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Original article

Boundary composed of small Helmholtz resonators: asymptotic approach

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ABSTRACT We consider the solution of the two-dimensional Neumann problem for the Helmholtz equation in a complex region composed of a square resonator with large number of smaller square resonators connected to it through small apertures along one side. The sizes of the apertures and distances between the neighbour apertures tend to zero. We use the method of matching of asymptotic expansions of solutions. By directing the number of attached small resonators to infinity, we obtain a problem for the Laplacian in the main square with energy-dependent boundary condition.

KEYWORDS eigenfunction, Helmholtz equation, boundary problem, asymptotics

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1. Introduction

Changes of the Laplacian spectrum under small perturbation of the domain is important and widely studied problem. It is well-known that sufficiently regular perturbation leads to the situation when $n-$th eigenvalue of the perturbed operator tends to $n-$th eigenvalue of the unperturbed operator if the perturbation decreases [1, 2]. However, if the perturbation is not so regular the situation becomes more complicated [3, 4]. An interesting case takes place if one consider a domain (resonator) with small coupled resonator or several resonators [5]. This is especially interesting if the number of coupled resonators tends to infinity simultaneously with the reducing of its sizes. Under these assumptions, the problem is analogous to homogenization problem [6–15]. This situation was studied from variational point of view in [16]. As a limiting result, one obtains the problem in main domain with a specific energy-dependent boundary condition (such conditions are interesting both from mathematical and physical points of view [17]). The result depends on the details of the limiting procedure (relation between "rooms" and "passages" (resonators and coupling channels)). A particular case of the problem was considered in [18] in the framework of the operator extensions theory model for coupled resonators. The role of the coupling windows shape was described in [19]. Numerical results are in [20]. Two close windows were considered in [21, 22] in the framework of the point-like window model. In the present paper, we consider the problem for the system shown in Fig. 1. We deal with asymptotics of the solution of the boundary problem in respect to two small parameters: apertures width and distance between neighbour apertures. Matching of asymptotic expansions of solutions (see, e.g., [23, 24] is used.

Recently, an additional interest to the problem was inspired by investigations towards creation of acoustic metamaterials, i.e. a form of man-made materials that can be specifically developed to have a sub-wavelength periodic structures with extraordinary characteristics not found in nature [25]. Physicists try to find such structures using the Helmholtz resonators, membrane-type structures, locally resonant and space-coiled structures. The particular question in the field is as follows: Can one create an unusual boundary condition by a specific geometry of the boundary? For example, physicists construct different combinations of small acoustic resonators (see, e.g., [26]).

2. The model

We consider the Laplace operator in the two-dimensional domain composed of a square (main resonator) Ω^- (the length of edge equals d) and a chain of N identical square resonators $\Omega_i^+, i = 1, 2, ...N$, coupled to one edge of the square through small apertures of widths $2ad$, where a is a dimensionless small parameter (see Fig. 1). In this domain, we deal with the Laplace operator $-\Delta$ defined on functions from the Sobolev space $W_2^2(\Omega)$, $\Omega = \Omega^- \cup \Omega_1^+ \cup \Omega_2^+ \cup ... \cup \Omega_N^+$, satisfying the Neumann boundary conditions at $\partial\Omega$ and the Meixner condition at edge points of the apertures (to ensure

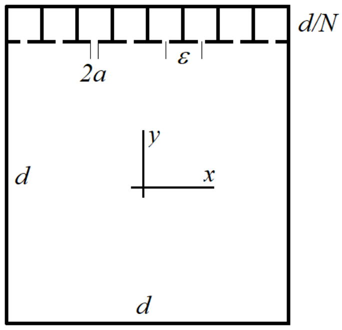

FIG. 1. Geometry of the system

the uniqueness of the solution). Correspondingly, in spectral problem one has the Helmholtz equation with the Neumann boundary conditions:

$$\Delta\psi + k^2\psi = 0, \quad \frac{\partial\psi}{\partial n}\Big|_{\partial\Omega} = 0. \tag{1}$$

The Green function for the Helmholtz operator for the square having side d) with the Neumann boundary condition has the form:

$$G(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\chi_{nm}(x,y)\chi_{nm}(x',y')}{\left(k^2 - \frac{\pi^2 n^2}{d^2} - \frac{\pi^2 m^2}{d^2}\right)(\delta_{nm}+1)}, \tag{2}$$

where $X = (x, y)$, $X' = (x', y')$, δ_{nm} is the Kronecker symbol ($\delta_{nm} = 1$ if $n^2 + m^2 = 0$, else $\delta_{nm} = 0$),

$$\chi_{nm}(x,y) = \frac{2}{d}\cos\frac{\pi n x}{d}\cos\frac{\pi m y}{d}. \tag{3}$$

The Green function allows one to transform equation (1) in the main square to the following form:

$$\psi(X) = \int\limits_{\partial\Omega^-} G(X, X', k)\frac{\partial\psi}{\partial n}(X')dX'. \tag{4}$$

The analogous presentation is valid for each small coupled resonator.

