NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Sobirov Z.A., et al. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (5), 586–596.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2024-15-5-586-596

Original article

Inverse source problem for the subdiffusion equation with edge-dependent order of time-fractional derivative on the metric star graph

Zarifboy A. Sobirov[1,2,a], Ariukhan A. Turemuratova[1,3,b]

[1]National University of Uzbekistan, 100174, Tashkent, Uzbekistan

[2]V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Science, 100174, Tashkent, Uzbekistan

[3]Branch of Russian Economic University named after G. V. Plekhanov in Tashkent, 100164, Tashkent, Uzbekistan

[a]z.sobirov@nuu.uz, [b]ariuxanturemuratova@gmail.com

Corresponding author: Z. A. Sobirov, z.sobirov@nuu.uz

PACS 02.30.Jr, 02.30.Zz

ABSTRACT The paper discusses the inverse source problem for the subdiffusion equation in the Sobolev space. The direct and inverse problems are transformed into operator equations to derive solutions. The uniqueness and existence of a strong solution to the direct problem are proven. The inverse problem is reduced to an operator equation, and the well-definedness and continuity of the corresponding resolvent operator are proven.

KEYWORDS subdiffusion equation, star metric graph, inverse problem, generalized solution, resolvent operator

FOR CITATION Sobirov Z.A., Turemuratova A.A. Inverse source problem for the subdiffusion equation with edge-dependent order of time-fractional derivative on the metric star graph. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (5), 586–596.

1. Introduction

The theory of differential equations on graphs has significant implications in various fields, including mathematics, physics, biology and engineering. There are numerous studies dedicated to the differential operators and differential equations on branched structures and networks [1–5]. In [6], a metric graph model is developed for Stokes flow in three-dimensional spaces, addressing the complexities of fluid dynamics in networks with varying viscosity and density. The study examines quantum graphs, where the absence of reflection is linked to the transparency of the graph vertices [7–9]. In [7], the researchers investigated the issue of reflectionless soliton transport in network branching points by modeling soliton dynamics in networks using the nonlinear Schrödinger equation on metric graphs. Paper [9] emphasizes the importance of boundary conditions in quantum graphs and proposes a method to achieve low reflection, a crucial factor for effective electron transport in nanosystems. In graphs, various differential equations are studied with a set of methods of solving. In particular, the linearized KdV equation [10], a pseudo-subdiffusion equation involving the Hilfer time-fractional derivative [11], the Schrödinger operator on the quantum graph [12], the Fokas method for the heat equation [13] are investigated. In [14], authors consider construction of the matrix-Green's functions of initial-boundary value problems for the time-fractional diffusion equation on the metric star graph with equal bonds. It is investigated how magnetic boundary control can be utilized to solve inverse problems for Schrödinger operators on metric graphs in [15].

The study of fractional diffusion equations on metric graphs is motivated by practical and important problems such as anomalous heat transport in mesoscopic networks, subdiffusion processes in nanoscale network structures, molecular wires, different lattices and discrete structures. The interest in fractional diffusion equations is fueled by the close connection between anomalous diffusion and fractional derivatives. This link has been explored in the works of Luchko [16] and Metzler and Klafter [17], among others. The well-posedness of some time-fractional parabolic equations has been investigated, as referenced in the works of Sakamoto and Yamamoto [18], Kubica and Yamamoto [19]. Additionally, space-time fractional diffusion equations have also been the subject of investigation by various authors.

The application of fractional derivatives in fractional Sobolev spaces is essential for introducing weak or generalized solutions to fractional differential equations. Just as in the theory of partial differential equations, various approaches can be employed to address this problem. In [20], Gorenflo et. al investigated the maximal regularity of solutions to the time-fractional diffusion equation with the Caputo derivative in the fractional Sobolev spaces. We mention also the recent work [21], it provides rigorous treatments for time-fractional derivatives in the Sobolev spaces and solutions to initial boundary value problems for time-fractional partial differential equations.

Inverse problems are recognized as one of the most important mathematical challenges in science and mathematics. This area of study has been the focus of extensive research by numerous scholars, leading to significant advancements in understanding and solving inverse problems in differential equations. In a specific instance, researchers have investigated the uniqueness and stability of the solution to the inverse problem of determining the order of the Caputo time-fractional derivative for a subdiffusion equation, as documented in [22]. Additionally, R. Ashurov et al. [23] have concentrated on an inverse problem related to determining the orders of systems of fractional pseudo-differential equations.

The integral overdetermination condition has been leveraged in various research works to address inverse problems associated with differential equations. Notably, Kamynin utilized the integral overdetermination condition for the solution of the inverse problem related to a degenerate parabolic equation, as referenced in [24,25]. Also, we refer to work [26].

2. Preliminary materials

In this section, we introduce some notations, provide the function spaces necessary for studying our problem.

2.1. Fractional integrals and derivatives

Definition 1. *[27] The left and right fractional integrals of order* $0 < \alpha < 1$ *for a function* $y(t) \in L_1(0,T)$ *are, respectively, defined by*

$$\mathrm{I}^{\alpha}_{0,t}y(t) \equiv \mathrm{D}^{-\alpha}_{0,t}y(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \frac{y(\tau)}{(t-\tau)^{1-\alpha}}d\tau,$$
$$\mathrm{I}^{\alpha}_{t,T}y(t) \equiv \mathrm{D}^{-\alpha}_{t,T}y(t) := \frac{1}{\Gamma(\alpha)}\int_t^T \frac{y(\tau)}{(\tau-t)^{1-\alpha}}d\tau.$$

Definition 2. *[27] The left and right Caputo fractional derivatives of order* $0 < \alpha < 1$ *for a function* $y(t)$ *on* $[0,T]$ *are, respectively, defined by*

$$d^{\alpha}_{0,t}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{y'(\tau)}{(t-\tau)^{\alpha}}d\tau, \qquad d^{\alpha}_{t,T}y(t) = \frac{-1}{\Gamma(1-\alpha)}\int_t^T \frac{y'(\tau)}{(\tau-t)^{\alpha}}d\tau,$$

provided that the integrals in the right-hand sides of these expressions exist.

Definition 3. *[27] The left and right Riemann-Liouville fractional derivatives of order* $0 < \alpha < 1$ *for a function* $y(t)$ *on* $[0,T]$ *are, respectively, defined by*

$$\mathrm{D}^{\alpha}_{0,t}y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{y(\tau)}{(t-\tau)^{\alpha}}d\tau,$$
$$\mathrm{D}^{\alpha}_{t,T}y(t) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{y(\tau)}{(\tau-t)^{\alpha}}d\tau,$$

provided that the integrals in the right-hand sides of these expressions exist.

