acceleration of the soliton dynamics described in terms of the nonlinear Schrödinger equation and tunneling in quantum regime [4]. Different modifications and application of the protocol have been presented later in [4–8]. One of the advantages of the Masuda–Nakamura protocol is its effective application to so-called adiabatic quantum processes. These processes are those occurring in very slowly evolving quantum systems. An interesting problem in this case is so-called quantum short cuts, implying the shortest path (in time) to the end of the (adiabatic) processes among others. In [2, 3] the Masuda–Nakamura protocol was adopted to the problem of short cuts. In the literature, the problem called "short cuts to adiabaticity" [14–18] (called also "transitionless quantum driving" by Berry [19]). One should also note successful application of the fast forward protocol to the problem of stochastic [8] and classical [20] heat engine.

This paper is organized as follows. In the next section, we present a brief description of the fast-forward protocol following [1]. Section 3 presents an application of the fast forward protocol to harmonic oscillator with time-dependent frequency and quantum box with moving wall. In Section 4, we calculate energetic cost of fast forward protocol for the harmonic oscillator with time-dependent frequency and compare it with the cost of inverse engineering protocol. In addition, Section 4 presents the results for quantum box with moving boundary conditions. Finally, the section 5 presents some concluding remarks.

2. Fast forward protocol for adiabatic quantum dynamics

Here we briefly recall the fast forward prescription for adiabatic quantum dynamics and it's application to the harmonic oscillator with time-dependent frequency [7]. Consider the dynamics of a wave function ψ_0 under the dynamical confinement $V_0 = V_0(x, R(t))$ which varies adiabatically (slowly). This adiabatic change is characterized by slowly-varying control parameter $R = R(t)$ which is given by

$$R(t) = R_0 + \epsilon t, \tag{1}$$

with the growth rate $\epsilon \ll 1$. The time-dependent 1D Schrödinger equation (1D TDSE) for ψ_0 is given as:

$$i\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2m}\partial_x^2\psi_0 + V_0(x, R(t))\psi_0, \tag{2}$$

where the coupling with electromagnetic field is assumed to be absent. If $R = const$ the problem reduces to an eigenvalue problem for stationary bound state ϕ_0 which satisfies the time-independent Schrödinger equation

$$E\phi_0 = \hat{H}_0\phi_0 \equiv \left[-\frac{\hbar^2}{2m}\partial_x^2 + V_0(x, R)\right]\phi_0. \tag{3}$$

With use of the eigenstate $\phi_0 = \phi_0(x, R)$, one can conceive the corresponding time-dependent state to be a product of ϕ_0 and a dynamical factor as

$$\phi_0(x, R(t)) = \bar{\phi}_0(x, R(t))e^{i\eta(x,R(t))}, \tag{4}$$

As it stands, however, ψ_0 does not satisfy TDSE in Eq. (2). To overcome this difficulty, we introduce a regularized wave function

$$\begin{aligned} \psi_0^{reg} &\equiv \phi_0(x, R(t))e^{i\epsilon\theta(x,R(t))}e^{-\frac{i}{\hbar}\int_0^t E(R(t'))dt'} \\ &\equiv \phi_0^{reg}(x, R(t))e^{-\frac{i}{\hbar}\int_0^t E(R(t'))dt'} \end{aligned} \tag{5}$$

and regularized potential

$$V_0^{reg} \equiv V_0(x, R(t)) + \epsilon\tilde{V}(x, R(t)). \tag{6}$$

Regularized wave function ψ_0^{reg} should satisfy the TDSE for regularized system,

$$i\hbar\frac{\partial\psi_0^{reg}}{\partial t} = -\frac{\hbar^2}{2m}\partial_x^2\psi_0^{reg} + V_0^{reg}\psi_0^{reg}, \tag{7}$$

up to the order of ϵ.