We will seek the eigenvalue k^2 close to $\sqrt{2}\pi/d$. The term corresponding to $n = m = 1$ in (2) gives a singularity in respect to the spectral parameter of the Green function. We use the method of matching asymptotic expansions of solutions [23, 24]. In the present paper, we deal with the problem containing two small parameters. We consider asymptotic expansions in small parameter a, that corresponds to the apertures radius and $a \to 0$. However, at the same time, the number of resonators attached tends to infinity ($N \to \infty$) so that the ε (distance between adjacent holes) tends to zero. We assume that these two small parameters are related in accordance with the following formula:

$$\varepsilon = |x_i - x_{i-1}| = ma^\delta = \frac{d}{N}, \quad \delta \in (0, 1). \tag{5}$$

Keeping in mind that the integrand in (4) vanishes on the part of the boundary outside the apertures, one can see that the integral is over apertures only. Correspondingly, one can find the main terms of the asymptotics in a in the following form which is analogous to that in [23]. Particularly, the solution of equation (1) near i-th aperture has the form:

$$\psi(X) = \begin{cases} \left(k^2 - \dfrac{2\pi^2}{d^2}\right)\alpha_i G_i^+(X, (x_i, 0), k), & X \in \Omega^+ \setminus S((x_i, 0), r(a)); \\ v_0^i(x/a) + v_1^i(x/a)\ln^{-1} a + o(\ln^{-1} a), & X \in S((x_1, 0), 2r(a)); \\ -\left(k^2 - \dfrac{2\pi^2}{d^2}\right)\displaystyle\sum_{j=1}^{N}\alpha_j G^-(X, (x_j, 0), k), & X \in \Omega^- \setminus S((x_i, 0), r(a)), \end{cases} \tag{6}$$

where $S(X, r)$ is sphere with center X and radius r, radius $r(a)$ is chosen in such a way that

$$ad < r(a) < 2r(a) < \varepsilon/2.$$

The asymptotic expansion for the deviation of k^2 from $\dfrac{2\pi^2}{d^2}$ is chosen in accordance with the following formula ($a \to 0$):

$$k^2 - \frac{2\pi^2}{d^2} = \tau_1 \ln^{-1} a + \tau_2 \ln^{-2} a + o\left(\ln^{-2} a\right). \tag{7}$$

Here coefficients τ_1, τ_2 are to be determined.

3. Matching of asymptotic expansions

3.1. Asymptotics of solution for coupled resonators

The asymptotic behavior (when $X \to (x_i, 0)$) of the Green function for each small coupled resonator is as follows:

$$G_N^+((x,0),(x_i,0),k) = -\frac{1}{\pi}\ln a + \frac{4N^2\cos(N\pi x_i/d)\cos(N\pi x/d)}{d^2\left(k^2 - 2N^2\pi^2/d^2\right)} + g_1^+(X) - \frac{1}{\pi}\ln|\xi| = -\frac{1}{\pi}\ln a + \frac{4\cos(\pi x_i/\varepsilon)\cos(\pi x/\varepsilon)}{k^2\varepsilon^2 - 2\pi^2} + g_1^+(X) - \frac{1}{\pi}\ln|\xi|, \quad (8)$$

where $X = (x, y)$, $\xi = \dfrac{x}{a}$, g_1^+ is bounded function. Note that the second term is bounded for small ε. Therefore, we attach it to the function $g_1^+(X)$ and call it g^+:

$$G_N^+((x,0),(x_i,0),k) = -\frac{1}{\pi}\ln a + g^+(X) - \frac{1}{\pi}\ln|\xi|, \quad (9)$$

where g^+ is bounded function.