2.2. Fractional Sobolev spaces and weak Caputo derivative

We introduce some functional spaces that are helpful in solving the studied problem. The fractional derivatives defined above are pointwise derivatives. It's necessary to provide a definition of the generalized (weak) fractional derivative, which is well-defined within a subspace of a fractional order Sobolev space. This particular derivative is defined in [21].

Following to [21], by $H^{\alpha}(0,T),\ 0 < \alpha < 1$, we denote the fractional Sobolev - Slobodeskii space governed by the norm (see [20], [21])

$$\|u\|_{H^{\alpha}(0,T)} := \left(\|u\|^2_{L_2(0,T)} + \int_0^T\int_0^T \frac{|u(t)-u(s)|^2}{(t-s)^{1+2\alpha}}dsdt\right)^{\frac{1}{2}}.$$

We put ${}_0H^{\alpha}(0,T) = \{u \in H^{\alpha}(0,T) : u(0) = 0\}$ for $\frac{1}{2} < \alpha \le 1$,

$$H_{\alpha}(0,T) = \begin{cases} H^{\alpha}(0,T), & 0 \le \alpha < \frac{1}{2}, \\ \{v \in H^{\frac{1}{2}}(0,T) : \int_0^T \frac{|v(t)|^2}{t}dt < \infty\}, & \alpha = \frac{1}{2}, \\ {}_0H^{\alpha}(0,T), & \frac{1}{2} < \alpha \le 1. \end{cases}$$

The space $H_{\alpha}(0,T)$ is a Banach space with the norm [21]

$$\|v\|_{H_{\alpha}(0,T)} = \begin{cases} \|v\|_{H^{\alpha}(0,T)}, & 0 < \alpha < 1,\ \alpha \ne \frac{1}{2}, \\ \left(\|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{t}dt\right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

According to [21], the space ${}_0C^1[0,T] = \{v \in C^1[0,T] : v(0) = 0\}$ is dense in $H_{\alpha}(0,T)$.

Theorem 1. *[21] Let* $0 < \alpha < 1$.

(i) $I^{\alpha}_{0,t} : L_2(0,T) \to H_{\alpha}(0,T)$ *is injective and surjective.*

(ii) There exists a constant $C > 0$ *such that*

$$C^{-1}\|I^{\alpha}_{0,t}u\|_{H_{\alpha}(0,T)} \le \|u\|_{L_2(0,T)} \le C\|I^{\alpha}_{0,t}u\|_{H_{\alpha}(0,T)}$$

for all $u \in L_2(0,T)$.

From Theorem 1, it follows that the inverse operator $(I_{0,t}^{\alpha})^{-1}$ exists. We put $I_{0,t}^{-\alpha} = (I_{0,t}^{\alpha})^{-1}$.

Corollary 1. *[21] Let* $0<\alpha<1$. *Then*

$$I_{0,t}^{-\alpha} I_{0,t}^{\alpha} u = u, \quad u \in L_2(0,T),$$

and

$$I_{0,t}^{\alpha} I_{0,t}^{-\alpha} u = u, \quad u \in H_{\alpha}(0,T).$$

Definition 4. *[21] For* $0 \le \alpha \le 1$, *we set*

$$\partial_{0,t}^{\alpha} u := I_{0,t}^{-\alpha} u, \quad u \in H_{\alpha}(0,T)$$

with the domain $\mathcal{D}(\partial_{0,t}^{\alpha}) = H_{\alpha}(0,T)$.

We consider the classical Caputo derivative $d_{0,t}^{\alpha}$ as an operator from $\mathcal{D}(d_{0,t}^{\alpha}) = {}_0C^1[0,T] \subset L_2(0,T)$ to $L_2(0,T)$. By $\overline{d_{0,t}^{\alpha}}$, we denote the closure in $L_2(0,T)$ of $d_{0,t}^{\alpha}$ with $\mathcal{D}(d_{0,t}^{\alpha}) = {}_0C^1[0,T]$ which is the smallest closed extension of $d_{0,t}^{\alpha}$.

Theorem 2. *[21] We have* $\mathcal{D}(\overline{d_{0,t}^{\alpha}}) = H_{\alpha}(0,T)$, *and*

$$\overline{d_{0,t}^{\alpha}} = \partial_{0,t}^{\alpha} = D_{0,t}^{\alpha}, \quad on\ H_{\alpha}(0,T).$$

This theorem means that our definition of $\partial_{0,t}^{\alpha}$ *is consistent with the classical Caputo derivative by considering the closure of the operator.*

Furthermore, the following norms are equivalent in $H_{\alpha}(0,T)$ (see [21])

$$\|\partial_{0,t}^{\alpha} v\|_{L_2(0,T)} \sim \|v\|_{H_{\alpha}(0,T)}.$$

We notice, that in the case $\frac{1}{2} < \alpha < 1$ for any $v(t) \in H^{\alpha}(0,T)$, the weak Caputo derivative can be defined by the equality $\partial_{0,t}^{\alpha} v(t) = \partial_{0,t}^{\alpha}(v(t) - v(0))$ (see [21]).

2.3. Metric star graph. The star metric graph Γ is a graph with n bonds, consisting of a finite set of vertices $V = \{\nu_i\}_0^n$ and a finite set of edges $E = \{e_i\}_1^n$, where e_i connects the vertices ν_0 and ν_i, $i = \overline{1,n}$ [28]. Each bond e_i is assigned the interval $(0, l_i)$, and coordinates x_i are defined on each bond. The vertex ν_0 of the graph has a coordinate of 0 on each bond. Further, without loss of generality, we will use x instead of x_i. For the function, $u : \Gamma \to R$, defined on the graph, we put $u|_{e_i} = u_i$.

We define some functional spaces in the metric star graph. For the functions defined on the graph, we also use vector-type notations $u = (u_1, ..., u_n)$, $u_x = \left(\frac{\partial u_1}{\partial x}, ..., \frac{\partial u_n}{\partial x}\right)$, $u_{xx} = \left(\frac{\partial^2 u_1}{\partial x^2}, ..., \frac{\partial^2 u_n}{\partial x^2}\right)$, $\int\limits_{\Gamma} u d\Gamma = \sum\limits_{i=1}^{n} \int\limits_0^{l_i} u_i dx$. For $u : \Gamma \to R, v : \Gamma \to R$ we put $uv = (u_1v_1, u_2v_2, \ldots, u_nv_n)$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $0 < \alpha_i < 1$, $i = \overline{1,n}$, $G_{\tau} = \{(x,t) : x \in \Gamma, t \in (0,\tau]\}$, $0 < \tau \le T$. We put $\partial_{0,t}^{\alpha} u = \left(\partial_{0,t}^{\alpha_1} u_1, \partial_{0,t}^{\alpha_2} u_2, \ldots, \partial_{0,t}^{\alpha_n} u_n\right)$.