The potential $\tilde{V}$ is determined as

$$\tilde{V} = -\hbar\cdot\mathrm{Im}\left[\frac{\partial\phi_0}{\partial R}/\phi_0\right] - \frac{\hbar^2}{m}\cdot\mathrm{Im}\left[\frac{\nabla\phi_0}{\phi_0}\right]\cdot\nabla\theta. \tag{8}$$

Rewriting $\phi_0(x, R(t))$ in terms of the real positive amplitude $\overline{\phi}_0(x, R(t))$ and phase $\eta(x, R(t))$ as

$$\phi_0(x, R(t)) = \bar{\phi}_0(x, R(t))e^{i\eta(x,R(t))}, \tag{9}$$

we see θ to satisfy

$$\partial_x(\bar{\phi}_0^2\partial_x\theta) = -\frac{m}{\hbar}\partial_R\bar{\phi}_0^2. \tag{10}$$

Integrating Eq. (10) over x, we have

$$\partial_x\theta = -\frac{m}{\hbar}\frac{1}{\bar{\phi}_0^2}\int^x \partial_R\bar{\phi}_0^2 dx', \tag{11}$$

which is the core equation of the regularization procedure. We shall now accelerate the quasi-adiabatic dynamics of ψ_0^{reg} in Eq. (5) by applying the external driving potential (fast forward potential). For this purpose, we introduce the fast-forward version of ψ_0^{reg} as

$$\psi_{FF} = \bar{\phi}_0(x, R(\Lambda(t)))e^{i\eta(x,R(\Lambda(t)))}e^{iv(t)\theta(x,R(\Lambda(t)))}e^{-\frac{i}{\hbar}\int_0^t E(R(\Lambda(s)))ds}. \tag{12}$$

For accelerated system, control parameter R now can be rewritten as

$$R(\Lambda(t)) = R_0 + \epsilon\Lambda(t), \tag{13}$$

where $\Lambda(t)$ is the future or advanced time

$$\Lambda(t) = \int_0^t \alpha(\mathrm{t}')\,\mathrm{d}t'. \tag{14}$$

The wave function of fast forward state given by Eq. (12) satisfies TDSE with a fast-forward Hamiltonian H_{FF}:

$$i\hbar\frac{\partial\psi_{FF}}{\partial t} = H_{FF}\psi_{FF} \equiv \left(-\frac{\hbar^2}{2m}\partial_x^2 + V_0 + V_{FF}\right)\psi_{FF}. \tag{15}$$

Here $V_0 = V_0(x, R(\Lambda(t)))$ and V_{FF} is given by

$$V_{FF} = -\frac{\hbar^2}{m}v(t)\partial_x\theta\cdot\partial_x\eta - \frac{\hbar^2}{2m}(v(t))^2(\partial_x\theta)^2 - \hbar v(t)\partial_R\eta - \hbar\dot{v}(t)\theta - \hbar(v(t))^2\partial_R\theta. \tag{16}$$

3. Application to adiabatic dynamical confinement

Masuda–Nakamura's fast forward protocol presented in the previous section can be applied to the simplest time-dependent system such as one-dimensional quantum box with a moving wall and one-dimensional harmonic oscillator with time-dependent frequency, evolving in the adiabatic regime. The main result for such a task is analytically or numerically calculated wave function Ψ_{FF} of the fast forwarded system and fast forwarding (driving) potential, V_{FF}.

3.1. Time-dependent harmonic oscillator

Consider first the harmonic oscillator with time-dependent frequency. The evolution of such system is described in terms of the following non stationary Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi_0(x, R(t)) = -\frac{\hbar^2}{2m}\partial_x^2\psi_0(x, R(t)) + \frac{1}{2}m\omega^2(R(t))x^2\psi_0(x, R(t)), \tag{17}$$

where time dependence of the frequency $\omega(t)$ is caused by adiabatically changing parameter $R(t)$ defined as $R(t) = \sqrt{1/\omega(t)}$. For adiabatic regime of evolution, the eigenvalue problem can be written in terms of the following Schrödinger equation:

$$H_0(x, R)\phi = E(R)\phi, \tag{18}$$

that gives one the eigenvalue and the eigenstate as

$$\begin{aligned} E_n &= \left(n + \frac{1}{2}\right)\hbar\omega(R), \\ \phi_n &= \left(\frac{m\omega(R)}{\pi\hbar}\right)^{1/4}\frac{1}{(2^n n!)^{1/2}}e^{-\frac{m\omega(R)}{2\hbar}x^2}H_n\left(\left(\frac{m\omega(R)}{\hbar}\right)^{1/2}x\right) \end{aligned}$$

with $n = 0, 1, 2, \ldots$. Here $H_n(\cdot)$ are the Hermite polynomials.

