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Original article

Quantum graph as a benchmark for persistent current

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ABSTRACT The problem of persistence current in nanosystems is studied. We demonstrate some simple theoretical observation which allows one to construct a benchmark for the persistence current. It can be used for improvement of the persistence current measurement procedure. The consideration is based on the quantum graph model. The benchmark is given by a graph with finite number of rings touching at one point with a lead attached to this point. It is assumed that the graph is plane and there exists a magnetic field orthogonal to the rings.

KEYWORDS quantum graph; persistent current

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1. Introduction

Electronic properties at low dimensions with various geometries have continued to fascinate the scientific community over the years. Among such structures quantum rings are widely celebrated due to their peculiar electronic properties. The fabrication of nanoscale quantum rings [1, 2] in semiconductor heterostructures has aided in the understanding of the theoretical results on the subject [3, 4]. Later, more complicated ring structures were studied, e.g., the Möbius strip [5–7]. An electrical current induced in a resistive circuit will rapidly decay in the absence of an applied voltage. This decay reflects the tendency of the circuit's electrons to dissipate energy and relax to their ground state. However, quantum mechanics predicts that the electrons' many-body ground state (and, at finite temperature, their thermal equilibrium state) may itself contain a "persistent" current which flows through the resistive circuit without dissipating energy or decaying. A quantum ring can host persistent current [8] when it is threaded by a magnetic flux. A magnetic flux threading the ring breaks time-reversal symmetry, allowing the persistent current to flow in a particular direction around the ring. It is a quantum effect. This current exists although the metal of the ring is resistive. Due to the small size of the ring, the electron moves as a ballistic one even for non-zero temperature. Calculations [8] show that a micron-diameter ring will support a persistent current of about 1 nA at temperatures less than 1 K. This persistent current is closely related to the Aharonov- Bohm effect [9]. A reasonable number of studies [10–23] has been devoted to confirming the existence and properties of the persistent current in ringlike quantum structures.

However, measuring the persistent current is challenging for a number of reasons [24]. For example, the persistent current flows only within the ring and so cannot be measured using a conventional ammeter. Experiments to date have mostly used SQUIDs to infer the persistent current from the magnetic field it produces. A SQUID (superconducting quantum interference device) is a very sensitive magnetometer used to measure extremely weak magnetic fields, based on superconducting loops containing Josephson junctions. Interpretation of these measurements has been complicated by the SQUIDs' low signal-to-noise ratio and the uncontrolled back action of the SQUID's ac Josephson oscillations, which may drive non-equilibrium currents in the rings.

In the present paper, we demonstrate some simple theoretical observation which allows one to construct a benchmark for the persistence current. It can be used for improvement of the persistence current measurement procedure. We consider the quantum rings in the framework of quantum graph model which shows its usefulness for description of nanosystems (see, e.g. [25–31]). The magnetic field is assumed to be orthogonal to the plane of the rings. The benchmark is given by a graph with finite number of rings touching at one point with a lead attached to this point (see Fig. 1).

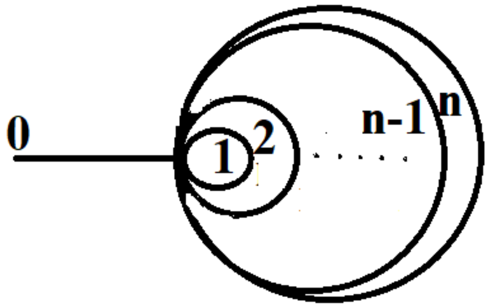

FIG. 1. Quantum graph structure. Edge numbering is shown.

2. Model

Consider the quantum graph Γ presented in Fig. 1. It consists of several rings $\Gamma_j, j = 1, 2, ...n$, and straight lead Γ_0. We assume that there is a homogeneous magnetic field $\mathbf{B}$ orthogonal to the plane of the graph. We investigate the ballistic electron in this graph. Correspondingly, the model is given by the following operator acting in the space $W_2^2(\Gamma)$:

$$\begin{aligned} H\psi_0 &= -\frac{d^2}{dx^2}, \qquad\qquad x \in \Gamma_0, \\ H\psi_j &= (i\frac{d}{dx} + B\pi R_j \Phi_0^{-1})^2, \quad j = 1, 2, ...n, x \in \Gamma_j, \end{aligned} \tag{1}$$

where $\Phi_0 = 2\pi\hbar c/|e|$ is the magnetic flux quantum playing a role of a unit for the magnetic flux in the system, c is the speed of light, $|e|$ is the electron charge, x is the length of the arc starting with the graph vertex (counter clock wise), $\hbar$ is the Planck constant, R_j is the radius of j-th ring, B is the magnetic field,

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ ... \\ \psi_n \end{pmatrix}.$$

As for the graph vertex x_0, we pose here the magnetic Kirchhoff conditions named also the Griffith conditions:

$$\begin{cases} \psi_j|_{x_0} = \psi_p|_{x_0}, \quad j \neq p, \\ \frac{d\psi_0}{dx}|_{x_0} + \sum_{j=1}^{n} (-1)^{\sigma_j} (\frac{d}{dx} - iB\pi R_j \Phi_0^{-1})\psi_j|_{x_0} = 0, \end{cases} \tag{2}$$

where $\sigma_j = 0$ for outgoing edge and $\sigma_j = 1$ for incoming edge.