Definition 5. *[28] The space* $L_2(\Gamma)$ *on* Γ *consists of functions that are measurable and square-integrable on each edge* e_i, $i = \overline{1,n}$ *with the scalar product and the norm:*

$$(u(x), v(x))_{L_2(\Gamma)} = \int\limits_{\Gamma} u(x) \cdot v(x) d\Gamma,$$

$$\|u\|_{L_2(\Gamma)}^2 = \sum_i \|u\|_{L_2(e_i)}^2.$$

In other words, $L_2(\Gamma)$ *is the orthogonal direct sum of spaces* $L_2(e_i)$, $i = \overline{1,n}$.

Definition 6. *The Hilbert space* $W_2^l(\Gamma)$, $l = 1,2$ *defined by*

$$W_2^l(\Gamma) = \bigoplus_{i=1}^{n} W_2^l(e_i), \ l = 1,2,$$

and with the scalar products

$$(u,v)_{W_2^1(\Gamma)} = \int\limits_{\Gamma} (uv + u_x v_x)\, d\Gamma,$$

$$(u,v)_{W_2^2(\Gamma)} = \int\limits_{\Gamma} (uv + u_x v_x + u_{xx} v_{xx})\, d\Gamma.$$

Definition 7. *Let the space* $Q = \{u : u_i(x) \in C^{\infty}(\bar{e}_i), u|_{\nu} = 0, \nu \in \partial\Gamma\}$. $\overset{\circ}{W}{}_2^1(\Gamma)$ *is a subspace of the space* $W_2^1(\Gamma)$ *that is the closure of* Q *with respect to the norm* $\|u\|_{W_2^1(\Gamma)} = \sqrt{(u,u)_{W_2^1(\Gamma)}}$.

Definition 8. *Let* $L_2(G_{\tau}) = L_2(0,\tau; L_2(\Gamma))$.

$W_2^{1,\alpha}(G_\tau)=\{u: u(x,\cdot)\in W_2^1(\Gamma), u, u_x, \partial_{0,t}^\alpha u\in L_2(G_\tau)\}$ *is a subspace of* $L_2(G_\tau)$ *with the scalar product*

$$(u,v)_{W_2^{1,\alpha}(G_\tau)}=\int_0^\tau\int_\Gamma\left(uv+u_xv_x+\partial_{0,t}^\alpha u\partial_{0,t}^\alpha v\right)d\Gamma dt,$$

and with the norm

$$\|u\|_{W_2^{1,\alpha}(G_\tau)}=\left(\int_0^\tau\int_\Gamma\left(u^2+u_x^2+(\partial_{0,t}^\alpha u)^2\right)d\Gamma dt\right)^{\frac{1}{2}}.$$

$W_{2,0}^{1,\alpha}(G_\tau)=\{u\in W_2^{1,\alpha}(G_\tau): u|_\nu=0, \nu\in\partial\Gamma\}$ *is a subset of* $W_2^{1,\alpha}(G_\tau)$.

Definition 9. $W_2^{2,\alpha}(G_\tau)$ *is the Hilbert space consisting of all elements of* $L_2(G_\tau)$ *that have generalized derivatives* $\partial_{0,t}^\alpha u$, u_x *and* u_{xx} *from* $L_2(G_\tau)$*. The scalar product in it is defined by the equality*

$$(u,v)_{W_2^{2,\alpha}(G_\tau)}=\int_0^\tau\int_\Gamma\left(uv+u_xv_x+\partial_{0,t}^\alpha u\partial_{0,t}^\alpha v+u_{xx}v_{xx}\right)d\Gamma dt$$

and the norm is denoted as follows: $\|\cdot\|_{W_2^{2,\alpha}(G_\tau)}$.

Definition 10. $W_{2,0}^{2,\alpha}(G_\tau)$ *is a subspace of* $W_2^{2,\alpha}(G_\tau)$*, which is the intersection of* $W_2^{2,\alpha}(G_\tau)$ *with* $W_{2,0}^{1,\alpha}(G_\tau)$.

Also, we need arithmetic inequality

$$|ab|\le\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2,\ \text{ for all } \varepsilon>0. \tag{1}$$

3. Problem setting

The main physical purpose for adopting and investigating fractional order diffusion equations to describe phenomena of anomalous diffusion usually met in transport processes through complex and/or disordered systems including fractal media [29]. Fractional diffusion equations appears in nano-sized systems due to the fact, that majority of such systems demonstrate anomalous flow through the branched nanotubes with memory and viscoelasticity effects [5,6,29].

In this paper, we consider the time fractional diffusion (subdiffusion) equation on the graph Γ. The correct formulation of the problem on the metric graph requires consideration of the flux (current) conservation rule on the branching points. For the subdiffusion equation

$$\partial_{0,t}^{\alpha_i}U_i(x,t)=U_{i,xx}(x,t)+\mathcal{F}_i(x,t),\ i=\overline{1,n},$$

where $\partial_{0,t}^{\alpha_i}$ denotes the Caputo fractional derivative of order $\alpha_i\in(0,1)$ defined by Definition 4, the flux is defined as follows

$$F[U_i(x,t)]=-\frac{\partial}{\partial x}\left(D_{0,t}^{1-\alpha_i}U_i(x,t)\right).$$

So, the vertex conditions should be defined as

$$\sum_{i=1}^n D_{0,t}^{1-\alpha_i}U_{i,x}(0,t)=0,\ U_i(0,t)=U_j(0,t),\ i\ne j,\ i,j=\overline{1,n},\ \ t\in(0,T]. \tag{2}$$

Condition (2) is disadvantageous due to the presence of a mixed derivative. To exclude of the mixed derivative in condition (2), we put $U_i=I_{0,t}^{1-\alpha_i}u_i(x)$ and obtain

$$\sum_{i=1}^n u_{i,x}(0,t)=0,\quad I_{0,t}^{1-\alpha_i}u_i(0,t)=I_{0,t}^{1-\alpha_j}u_j(0,t),\ i\ne j,\ i,j=\overline{1,n},\ \ t\in(0,T], \tag{3}$$

where $I_{0,t}^{1-\alpha}u=\left(I_{0,t}^{1-\alpha_1}u_1,I_{0,t}^{1-\alpha_2}u_2,\dots,I_{0,t}^{1-\alpha_n}u_n\right)$. The vertex conditions (3) are more advantageous for our further consideration.