Fast forward state and fast forward potential for such system can be calculated analytically. It is given by (see [7] for details):

$$\psi_{FF} = \phi_n(x, R(\Lambda(t))e^{i\frac{m}{2\hbar}\frac{\dot{R}}{R}x^2}e^{-\left(n+\frac{1}{2}\right)i\int_0^t\omega(R(\Lambda(t')))dt'} \equiv \langle x|n\rangle$$

and

$$V_{FF} = -\frac{m\ddot{R}}{2R}x^2. \tag{19}$$

3.2. Time-dependent quantum box

Now, let us investigate 1D quantum box with a moving wall. The dynamics of a particle is governed by

$$i\hbar\frac{\partial\psi}{\partial t} = H_0\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi \tag{20}$$

with the time-dependent box boundary conditions as $\psi(x = 0, t) = 0$ and $\psi(x = L(t), t) = 0$. $L(t)$ is assumed to change adiabatically as $L(t) = L_0 + \epsilon t$. Length of the wall $L(t)$ is to be assumed as control parameter of the confinement.

The adiabatic eigenvalue problem related to Eq. (20) gives one eigenvalues and eigenstates as follows

$$\begin{aligned} E_n &= \frac{\hbar^2}{2m}\left(\frac{\pi n}{L(t)}\right)^2, \\ \phi_n &= \sqrt{\frac{2}{L(t)}}\sin\left(\frac{\pi n}{L(t)}x\right). \end{aligned} \tag{21}$$

The phase θ which the regularized state acquires is given using the formula (11), as

$$\begin{aligned} \partial_x\theta &= -\frac{m}{\hbar}\frac{1}{\phi_n^2}\partial_L\int_0^x \phi_n^2 \mathrm{d}x = \frac{m}{\hbar}\frac{x}{L(t)}, \\ \theta &= \frac{m}{2\hbar}\frac{x^2}{L(t)}. \end{aligned} \tag{22}$$

Due to the real nature of ϕ_n, we find $\eta = 0$ in Eq. (8) and see that $\tilde{V}$ vanishes.

Now, we apply the fast forward scheme from Section 2. This will be done by changing time t by future time $\Lambda(t)$ in control parameter $L(t)$. By taking the asymptotic limit ($\epsilon \to 0, \bar{\alpha} \to \infty$ with $\epsilon\alpha = v(t)$), the fast forward state can be written as:

$$\psi_{FF} = \phi_n\left(x, L(\Lambda(t))\right) e^{i\frac{m\dot{L}(\Lambda(t))}{2\hbar L(\Lambda(t))}x^2} e^{-i\frac{\hbar}{2m}(\pi n)^2 \int_0^t \frac{\mathrm{d}t'}{L^2(\Lambda(t'))}}, \tag{23}$$

where $L\left(\Lambda(t)\right) = L_0 + \int_0^t v(t')\mathrm{d}t'$ with the time scaling factor $v(t)$. In accordance with Eq. (16), the fast forward potential is given by

$$V_{FF} = -\frac{m}{2}\frac{\ddot{L}(\Lambda(t))}{L(\Lambda(t))}x^2. \tag{24}$$

In the next section, we compute energy cost needed for realization of the above models, i.e. for fast forwarding of the quantum evolution in time-dependent box and time-dependent harmonic oscillator.

4. Energy cost needed for fast forwarding of the evolution of a quantum system

The practical application of the above (or any other) fast forward evolution prescription is closely related to the question: how much energy one needs to use to apply the prescription. In other words, the effectiveness of the fast forward protocol depends on the cost of energy to be spent: as smaller the energy cost as effective the protocol. Here we consider the problem for estimation of the energy cost needed for application of the Masuda–Nakamura's fast forward protocol. According to the Ref. [11] energy cost to be paid for a given fast forward protocol is determined as:

$$C = \frac{1}{T_{FF}}\int_0^{T_{FF}} ||H||dt, \tag{25}$$

where $||\hat{A}||$ denotes the Frobenius norm defined as follows: $||\hat{A}|| = \sqrt{\mathrm{Tr}\left[\hat{A}^\dagger\hat{A}\right]}$ and H is the total Hamiltonian of the system, which is given by

$$H = H_0 + V_{FF},$$

where H_0 is the Hamiltonian of the standard system (to be fast forwarded) and V_{FF} is the fast forward potential given by Eq. (16). Here we estimate the energy cost for two systems adiabatically evolving under the dynamical confinement. Namely, we consider the time-dependent harmonic oscillator and the quantum box with moving wall described above. Let us start with the time-dependent harmonic oscillator given by Eq. (17).