Taking into account (7), one comes to the following lemma

Lemma 1. *The first terms of the asymptotic expansion of the solution (6) in the upper i-th resonator has the form:*

$$\psi^+(x,0) = \alpha_i\left(k^2 - \frac{2\pi^2}{d^2}\right)\left[-\frac{1}{\pi}\ln a + g^+(X) - \frac{1}{\pi}\ln\xi\right] = -\frac{1}{\pi}\alpha_i\tau_1 + o(1), \quad (10)$$

where $x \to x_i$, $a \to 0$.

3.2. Asymptotics of solution for the main resonator

For the main cavity, if $N \to \infty$, then each aperture contracts and, simultaneously, the apertures are getting closer to each other (correspondingly, the number of coupled resonators increases). Hence, in addition to the asymptotics of the solution for $a \to 0$, one must also take into account the asymptotics of it for $\varepsilon = |x_i - x_{i-1}| \to 0$. The asymptotics of the Green function near the i-th hole for $X \to (x_i, 0)$ is analogous to that for the coupled resonators (9). The asymptotic behavior of the Green function near the hole (when $\varepsilon \to 0$) depends on the speed of approaching of the holes. Particularly, for the chosen relation (5) it has a conventional form:

$$G^-((x,0),(x_i,0),k) = -\frac{1}{\pi}\ln\varepsilon + \frac{4\cos(\pi x/d)\cos(\pi x_i/d)}{d^2\left(k^2 - \frac{2\pi^2}{d^2}\right)} + h^-(x - x_i) - \frac{1}{\pi}\ln|\xi|, \quad (11)$$

where h^- is bounded function depending on (5). For the main resonator, the asymptotic behavior of the solution (6) near the i-th aperture has the form:

$$\psi^-(x,0) = -\alpha_i\left[-\frac{\tau_1}{\pi} + \frac{4\cos^2(\pi x/d)}{d^2}\right] - \alpha_i\ln^{-1}a\left[-\frac{\tau_2}{\pi} + \tau_1\left(g^-(X) - \frac{1}{\pi}\ln|\xi|\right)\right] - \left(k^2 - \frac{2\pi^2}{d^2}\right)\sum_{j=1,j\neq i}^{N}\alpha_j\left[-\frac{\delta}{\pi}\ln a + \frac{4\cos(\pi x_j/d)\cos(\pi x_i/d)}{d^2(k^2 - 2\pi^2/d^2)} + h^-(x - x_j)\right] + o(1). \quad (12)$$

Keeping in mind that $x \to x_i$, one comes to the following lemma

Lemma 2. *The asymptotics of the solution to (6) in the main resonator has the following form:*

$$\psi^-(X) = -\alpha_i\left[-\frac{\tau_1}{\pi} + \frac{4}{d^2}\cos^2(\pi x_i/d)\right] - \sum_{j=1,j\neq i}^{N}\alpha_j\left[\frac{4}{d^2}\cos(\pi x_i/d)\cos(\pi x_j/d) - \frac{\delta\tau_1}{\pi} + \right] + o(1). \quad (13)$$

3.3. Matching

To ensure the consistency of the asymptotic expansions in the both regions (6), the coincidence of the zero-order terms is necessary. Correspondingly, the matching function $v_0^i(x/a)$ can be chosen as a constant. Equating the terms of order a^0, one obtains:

$$-\alpha_i\frac{\tau_1}{\pi} = -\alpha_i\left[-\frac{\tau_1}{\pi} + \frac{4}{d^2}\cos^2(\pi x_i/d)\right] - \sum_{j=1,j\neq i}^{N}\alpha_j\left[\frac{4}{d^2}\cos(\pi x_i/d)\cos(\pi x_j/d) - \frac{\delta\tau_1}{\pi}\right]. \quad (14)$$

One has such relation for each aperture. Hence, we obtain the system of linear equations for α_i:

$$\alpha_i\left[-\frac{2\tau_1}{\pi}+\frac{4}{d^2}\cos^2(\pi x_i/d)\right]-\sum_{j=1,j\neq i}^{N}\alpha_j\left[\frac{4}{d^2}\cos(\pi x_i/d)\cos(\pi x_j/d)-\frac{\delta\tau_1}{\pi}\right]=0,\quad i=1,2,...N. \tag{15}$$

Let us make designations for the convenience of calculating the determinant of the system. Then we get the following system of linear equations:

$$\alpha_i\left(\cos^2 x_i'-\beta\right)+\sum_{j=1,j\neq i}^{N}\alpha_j\left(\cos x_i'\cos x_j'-\delta'\beta\right)=0, \tag{16}$$

where $\beta=d^2\tau_1/2\pi$, $x_i'=\pi x_i/d$, $\delta'=-\delta/2$.