So, we investigate subdiffusion equation on each edge of the graph Γ

$$\partial_{0,t}^{\alpha_i}u_i(x,t)-u_{i,xx}(x,t)=f(t)g_i(x,t)+h_i(x,t),\quad x\in e_i,\ \ t\in(0,T],\ i=\overline{1,n}. \tag{4}$$

We need to establish the following initial conditions

$$u_i(x,0)=0,\quad x\in\bar{e}_i,\quad i=\overline{1,n}, \tag{5}$$

the vertex conditions (3) and the boundary conditions

$$u_i(l_i,t)=0,\ t\in(0,T], \tag{6}$$

where $g_i(x,t)$, $h_i(x,t), i=\overline{1,n}$, are given functions, $f(t)$ is an unknown function.

The main aim is to find the pair of functions $\{u(x,t),f(t)\}$. To find the solution $f(t)$ to the inverse problem, an additional condition is needed. Therefore, we introduce an additional integral overdetermination condition in the form of

$$\int\limits_\Gamma \eta(x)I_{0,t}^{1-\alpha}u(x,t)d\Gamma=\psi(t),\quad t\in[0,T], \tag{7}$$

where $\eta_i(x)$, $i=\overline{1,n}$, and $\psi(t)$ are known functions.

Here we should mention, that by Mehandiratta et al. [30], the existence and uniqueness of the weak solution of the time-fractional diffusion equation on a metric star graph were investigated. This equation involved the same fractional orders of derivatives in each of the edges. The Cauchy problem for the subdiffusion equation with edge-dependent order of the Riemann-Liouville time-fractional derivatives on a metric star graph with semi-infinite bonds was studied in another work [31].

In the current work, the focus is on the strong solution for initial-boundary value problem in the star graph. The approach to solving this problem involves the method introduced by Ladyzhenskaya [32], which entails reducing the given problem to an operator equation and using a-priory estimates.

We represent the solution of the problem as $u(x,t) = v(x,t) + w(x,t)$, where v satisfies the following equation

$$\partial_{0,t}^{\alpha_i} v_i(x,t) - v_{i,xx}(x,t) = h_i(x,t)$$

with (3), (5), (6) conditions. This problem we call *the direct problem*. Also, we define *the inverse problem* looking for $f(t)$ in the following way

$$\partial_{0,t}^{\alpha_i} w_i(x,t) - w_{i,xx}(x,t) = f(t)g_i(x,t), \tag{8}$$

with (3), (5), (6) conditions and an additional condition given by

$$\int\limits_{\Gamma} \eta(x) I_{0,t}^{1-\alpha} w(x,t)d\Gamma = E(t) = \psi(t) - \int\limits_{\Gamma} \eta(x) I_{0,t}^{1-\alpha} v(x,t)d\Gamma, \quad t \in (0,T]. \tag{9}$$

4. Well-posedness of the direct problem

Theorem 3. Let $h(x,t) \in L_2(G_T)$ and $g(x,t) = 0$. Then the problem (3)–(6) has a unique strong solution in $W_{2,0}^{2,\alpha}(G_T)$.

Proof. According to [32], the direct problem can be represented as the problem of solving the operator equation $Av = h$. The domain of A is denoted as

$$D(A) = \left\{ \begin{array}{l} v \in W_{2,0}^{2,\alpha}(G_T) : v_i|_{t=0} = 0, \ \sum\limits_{i=1}^{n} v_{i,x}(0,t) = 0, \\ I_{0,t}^{1-\alpha_i} v_i(x,t) = I_{0,t}^{1-\alpha_j} v_j(x,t), \ i \neq j, \ i,j = \overline{1,n} \end{array} \right\}$$

and the range $R(A) \subset L_2(G_T)$.

Proposition 1. *Operator $A : D(A) \to L_2(G_T)$ is continuous.*

Proof. Continuity of the operator A is a consequence of the following inequality

$$\|Av\|_{L_2(G_T)} = \left\|\partial_{0,t}^{\alpha} v - v_{xx}\right\|_{L_2(G_T)} \leq \|\partial_{0,t}^{\alpha} v\|_{L_2(G_T)} + \|v_{xx}\|_{L_2(G_T)} \leq \|v\|_{W_{2,0}^{2,\alpha}(G_T)}. \tag{10}$$

Further, we suppose that the space $R(A)$ is equipped with the norm of $L_2(G_T)$.

Proposition 2. *The inverse operator $A^{-1} : R(A) \to W_{2,0}^{2,\alpha}(G_T)$ is well-defined and continuous.*

Proof. We square Av and integrate over Γ:

$$\int_{\Gamma} (Av)^2 \, d\Gamma = \int_{\Gamma} \left(\partial_{0,t}^{\alpha} v - v_{xx}\right)^2 d\Gamma = \int_{\Gamma} \left[\left(\partial_{0,t}^{\alpha} v\right)^2 + (v_{xx})^2\right] d\Gamma$$

$$-2 \int_{\Gamma} \partial_{0,t}^{\alpha} v \cdot (v_{xx}) \, d\Gamma = \int_{\Gamma} \left[\left(\partial_{0,t}^{\alpha} v\right)^2 + (v_{xx})^2\right] d\Gamma$$

$$-2 \sum_{i=1}^{n} \frac{\partial}{\partial t} I_{0,t}^{1-\alpha_i} v_i \cdot v_{i,x} \Big|_{x=0}^{x=l_i} + 2 \int_{\Gamma} v_x \partial_{0,t}^{\alpha} v_x d\Gamma.$$