According to [12] the energy cost needed to fast forward a quantum system with an unbound (discrete) spectrum (which is the case for our system) can be rewritten as

$$C_{FF} = \frac{1}{T_{FF}}\int_0^{T_{FF}} \bar{U}dt, \tag{26}$$

where T_{FF} shortened or fast-forward time and U is the internal energy given by the following formula

$$\bar{U} = \mathrm{Tr}(\rho\hat{H}_{FF}), \tag{27}$$

with density matrix ρ defined as follows:

$$\rho\left(t\right) = \sum_{n=0}^{\infty} \left|n\right\rangle f_n \left\langle n\right|, \tag{28}$$

where $\{|n\rangle\}$ is the exact solution of TDSE (15) and f_n is the Fermi–Dirac distribution having the form

$$f_n = \frac{1}{e^{\beta(E_n(L(\Lambda(t)))-\mu)} + 1}.$$

For the time-dependent harmonic oscillator, the expression of U can be written as (see [7] for details):

$$\bar{U} = A\left(\frac{\hbar^2}{4mL^2} - \frac{m}{8}L\ddot{L} + \frac{m}{8}\dot{L}^2\right) \tag{29}$$

with

$$A = N^2 \left[1 + \frac{4\pi^2}{3} L_0^2 \left(\frac{mkT}{\hbar^2} \right)^2 \left(\frac{N}{L_0} \right)^{-2} + \cdots \right]$$

and

$$L = L_0 + \bar{v} \left(\frac{1}{2} \frac{t^2}{T_{FF}} - \frac{1}{3} \frac{t^3}{T_{FF}^2} \right), \tag{30}$$

$$\dot{L} = \bar{v} \left(\frac{t}{T_{FF}} - \frac{t^2}{T_{FF}^2} \right). \tag{31}$$

Thus, for the cost, we have the formula

$$\begin{aligned} C_{FF} &= \frac{1}{T_{FF}} \int_0^{T_{FF}} \bar{U} dt \\ &= \frac{A\hbar^2}{4mT_{FF}} \int_0^{T_{FF}} \frac{1}{L^2} dt - \frac{mA}{8T_{FF}} \int_0^{T_{FF}} (L\ddot{L} - \dot{L}^2) dt \\ &= \frac{A\hbar^2}{4mT_{FF}} \int_0^{T_{FF}} \frac{1}{L^2} dt + \frac{mA}{8T_{FF}} \int_0^{T_{FF}} (-\frac{d}{dt}(\dot{L}L) + 2\dot{L}^2) dt \\ &= \frac{A\hbar^2}{4mT_{FF}} \int_0^{T_{FF}} \frac{1}{L^2} dt - \frac{mA}{4T_{FF}} \int_0^{T_{FF}} \dot{L}^2 dt. \end{aligned}$$

Using Eq. (31), one obtains

$$\begin{aligned} mA \frac{1}{4T_{FF}} \int_0^{T_{FF}} \dot{L}^2 dt &= mA \frac{1}{4T_{FF}} \int_0^{T_{FF}} \bar{v}^2 \left(\frac{t}{T_{FF}} - \frac{t^2}{T_{FF}^2} \right)^2 dt \\ &= mA \frac{\bar{v}^2}{4T_{FF}} \left(\frac{1}{3} T_{FF} - \frac{1}{2} T_{FF} + \frac{1}{5} T_{FF} \right) = \frac{mA}{120} \bar{v}^2. \end{aligned}$$

Then

$$C_{FF} = \frac{1}{T_{FF}} \int_0^{T_{FF}} \bar{U} dt = \frac{A\hbar^2}{4mT_{FF}} \int_0^{T_{FF}} \frac{1}{L^2} dt + \frac{mA}{120} \bar{v}^2. \tag{32}$$