The system (16) has a nontrivial solution if its determinant is zero. Calculation of the determinant (see, e.g., [27,28]) gives one the following relation:

$$\det\begin{pmatrix}\cos^2 x_1-\beta & \cos x_1\cos x_2 & \cos x_1\cos x_3 & ... & \cos x_1\cos x_N\\ \cos x_2\cos x_1 & \cos^2 x_2-\beta & \cos x_2\cos x_3 & ... & \cos x_2\cos x_N\\ ... & & & & \\ \cos x_N\cos x_1 & \cos x_N\cos x_2 & \cos x_N\cos x_3 & ... & \cos^2 x_N-\beta\end{pmatrix}=$$

$$=\beta^{N-1}(1+\delta')^{N-2}\delta' N\left[-\frac{1}{2}N+(1+\delta')\beta+o(1)\right]=0, \tag{17}$$

whence we get

$$\beta=\frac{N}{2+\delta}+o(N),\quad N\to\infty. \tag{18}$$

Then we can get the first coefficient in the expansion (7) for k^2.

Lemma 3.

$$\tau_1=\frac{2\pi}{d^2(2-\delta)}N+o(N). \tag{19}$$

Let us express α_i from each equality (15) and substitute the expression to (6). We obtain an expression for ψ in the main cavity using (19) and (15):

$$\psi^-(x,0)=-\left(k^2-\frac{2\pi^2}{d^2}\right)\sum_{i=1}^{N}G^-((x,0),(x_i,0),k)\left[\frac{d^2\tau_1}{2\pi}-\cos^2\left(\frac{\pi x_i}{d}\right)\right]^{-1}\times$$
$$\sum_{j=1,j\neq i}^{N}\alpha_j\left(\cos(\pi x_i/d)\cos(\pi x_j/d)-\frac{d^2\delta\tau_1}{4\pi}\right)=$$
$$-\left(k^2-\frac{2\pi^2}{d^2}\right)\frac{2-\delta}{N}\sum_{i=1}^{N}G^-((x,0),(x_i,0),k)\left[\frac{\alpha_i}{2(2-\delta)}N-\frac{d^2}{4}\psi^-(x_i,0)+o(1)\right]=$$
$$\frac{1}{2}\psi^-(x,0)+\frac{d}{4}(2-\delta)\left(k^2-\frac{2\pi^2}{d^2}\right)\sum_{i=1}^{N}G^-((x,0),(x_i,0),k)\big(\psi^-(x_i)+o(1)\big)\cdot\frac{d}{N}. \tag{20}$$

Finally for $\psi^-(x)$ one obtains the following expression:

$$\psi^-(x)=\frac{d}{2}(2-\delta)\left(k^2-\frac{2\pi^2}{d^2}\right)\sum_{i=1}^{N}G^-((x,0),(x_i,0),k)\big(\psi^-(x_i)+o(1)\big)\cdot\frac{d}{N} \tag{21}$$

which presents the integral sum. Performing the limiting transition for $N\to\infty$, one obtains the integral equation for the eigenfunction ψ.

Lemma 4. *Eigenfunction for the limit problem satisfies the following integral equation:*

$$\psi^-(X)=\frac{d}{2}(2-\delta)\left(k^2-\frac{2\pi^2}{d^2}\right)\int_{\Gamma}G^-(X,X',k)\psi^-(X')dX'. \tag{22}$$

4. Conclusion

Taking into account (4), one can determine the boundary condition corresponding to the obtained integral eq. (22).

Theorem 4.1. *Eigenfunctions of the Laplacian corresponding to eigenvalue close to* $\frac{2\pi^2}{d^2}$ *for the system with many coupled resonators converges to the eigenfunctions of the Laplacian with the following boundary condition on the edge of the main square:*

$$\frac{\partial \psi}{\partial n}\Big|_{\partial\Omega} = \frac{d}{2}(2-\delta)\left(k^2 - \frac{2\pi^2}{d^2}\right)\psi\Big|_{\partial\Omega}. \tag{23}$$

The dependence of the boundary condition on the spectral parameter correlates with the results of [16]. We considered the eigenvalue close to one of the eigenvalues of the Neumann Laplacian in the square. Evidently, the same can be made for any eigenvalue. Square domain was taken for simplicity only. Really, we use only the fact that the part of the boundary with coupled resonators is straight. The result can be generalized to the case of any smooth boundary.

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