Let us determine the persistent current in the rings. Consider the scattering problem for the graph. The electron wave function ψ in semi-infinite lead "0" has the form

$$\psi_0 = e^{ikx} + re^{-ikx}, \tag{3}$$

where k is the wave number of the electron, r is the reflection coefficient. The magnetic field has no influence on the electron in this one-dimensional straight lead. Naturally, the situation changes in the rings. Let the $j-$th ring have radius R_j. Then, the wave function has the following form

$$\psi_j = a_j e^{i(k+B\pi R_j \Phi_0^{-1})x} + b_j e^{i(-k+B\pi R_j \Phi_0^{-1})x}, \quad j = 1, 2, ...n, \tag{4}$$

Conditions (2) gives one the following system for coefficients r, a_j, b_j, $j = 1, ...n$:

$$\begin{cases} 1 + r = a_j + b_j, \quad j = 1, 2...n, \\ 1 + r = a_j e^{i(k+B\pi R_j \Phi_0^{-1})2\pi R_j} + b_j e^{i(-k+B\pi R_j \Phi_0^{-1})2\pi R_j}, \quad j = 1, 2...n, \\ ikr - ik = ik \sum_{j=1}^{n} \left(a_j \left(1 - e^{i(k+B\pi R_j \Phi_0^{-1})2\pi R_j} \right) - b_j \left(1 - e^{i(-k+B\pi R_j \Phi_0^{-1})2\pi R_j} \right) \right). \end{cases} \tag{5}$$

Transformation of the first j pairs of equations from (5) gives one:

$$\begin{cases} a_j = -\dfrac{1-e^{i(-k+B\pi R_j\Phi_0^{-1})2\pi R_j}}{e^{i2B\pi^2R_j^2}2i\sin(2\pi kR_j)}(1+r), \quad j=1,2...n, \\ b_j = \dfrac{1-e^{i(k+B\pi R_j\Phi_0^{-1})2\pi R_j}}{e^{i2B\pi^2R_j^2}2i\sin(2\pi kR_j)}(1+r), \quad j=1,2...n, \\ ikr-ik = ik\sum_{j=1}^{n}\left(a_j\left(1-e^{i(k+B\pi R_j\Phi_0^{-1})2\pi R_j}\right)-b_j\left(1-e^{i(-k+B\pi R_j\Phi_0^{-1})2\pi R_j}\right)\right). \end{cases} \tag{6}$$

The persistent current in the j-th ring is determined as follows [23]:

$$I_j = \frac{1}{2ki}\left(\overline{\psi_j}\frac{d}{dx}\psi_j - \psi_j\overline{\frac{d}{dx}\psi_j} - i\frac{2B\pi R_j}{\Phi_0}\overline{\psi_j}\psi_j\right). \tag{7}$$

By inserting expressions (4) in (7), one obtains:

$$I_j = |a_j|^2 - |b_j|^2. \tag{8}$$

Expressions (6) gives one

$$\begin{cases} |a_j|^2 = \dfrac{\sin^2(-\pi kR_j + B\pi^2R_j^2(\Phi_0)^{-1}}{\sin^2(\pi kR_j)}|1+r|^2, \quad j=1,2...n, \\ |b_j|^2 = \dfrac{\sin^2(\pi kR_j + B\pi^2R_j^2(\Phi_0)^{-1}}{\sin^2(\pi kR_j)}|1+r|^2, \quad j=1,2...n. \end{cases} \tag{9}$$

Hence, the ratio of the persistent currents in j-th and s-th rings does not depend on the number n of rings but depends on their radii, the magnetic field and the electron energy:

$$\frac{I_j}{I_s} = \frac{\sin(2B\pi^2R_j^2(\Phi_0)^{-1})\sin(2\pi kR_s)}{\sin(2B\pi^2R_s^2(\Phi_0)^{-1})\sin(2\pi kR_j)}. \tag{10}$$

It allows one to construct “a benchmark” for the persistent current. One can add or remove rings without changing the ratio (10) for other pairs of rings. If one consider $\ln I_j$ then the difference $\ln I_j - \ln I_s$ does not depend on the number of rings. One can collect rings to construct a benchmark with fixed step of $\ln I_j$ between the neighbour rings ($\ln I_{j+1} - \ln I_j$). Moreover, if one chooses some pair of rings and modifies the graph by adding a ring then the step of the logarithm of the persistent current in the chosen pair of rings ($\ln I_{j+1} - \ln I_j$) does not change. It means that one can extend the benchmark by adding rings without destroying the scale of the initial benchmark.

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