The sum in the final line equals zero according to the vertex conditions (3) and boundary conditions (6). Integrating the last equality with respect to t and taking into account the inequality [33]

$$\int_{\Gamma} y \partial_{0,t}^{\alpha} y d\Gamma \geq \frac{1}{2} \int_{\Gamma} \partial_{0,t}^{\alpha} y^2 d\Gamma, \tag{11}$$

we obtain

$$\int_0^t \int_{\Gamma} (Av)^2 \, d\Gamma d\tau = \int_0^t \int_{\Gamma} \left[\left(\partial_{0,\tau}^{\alpha} v\right)^2 + (v_{xx})^2\right] d\Gamma d\tau + 2 \int_0^t \int_{\Gamma} v_x \partial_{0,\tau}^{\alpha} v_x d\Gamma d\tau$$

$$\geq \int_0^t \int_{\Gamma} \left[\left(\partial_{0,\tau}^{\alpha} v\right)^2 + (v_{xx})^2\right] d\Gamma d\tau + \int_0^t \int_{\Gamma} \partial_{0,\tau}^{\alpha} (v_x)^2 \, d\Gamma d\tau.$$

Then we obtain

$$\int_0^t \int_{\Gamma} \left[\left(\partial_{0,\tau}^{\alpha} v\right)^2 + (v_{xx})^2\right] d\Gamma d\tau + \int_{\Gamma} I_{0,t}^{1-\alpha} v_x^2 d\Gamma d\tau \leq \int_0^t \int_{\Gamma} (Av)^2 \, d\Gamma d\tau. \tag{12}$$

Now we estimate $\|v\|_{L_2(G_t)}$ and $\|v_x\|_{L_2(G_t)}$. By utilizing Proposition 2.1 in [34], we have $\|I_{0,t}^{\alpha} f\|_{L_p(0,T)} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L_p(0,T)}$, for all $f \in L_p(0,T)$ and $\alpha > 0$, $1 \leq p \leq \infty$. Using this inequality and the estimate (12), by the Definition 4. and Corollary 1. we obtain the following

$$\|v\|^2_{L_2(G_t)} = \sum_{i=1}^{n} \int_0^{l_i} \|I_{0,t}^{\alpha_i}(\partial_{0,t}^{\alpha_i} v_i)\|^2_{L_2(0,t)} dx$$

$$\leq \sum_{i=1}^{n} \left(\frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)^2 \|\partial_{0,t}^{\alpha_i} v_i\|^2_{L_2(G_t)} \leq C_1 \|\partial_{0,t}^{\alpha} v\|^2_{L_2(G_t)} \leq \int_0^t \int_\Gamma (Av)^2 d\Gamma d\tau, \tag{13}$$

where $C_1 = \max\limits_{0\leq i\leq n} \left\{ \left(\frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)^2 \right\}$. For each edge of the graph, taking into account (12), we come to the following estimate

$$\int_0^t \int_0^{l_i} v_{i,x}^2(x,\tau) dx d\tau = I_{0,t}^1 \int_0^{l_i} v_{i,x}^2(x,t) dx = I_{0,t}^{\alpha_i} \left(I_{0,t}^{1-\alpha_i} \int_0^{l_i} v_{i,x}^2 dx \right) \leq$$

$$\leq I_{0,t}^{\alpha_i} 1 \cdot \operatorname{ess}\sup_{0\leq\tau\leq t} I_{0,\tau}^{1-\alpha_i} \int_0^{l_i} v_{i,x}^2 dx = \frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} \operatorname{ess}\sup_{0\leq\tau\leq t} \sum_{i=1}^{n} I_{0,\tau}^{1-\alpha_i} \int_0^{l_i} v_{i,x}^2 dx \leq$$

$$\leq \frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} \operatorname{ess}\sup_{0\leq\tau\leq t} \int_0^{\tau} \int_\Gamma (Av(x,s))^2 d\Gamma ds = \frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} \int_0^t \int_\Gamma (Av(x,s))^2 d\Gamma ds.$$

Summarizing the above estimate, we obtain

$$\int_0^t \int_\Gamma v_x^2 d\Gamma d\tau \leq C_2 \int_0^t \int_\Gamma (Av)^2 d\Gamma d\tau, \tag{14}$$

where $C_2 = \sum\limits_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}$.

Finally, from (12), (13), (14), we conclude that

$$\|v\|_{W_{2,0}^{2,\alpha}(G_t)} \leq C_3 \|Av\|_{L_2(G_t)}.$$

The last estimate shows that the inverse operator A^{-1} exists and is continuous.

From Proposition 1. and Proposition 2, we can conclude the following result.

Corollary 2. The range $R(A)$ of the operator A is a closed linear subspace of $L_2(G_T)$.

To demonstrate the solvability of the direct problem, we need to establish that the range $R(A)$ of the operator A in $L_2(G_T)$ does not have an orthogonal complement.

Lemma 1. *If for some $\varphi \in L_2(G_T)$ it holds $(Av, \varphi) = 0$ for all $v \in D(A)$, then $\varphi = 0$.*

Proof. We must show that if for all $v \in D(A)$ the following equality

$$\int_0^t \int_\Gamma \left(\partial_{0,\tau}^{\alpha} v - v_{xx}\right) \varphi d\Gamma d\tau = 0, \tag{15}$$

is satisfied then $\varphi = 0$.

Below, we construct a test function v.

Let $0 < t_1 < T$, $v(x,t) = 0$ for $0 < t \leq t_1$. We consider the following auxiliary problem:

$$v_{i,xxt}(x,t) = \varphi_i(x,t),\ x \in e_i,\ t_1 < t < T,\ v_{xx}|_{t=t_1} = 0,\ x \in \bar{e}_i\ i = \overline{1,n},$$

$$v_i(l_i,t) = 0,\ \sum_{i=1}^{n} v_{i,x}(0,t) = 0,\ I_{0,t}^{1-\alpha_i} v_i(0,t) = I_{0,t}^{1-\alpha_j} v_j(0,t),\ i \neq j,\ i,j = \overline{1,n},\ < t_1 < t < T]. \tag{16}$$

The solution to this equation is in the following form

$$v_i(x,t) = \int_0^x (x-\xi)\tilde{\varphi}_i(x,t) d\xi + a_i x + b_i, \tag{17}$$

where $\tilde{\varphi}_i(x,t) = \int_{t_1}^{t} \varphi_i(x,\tau) d\tau$.