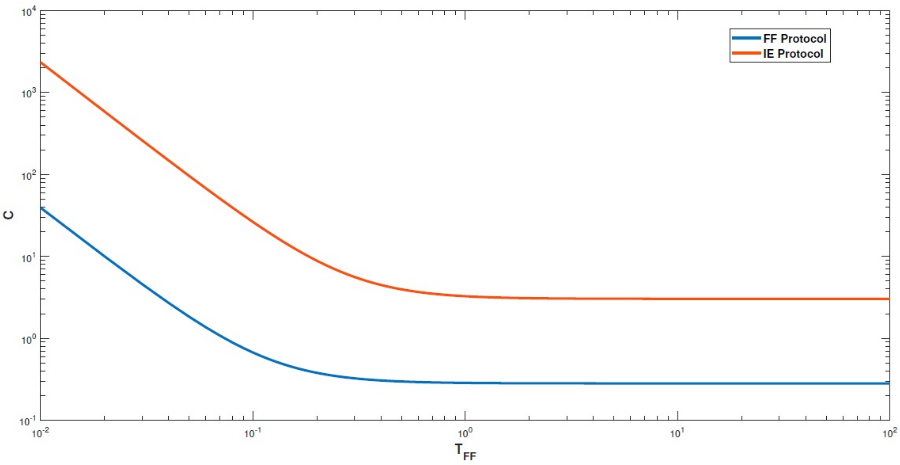

FIG. 1. Energy cost for the fast forward protocol (blue) with control parameter L given by Eq. (30) and the inverse engineering protocol (red) with the initial frequency $\omega_0 = 1$ and the final frequency $\omega_F = 10$

In the case of the control parameter $L(t)$ chosen in the form

$$L(t) = L_0 + \bar{v} \left(t - \frac{T_{FF}}{2\pi} \sin \frac{2\pi}{T_{FF}} t \right), \tag{33}$$

$$\dot{L} = \bar{v}\left(1 - \cos\frac{2\pi}{T_{FF}}t\right), \tag{34}$$

we have

$$C_{FF} = \frac{1}{T_{FF}}\int_0^{T_{FF}} \bar{U}dt = \frac{A\hbar^2}{4mT_{FF}}\int_0^{T_{FF}} \frac{1}{L^2}dt + \frac{3Am}{8}\bar{v}^2. \tag{35}$$

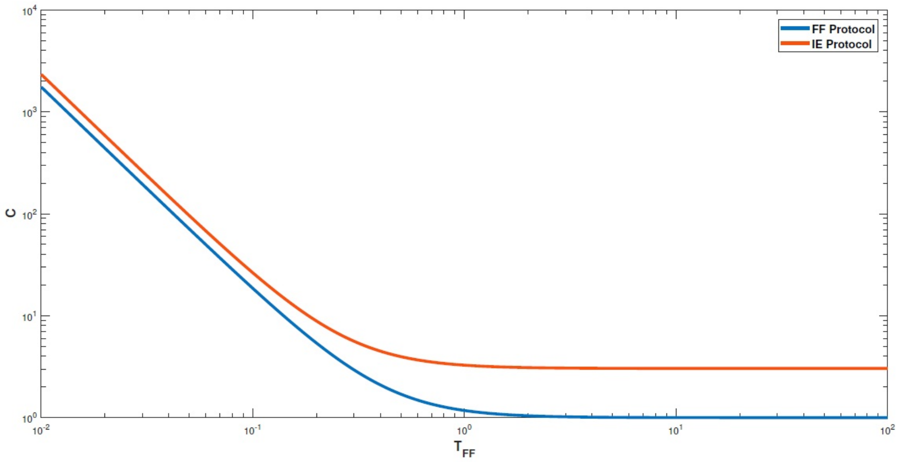

FIG. 2. Energy cost for the fast forward protocol (blue) with the control parameter L given by Eq.(30) and the inverse engineering protocol (red) with the initial frequency $\omega_0 = 1$ and the final frequency $\omega_F = 10$