The unknown coefficients $a_i = a_i(t),\ b_i = b_i(t)$ can be found using the boundary and vertex conditions in (16) as follows

$$\begin{cases} \int_0^{l_i} (l_i - \xi)\tilde{\varphi}(\xi, t)d\xi + a_i l_i + b_i = 0,\ i = \overline{1, n}, \\ I_{0,t}^{1-\alpha_1} b_1 = I_{0,t}^{1-\alpha_2} b_2 = \ldots = I_{0,t}^{1-\alpha_n} b_n, \\ \sum_{i=1}^{n} a_i = 0. \end{cases}$$

Without loss of generality, we take $\alpha_1 = \min_{1\le i\le n}\{\alpha_i\}$. So, we obtain

$$\begin{cases} a_i = \zeta_i(t) - \frac{1}{l_i} b_i,\ i = \overline{1, n}, \\ b_i = I_{0,t}^{\alpha_i - \alpha_1} b_1,\ i = \overline{2, n}, \end{cases} \tag{18}$$

$$\sum_{i=1}^{n} \frac{1}{l_i} I^{\alpha_i - \alpha_1} b_1(t) = \tilde{\zeta}(t). \tag{19}$$

where $\zeta_i(t) = -\frac{1}{l_i}\int_0^{l_i}(l_i - \xi)\tilde{\varphi}(\xi, t)d\xi,\ \tilde{\zeta}(t) = \sum_{i=1}^{n}\zeta_i(t)$. The equation (19) is called the generalized Abel integral equation. Following the results of [35], we use the following notations to describe the solution of the generalized Abel integral equation (19). We put

$$\mathrm{w}_\mu(t) = l_1 G_n^{\mu}\left(t; -\frac{l_1}{l_2}, \ldots, -\frac{l_1}{l_n}; \alpha_1, \ldots, \alpha_n\right),$$

where $\mu = \sum_{i=1}^{n}\mu_i,\ \mu_i > 0,$

$$G_n^{\mu}(t; \gamma_1, \ldots, \gamma_n; \rho_1, \ldots, \rho_n) = \int_0^{+\infty} e^{-\tau} S_n^{\mu}(t; \gamma_1\tau, \ldots, \gamma_n\tau; \rho_1, \ldots, \rho_n)d\tau,$$

$$S_n^{\mu}(t; z_1, \ldots, z_n; \rho_1, \ldots, \rho_n) = (y_1 * y_2 * \ldots * y_n)(t),$$
$$y_i = y_i(t) = t^{\mu_i - 1}\phi(\rho_i, \mu_i; z_i t^{\rho_i}).$$

By $\mathrm{f} * \mathrm{g}$, we denoted the Laplace convolution of functions defined by $(\mathrm{f} * \mathrm{g})(t) = \int_0^t \mathrm{f}(\tau)\mathrm{g}(t-\tau)d\tau$.

The function $\mathrm{w}_\mu(t)$ has the following properties.

Lemma 2. [35]

(a) The function $\mathrm{w}_\mu(t)$ is independent of the distribution of parameters μ_i but depends only on their sum μ.

(b) The function $\mathrm{w}_\mu(t)$ satisfies the following relations

$$\mathrm{w}_\mu(t) = O(t^{\mu-1}) \text{ as } t \to 0,\ D_{0,t}^{\beta}\mathrm{w}_\mu(t) = \mathrm{w}_{\mu-\beta}(t) \text{ for } \mu > \beta > 0.$$

According to [35], Theorem 6, the solution of equation (19) for $\tilde{\zeta}(t) \in L_1(0, T)$ is given by

$$b_1(t) = D_{0,t}^{\mu}(\tilde{\zeta}(t) * \mathrm{w}_\mu)(t), \tag{20}$$

where $\mu > 0$. Substituting the found function $b_1(t)$ into (18) we obtain the values of the coefficients in (17). So, we find the solution of the auxiliary problem.

Also, we notice that as $\varphi \in L_2(G_T)$, it follows that $\zeta(t) \in H^1(t_1, T)$. According to general theory of integral equations, from (19), it follows that the solution $b_1(t)$ smoothness is such as the right-hand side of the equation $\zeta(t)$.

This way, we construct the test function. Now, from (15), we obtain

$$0 = \int_{t_1}^{t}\int_{\Gamma}\left(\partial_{0,\tau}^{\alpha} v - v_{xx}\right) v_{xx\tau} d\Gamma d\tau = \int_{t_1}^{t}\int_{\Gamma}\partial_{0,\tau}^{\alpha} v \cdot v_{xx\tau} d\Gamma d\tau - \int_{t_1}^{t}\int_{\Gamma} v_{xx} v_{xx\tau} d\Gamma d\tau$$

$$= \int_{t_1}^{t}\sum_{i=1}^{n}\frac{\partial}{\partial\tau} I_{0,\tau}^{1-\alpha_i} v_i \cdot v_{i,x\tau}\Big|_{x=0}^{x=l_i} d\tau - \int_{t_1}^{t}\int_{\Gamma}\partial_{0,\tau}^{\alpha} v_x \cdot v_{x\tau} d\Gamma d\tau - \int_{t_1}^{t}\int_{\Gamma}\frac{1}{2}\frac{\partial}{\partial\tau}(v_{xx})^2 d\Gamma d\tau.$$

Considering the vertex conditions (3) and boundary conditions (6) for the second integral in the last equation, it is evident that the sum equals zero. From the construction of $v(x, t)$, it follows that $v_{xx}(x, t_1) = 0$. Hence, we obtain

$$0 = \mathcal{I}_1 + \frac{1}{2}\int_{\Gamma} v_{xx}^2\Big|_{\tau=t} d\Gamma, \tag{21}$$

where considering (11), and according to Theorem 1, Theorem 2, Corollary 1 and Definition 4, we can write as follows

$$\mathcal{I}_1 = \int_{t_1}^{t} \int_{\Gamma} \partial_{0,\tau}^{\alpha} v_x \cdot v_{x\tau} d\Gamma d\tau = \int_{t_1}^{t} \int_{\Gamma} I_{0,\tau}^{1-\alpha} v_{x\tau} v_{x\tau} d\Gamma d\tau$$

$$= \int_{t_1}^{t} \int_{\Gamma} I_{0,\tau}^{1-\alpha} v_{x\tau} \cdot \partial_{0,t}^{1-\alpha} \left(I_{0,\tau}^{1-\alpha} v_{x\tau} \right) d\Gamma d\tau \geq \frac{1}{2} \int_{t_1}^{t} \int_{\Gamma} \partial_{0,\tau}^{1-\alpha} \left(I_{0,\tau}^{1-\alpha} v_{x\tau} \right)^2 d\Gamma d\tau$$

$$= \frac{1}{2} \int_{\Gamma} \left(I_{0,t}^{\alpha} \left(I_{0,t}^{1-\alpha} v_{xt} \right)^2 - I_{0,t_1}^{\alpha} \left(I_{0,t}^{1-\alpha} v_{xt} \right)^2 \right) d\Gamma.$$

Substitute the final estimate for $\mathcal{I}_1$ into equation (21) to obtain the following result

$$0 \geq \frac{1}{2} \int_{\Gamma} I_{t_1,t}^{\alpha} \left(I_{0,t}^{1-\alpha} v_{xt} \right)^2 d\Gamma + \frac{1}{2} \int_{\Gamma} v_{xx}^2(x,t) d\Gamma.$$

We obtain from the last inequality that $v_{xx} = 0$, $I_{0,t}^{1-\alpha} v_{xt} = 0$ for $t_1 < t \leq T$, $x \in \Gamma$. Taking into consideration the construction of the function v, that t_1 is arbitrary number in $(0,T)$, we obtain $\varphi = v_{xxt} = 0$ in $L_2(G_T)$. The lemma is proven.