Now, let us do similar calculations for the quantum box with moving wall. Energy $\bar{U}$ for accelerated dynamics of the quantum box with moving wall looks as follows (for details see [7]):

$$\begin{aligned} \bar{U} &= \frac{\pi^2\hbar^2}{24m}\frac{N^3}{L^2}\left[1 + \frac{24}{\pi^2}\left(\frac{mkT}{\hbar^2}\right)^2\left(\frac{N}{L}\right)^{-4} + \cdots\right] \\ &- \frac{N}{6}(mL\ddot{L} - m\dot{L}^2)\left[1 + \frac{6}{\pi^2}\frac{1}{N^2}\left(1 + \frac{16}{3\pi^2}\left(\frac{mkT}{\hbar^2}\right)^2\left(\frac{N}{L}\right)^{-4} + \cdots\right)\right]. \end{aligned} \tag{36}$$

Here we also considered two cases: the first one takes place if the control parameter $L(t)$ is given as polynomial function, the second one takes place if $L(t)$ is trigonometric function. As for polynomial $L(t)$, it is as follows:

$$L = L_0 + \bar{v}\left(\frac{1}{2}\frac{t^2}{T_{FF}} - \frac{1}{3}\frac{t^3}{T_{FF}^2}\right). \tag{37}$$

The energetic cost is given by

$$C_{FF} = \frac{1}{T_{FF}}\int_0^{T_{FF}} \bar{U}dt = B_1\frac{1}{24T_{FF}}\int_0^{T_{FF}} \frac{1}{L^2} + B_2\frac{\bar{v}^2}{90}, \tag{38}$$

where B_1 and B_2 are the following constants:

$$B_1 = \frac{\pi^2\hbar^2N^3}{24m}\left[1 + \frac{24}{\pi^2}\left(\frac{mkT}{\hbar^2}\right)^2\left(\frac{N}{L}\right)^{-4} + \cdots\right], \tag{39}$$

$$B_2 = \frac{mN}{6}\left[1 + \frac{16}{3\pi^2}\left(\frac{mkT}{\hbar^2}\right)^2\left(\frac{N}{L}\right)^{-4} + \cdots\right]. \tag{40}$$

For the case of trigonometric control parameter, one has $L(t)$ in the form

$$L(t) = L_0 + \bar{v}\left(t - \frac{T_{FF}}{2\pi}\sin\frac{2\pi}{T_{FF}}t\right). \tag{41}$$

For the energetic cost, we obtain the expression

$$C_{FF} = \frac{1}{T_{FF}}\int_0^{T_{FF}} \bar{U}dt = B_1\frac{1}{24T_{FF}}\int_0^{T_{FF}} \frac{1}{L^2} + B_2\frac{\bar{v}^2}{2}. \tag{42}$$

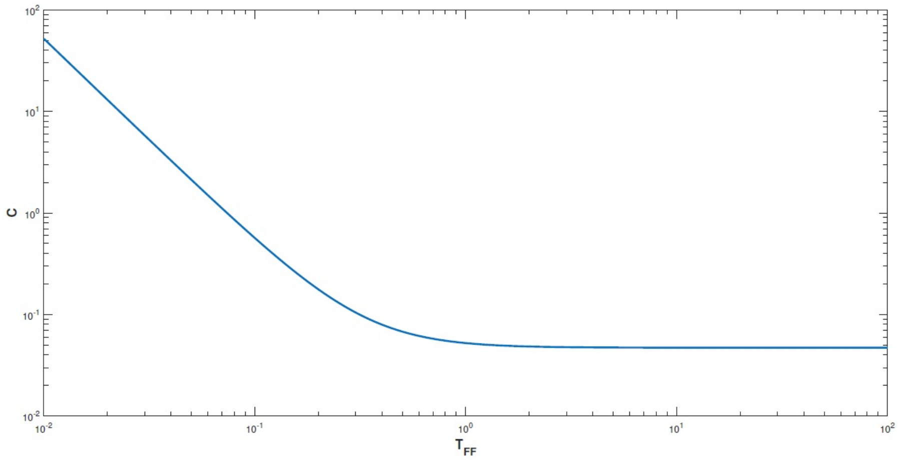

FIG. 3. Energy cost of the fast forward protocol with L given by Eq. (37) for the parameters $L_0 = 1$ and $L_F = 10$