We can conclude from this that the direct problem has a unique solution in $W_{2,0}^{2,\alpha}(G_T)$.

5. Solvability of the inverse problem

Now, we show that the inverse problem has a solution. Let the following conditions be satisfied

$$(K1) \quad g(x,t) \in L_\infty \left(0, T, L_2(\Gamma)\right),$$

$$\eta(x) \in W_2^1(\Gamma),\ \eta(x) \ is\ continuous\ in\ \Gamma,\ \ \eta|_\nu = 0,\ \nu \in \partial\Gamma,\ i = \overline{1,n},$$

$$\|\eta_x(x)\|_{L_2(\Gamma)} = m > 0,\ E(t) \in W_2^1(0,T),\ |g^*(t)| \geq q > 0,$$

where

$$g^*(t) = \int_{\Gamma} \eta(x) g(x,t) d\Gamma,\ \ t \in (0,T].$$

We multiply the both sides of equation (8) by the function $\eta(x)$ and integrate over Γ, we obtain

$$\int_{\Gamma} \eta(x) \partial_{0,t}^{\alpha} \omega(x,t) d\Gamma - \int_{\Gamma} \eta(x) \omega_{xx}(x,t) d\Gamma = f(t) \int_{\Gamma} g(x,t) \eta(x) d\Gamma.$$

Using conditions (3), (6), (7) and (K1), we obtain

$$\int_{\Gamma} \eta(x) \partial_{0,t}^{\alpha} \omega(x,t) d\Gamma + \int_{\Gamma} \eta_x(x) \omega_x(x,t) d\Gamma = f(t) g^*(t).$$

Taking into account (9), we can find $f(t)$ in such form

$$f(t) = (Bf)(t) + \frac{E_t(t)}{g^*(t)}, \tag{22}$$

where $E(t)$ is defined in (9) and

$$(Bf)(t) = \frac{1}{g^*(t)} \left\{ \int_{\Gamma} \eta_x(x) \omega_x(x,t) d\Gamma \right\},\quad B : L_2(0,T) \to L_2(0,T).$$

The operator B can be considered as a result of combining two operators: $Bf = M(Kf)$, where $\omega = Kf := A^{-1}(fg)$, where operator A and the proof of existence of its inverse can be found in the previous section, and

$$(M\omega)(t) := \frac{1}{g^*(t)} \left\{ \int_{\Gamma} \eta_x(x) \omega_x(x,t) d\Gamma \right\}.$$

If the conditions known as (K1) are met, then we can substitute the overdetermination condition represented by equation (9) with equation (22). This substitution is equivalent and does not change the meaning or outcome of the conditions.

Theorem 4. *Let conditions (K1) hold. If $h(x,t) \in L_2(G_T)$, then the inverse problem (3)–(7) has a unique generalized solution $\{u, f\} \in W_{2,0}^{2,\alpha}(G_T) \times L_2(0,T)$.*

Proof. It is sufficient to demonstrate that the resolvent operator $(\mathcal{I} - B)^{-1}$ is bounded and continuous when mapping from $L_2(0,T)$ to $L_2(0,T)$, where $\mathcal{I}$ is the identity operator. This then implies that f belongs to $L_2(0,T)$ and $h(x,t) + f(t)g(x,t) \in L_2(G_T)$. Consequently, according to Theorem 3., $u(x,t) \in W_{2,0}^{2,\alpha}(G_T)$.

First, we need to obtain some a-priory estimate. We multiply each of equations (8) by the corresponding $\partial_{0,t}^{\alpha}\omega(x,t)$ and integrate over G_t to get

$$\int_0^t\int_\Gamma \left(\partial_{0,\tau}^{\alpha}\omega(x,\tau)\right)^2 d\Gamma d\tau - \int_0^t\int_\Gamma \omega_{xx}(x,\tau)\partial_{0,\tau}^{\alpha}\omega(x,\tau)d\Gamma d\tau$$
$$= \int_0^t f(\tau)d\tau \int_\Gamma g(x,\tau)\partial_{0,\tau}^{\alpha}\omega(x,\tau)d\Gamma.$$

Integrating the second integral by parts, we obtain

$$\int_0^t\int_\Gamma \left(\partial_{0,\tau}^{\alpha}\omega(x,\tau)\right)^2 d\Gamma d\tau - \int_0^t\sum_{i=1}^{n}\left(\frac{\partial}{\partial\tau}I_{0,\tau}^{1-\alpha_i}\omega_i\right)\cdot(\omega_{i,x})\Bigg|_{x=0}^{x=l_i} d\tau$$
$$+\int_0^t\int_\Gamma \omega_x(x,\tau)\partial_{0,\tau}^{\alpha}\omega_x(x,\tau)d\Gamma d\tau = \int_0^t f(\tau)d\tau\int_\Gamma g(x,\tau)\partial_{0,\tau}^{\alpha}\omega(x,\tau)d\Gamma.$$

Based on conditions (3) and (6), the sum in the last equation becomes zero. Considering the inequality (11) and inequality (1) with $\varepsilon = 1$ on the right hand side of the equation, we can express it as

$$\|\partial_{0,t}^{\alpha}\omega\|^2_{L_2(G_t)} + \frac{1}{2}\sum_{i=1}^{n} I_{0,t}^{1-\alpha_i}\|\omega_x\|^2_{L_2(e_i)} \le \frac{1}{2}c^2\int_0^t f^2(\tau)d\tau + \frac{1}{2}\|\partial_{0,t}^{\alpha}\omega\|^2_{L_2(G_t)},$$

where $c = \text{ess}\sup\limits_{0\le t\le T}\|g(\cdot,t)\|_{L_2(\Gamma)}$. Then we have

$$\|\partial_{0,t}^{\alpha}\omega\|^2_{L_2(G_t)} + \sum_{i=1}^{n} I_{0,t}^{1-\alpha_i}\|\omega_x\|^2_{L_2(e_i)} \le c^2\int_0^t f^2(\tau)d\tau.$$