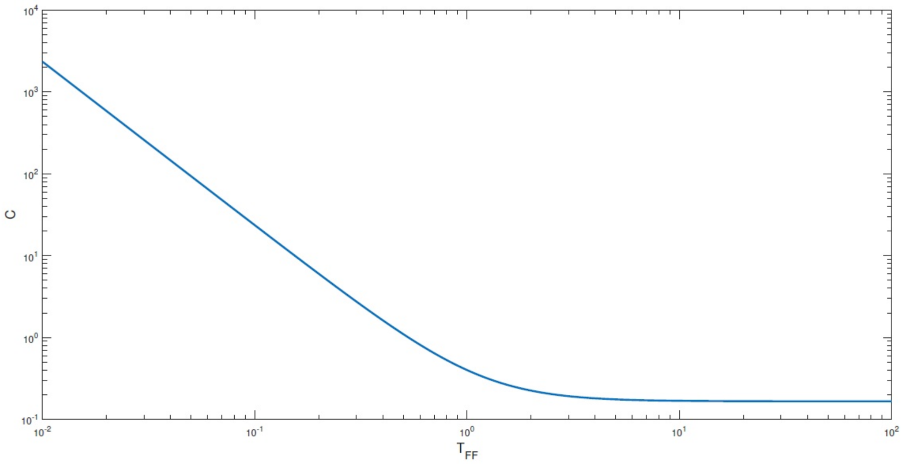

FIG. 4. Energy cost of the fast forward protocol with L given by Eq. (41) for the parameters $L_0 = 1$ and $L_F = 10$

Fig. 1 compares plots of the energy cost as function of time for Masuda–Nakamura's (blue line) and the inverse engineering (IE) (red line) fast forward protocols for the time-dependent harmonic oscillator (see [12] for details). Energy cost for IE protocol is obtained numerically by using the following IE Hamiltonian [12] :

$$\langle H_{IE}\rangle = \frac{1}{2}\Big[\frac{\dot{b}^2(t)}{2\omega_0} + \frac{\omega^2(t)b^2(t)}{2\omega_0} + \frac{\omega_0}{2b^2(t)}\Big]\coth\Big(\frac{\beta\omega_0}{2}\Big), \tag{43}$$

where $b(t)$ is the dimensionless function satisfying the Ermakov equation:

$$\ddot{b}(t) + \omega^2(t)b(t) = \omega_0/b^3(t). \tag{44}$$

The calculations were done for the control parameter $L(t)$ given by Eq. (30). As the plot shows, the cost for the Masuda–Nakamura protocol is much smaller than that for the inverse engineering one and the curves are almost parallel to each other. Fig. 2 presents similar plots for the form of $L(t)$ given by Eq. (33). It can be seen that the costs are completely different than that in Fig. 1, both qualitatively and quantitatively, i.e. the cost for Masuda–Nakamura's fast forward protocol in Fig. 1 is much smaller than that in Fig. 2. In addition, at the initial time the difference between the costs are much smaller than that at longer times. In Fig. 3, the time-dependence of the energy cost for the Masuda–Nakamura protocol is plotted for the quantum box with moving wall for the control parameter given by Eq. (30). Fig. 4 presents similar plot for the case when $L(t)$ is given by Eq. (33). Comparing plots of the costs presented in Figs.3 and 4 with those in Figs. 1 and 2, one can conclude that the costs for the time-dependent harmonic oscillator and for the quantum box with moving wall within the Masuda–Nakamura fast forward protocol are almost equal for the same systems (provided they are estimated for the same control parameter).

5. Conclusion

In this paper, we proposed two models where Masuda–Nakamura's fast forward protocol can be applied in the adiabatic regime and the energy cost for the implementation of such protocol can be computed. In particular, Masuda–Nakamura's method for the fast-forward evolution [1,2] is applied for the acceleration of the evolution of the time-dependent box with slowly moving wall and the harmonic oscillator with slowly varying time-dependent frequency. The quantitative comparison of the energetic cost of the Masuda–Nakamura protocol with the inverse engineering protocol is given. In particular, the plots of the energy cost for the Masuda–Nakamura and the inverse engineering protocols in Figs. 1 and 2 shows that the Masuda–Nakamura protocol requires less cost than the inverse engineering protocol. The results obtained in this paper can be used for further development of energy-efficient and resource-saving low-dimensional quantum devices.

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