From the last inequality, we obtain

$$\sum_{i=1}^{n} I_{0,t}^{1-\alpha_i}\|\omega_x\|^2_{L_2(e_i)} \le c^2\int_0^t f^2(\tau)d\tau. \tag{23}$$

Taking into account conditions (K1), we have

$$\|(Bf)(\tau)\|^2_{L_2(0,t)} = \int_0^t |(Bf)(t)|^2 d\tau \le \frac{m^2}{q^2}\int_0^t \|\omega_x\|^2_{L_2(\Gamma)}d\tau$$
$$= \frac{m^2}{q^2}\sum_{i=1}^{n} I_{0,t}^{1}\|\omega_x\|^2_{L_2(e_i)} = \frac{m^2}{q^2}\sum_{i=1}^{n} I_{0,t}^{\alpha_i}\left(I_{0,t}^{1-\alpha_i}\|\omega_x\|^2_{L_2(e_i)}\right). \tag{24}$$

If $f(t)\ge 0$, $t\in(0,T]$, $\rho > \beta$, then the following estimate holds

$$I_{0,t}^{\rho}f(t) = \frac{1}{\Gamma(\rho)}\int_0^t \frac{1}{(t-\tau)^{1-\rho}}f(\tau)d\tau = \frac{1}{\Gamma(\rho)}\int_0^t \frac{1}{(t-\tau)^{1-\beta}}f(\tau)(t-\tau)^{\rho-\beta}d\tau$$
$$\le \frac{\Gamma(\beta)}{\Gamma(\rho)}t^{\rho-\beta}I_{0,t}^{\beta}f(t) \le \frac{\Gamma(\beta)}{\Gamma(\rho)}T^{\rho-\beta}I_{0,t}^{\beta}f(t). \tag{25}$$

Accordingly, from (24) and (25), we obtain

$$\|(Bf)(t)\|^2_{L_2(0,t)} \le \frac{m^2}{q^2}\sum_{i=1}^{n}\frac{\Gamma(\alpha_{\min})}{\Gamma(\alpha_i)}T^{\alpha_i-\alpha_{\min}}I_{0,t}^{\alpha_{\min}}\left(I_{0,t}^{1-\alpha_i}\|\omega_x\|^2_{L_2(e_i)}\right),$$

or

$$\|(Bf)(t)\|^2_{L_2(0,t)} \le C_1 I_{0,t}^{\alpha_{\min}}\left(\sum_{i=1}^{n} I_{0,t}^{1-\alpha_i}\|\omega_x\|^2_{L_2(e_i)}\right), \tag{26}$$

where $\alpha_{min} = \min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, $C_1 = \dfrac{m^2}{q^2}\cdot\max\limits_{1\le i\le n}\{\dfrac{\Gamma(\alpha_{\min})}{\Gamma(\alpha_i)}T^{\alpha_i-\alpha_{\min}}\}$.

From (23) and (26), we come to

$$\|(Bf)(t)\|^2_{L_2(0,t)} \le C I_{0,t}^{\alpha_{\min}}\|f\|^2_{L_2(0,t)}, \tag{27}$$

where $C = c^2\cdot C_1$. Now, iterating the inequality (27) k times we obtain

$$\|(B^k f)(t)\|^2_{L_2(0,t)} \le C I_{0,t}^{\alpha_{\min}}\|B^{k-1}f\|^2_{L_2(0,t)} \le C^k I_{0,t}^{k\alpha_{\min}}\|f\|^2_{L_2(0,t)},\ k = 1,2,3,....$$

From the last inequality, taking into account that the function $\tilde{f}(t) = \|f\|^2_{L_2(0,t)}$ is a nonnegative and non-decreasing function on $t\in[0,T]$, we have

$$\|(B^k f)(t)\|^2_{L_2(0,t)} \le C^k\|f\|^2_{L_2(0,t)}\cdot I_{0,t}^{k\alpha_{\min}}1 = \frac{C^k t^{k\alpha_{\min}}}{\Gamma(k\alpha_{\min}+1)}\|f\|^2_{L_2(0,t)}.$$

Accordingly, we have

$$\|(\mathcal{I}-B)^{-1}f\|_{L_2(0,T)} \leq \sum_{k=0}^{+\infty} \frac{\left(\sqrt{CT^{\alpha_{\min}}}\right)^k}{\sqrt{\Gamma(k\alpha_{\min}+1)}} \|f\|_{L_2(0,t)}.$$

So, it follows that the resolvent operator $(\mathcal{I}-B)^{-1} : L_2(0,T) \to L_2(0,T)$ is bounded and continuous mapping and

$$f(t) = (\mathcal{I}-B)^{-1}\left[\frac{E_t(t)}{g^*(t)}\right].$$

6. Conclusion

In our research, we concentrated on investigating the direct and inverse source problems related to the subdiffusion equation on a metric graph with an edge-dependent order of time-fractional derivative. We achieved this by transforming the problem into the task of solving an operator equation. We were able to prove the existence and uniqueness of a generalized solution in the direct problem. Additionally, we showed that the resolvent operator is appropriately defined in the inverse source problem, as evidenced by the overdetermination condition. We are particularly excited about the potential applications of our research in modeling subdiffusion processes in branched nanostructures. The subdiffusion equation on a metric graph with an edge-dependent order of the time-fractional derivative provides a robust mathematical framework for simulating and understanding subdiffusion phenomena in these complex nanostructures.

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Submitted 8 August 2024; revised 22 September 2024; accepted 23 September 2024

Information about the authors:

Zarifboy A. Sobirov – National University of Uzbekistan, Universitet str., 4, 100174, Tashkent, Uzbekistan; V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Science, 100174, Tashkent, Uzbekistan; ORCID 0000-0002-6019-907X; z.sobirov@nuu.uz

Ariukhan A. Turemuratova – National University of Uzbekistan, Universitet str., 4, 100174, Tashkent, Uzbekistan; Branch of Russian Economic University named after G. V. Plekhanov in Tashkent, 100164, Tashkent, Uzbekistan; ORCID 0009-0003-5428-3958; ariuxanturemuratova@gmail.com

Conflict of interest: the authors declare no conflict of interest.