NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Muminov S., et al. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (6), 742–748.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2024-15-6-742-748

Original article

Qualitative properties of the mathematical model of nonlinear cross-diffusion processes

Sokhibjan Muminov[1,2,a], Praveen Agarwal[3,4,5,b], Dildora Muhamediyeva[6,c]

[1]Mamun university, Khiva, Uzbekistan

[2]Urgench state university, Urgench, Uzbekistan

[3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE

[4]Department of Mathematics, Anand International College of Engineering, Jaipur, India

[5]International Center for Basic and Applied Sciences, Jaipur, India

[6]Department of Software of Information Technologies, Tashkent University of information technologies named after Muhammad Al-Khwarizmi, Tashkent, Uzbekistan

[a]sokhibjan.muminov@mamunedu.uz, [b]praveen.agarwal@anandice.ac.in, [c]matematichka@inbox.ru

Corresponding author: S. Muminov, sokhibjan.muminov@mamunedu.uz

PACS 35A01, 35B44, 35K57, 35K65

ABSTRACT The work is devoted to developing a self-similar solution for a system of nonlinear differential equations that describe diffusion processes. Various techniques are used to examine the capacity for generating self-similar solutions that can estimate and predict system behavior under diffusion conditions. The focus is on developing and applying numerical algorithms, as well as using theoretical tools such as asymptotic analysis, to obtain more accurate and reliable results. The study's results can be applied to various scientific and technical fields, such as physics, chemistry, biology, and engineering, where diffusion processes play an essential role. The development of self-similar solutions for systems of nonlinear differential equations related to diffusion opens novel opportunities for modeling and analyzing complex systems and enhancing diffusion processes in various fields.

KEYWORDS nonlinear system, diffusion, self-similar solution, flow, model, algorithm, parabolic differential equation.

FOR CITATION Muminov S., Agarwal P., Muhamediyeva D. Qualitative properties of the mathematical model of nonlinear cross-diffusion processes. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (6), 742–748.

1. Introduction

The scientific community is actively studying self-similar solutions of nonlinear differential equations related to mutual diffusion. Interdiffusion refers to the movement of substances within a medium where components interact through diffusion flow. Analytical solutions to such systems are complicated due to nonlinearities. Therefore, self-similar methods are an effective tool for constructing approximate solutions and analysis of such systems.

Various methods for simulating systems of nonlinear differential equations that describe mutual diffusion are explored. The article is focused on the problem of finding weak solutions and analyzing the asymptotic properties of these equations. Particular attention is given to the study of regular, unbounded, and finite solutions to gain a more complete understanding of the system's behavior. Such models can be applied across various scientific and technical fields where mutual diffusion is essential. Understanding and modeling interdiffusion are of practical importance in biology, ecology, chemistry, and physics. The complexity of the analytical solution of systems of nonlinear differential equations describing mutual diffusion requires the development of effective numerical methods. The self-similar methods are considered as a practical approach for obtaining approximate solutions with complex interactions between components and diffusion conditions. This method makes it possible to apply numerical methods to solve the current type of nonlinear differential equations and provides the ability to simulate diffusion processes. Applying self-similar solutions in the prediction and analysis of mutual diffusion systems is extensive. They can be used to optimize processes, manage resources, prevent the spread of harmful substances and diseases, and understand fundamental principles.

Exploring the category of nonlinear differential equations and systems, particularly those that include the desired function and its derivative in power form, is intriguing in the study of real physical processes. These nonlinearities are commonly observed in reaction-diffusion, interdiffusion, and biological population problems [1–3].

Finding an analytical solution to nonlinear boundary value problems is challenging. Determining the new properties of the solution requires a significant energy and time. Their solving faces with several difficulties. The works of A.A.

Samarsky, V.A. Galaktionov, A.S. Kalashnikov, L.K. Martinson, R. Kershner, G.I. Barenblatt, B.F. Knerr, Chen Xingfu, Yu.W. Qi, J.S. Guo, I. Kombe, T. Kusano, T. Tanigawa, S.N. Dimov, M.M. Aripov, and A.T. Khaidarov are devoted to the study of the properties of solving the above problems. Sh.A. Sadullaeva et al. have shown the significance of self-similar solutions corresponding to specific parameter values [4–18], The critical curves of a degenerate parabolic equation were studied in the work of M.Aripov and J.Raimbekov [19].

Cross-diffusion processes are significant in many areas of nanoscience, and a deeper study of these processes contributes to the development of advanced technologies. For example, in nanoelectronics, the interdiffusion of atoms and particles in semiconductors significantly influences electronic devices' performance characteristics and reliability. Diffusion processes in the field of nanomaterials play a decisive role in the formation of nanostructures of materials and the optimization of their mechanical, optical, and electrical properties. In nano-optics and plasmonics, the propagation of light and collective oscillations of electrons in nanostructured materials depends on the diffusion processes, which expands the possibilities of light control and redirection. In nanomedicine and biomaterials, cross-diffusion in drug delivery systems ensures efficient distribution of drugs within cells and tissues, increasing treatment effectiveness. In nanocatalysis, the interdiffusion of reactants on the catalyst surface is a critical factor in controlling the rate and selectivity of chemical reactions. Also, diffusion processes in nanocomputer technologies affect the ability of nano-sized memory elements to store and process information. Thus, the study of mutual diffusion processes contributes to a deeper understanding of the physicochemical behavior of substances at the nanoscale and is also of fundamental importance in developing new materials and devices.

In article [20], the formation processes of Liesegang structures are studied using the Keller-Rubinow model. Liesegang structures are periodic layered structures formed by chemical reactions from diffusion and precipitation. The article examines a mathematical analysis of these processes using the Keller-Rubinow model, and the modeling results clearly show the main features of the formation of Liesegang structures. The research results are essential in creating nanoscale structures and understanding their formation mechanisms.

The article [21] examines the characteristics of the oriented ring of neurons based on the FitzHugh-Nagumo model. The FitzHugh-Nagumo model is widely used for mathematical modeling of neuronal activity dynamics and represents the transmission processes of nerve impulses. This study analyzes an oriented neuron loop's steady states and dynamic behavior. The article shows the interactions between neurons in the ring, the conditions of signal propagation, and the effect of these processes on the functional activity of neural networks. The research findings are important for understanding neurobiological systems and their modeling in artificial intelligence systems.

These articles [20, 21] cover the mathematical and physical approaches needed for modeling and analysis in nanotechnology and neurobiology and act as a bridge between theoretical research and experimental practice. This work is devoted to the study of constructing a self-similar solution to a system of nonlinear differential equations representing mutual diffusion problems.

2. Methods and models

We consider the following problem in a specified spatial region $\Omega = \{(x,t) : x \in \mathbb{R},\ t \in (0;T)\}$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(u^{\sigma_1}\dfrac{\partial u}{\partial x}\right) - v^{\beta_1}\left|\dfrac{\partial u}{\partial x}\right|^{P_1}, \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\left(v^{\sigma_2}\dfrac{\partial v}{\partial x}\right) - u^{\beta_2}\left|\dfrac{\partial v}{\partial x}\right|^{P_2}, \end{cases} \tag{1}$$

with initial conditions

$$u\,(x,0) = u_0\,(x)\,,\ v\,(x,0) = v_0\,(x)\,,\ x \in \mathbb{R}, \tag{2}$$

and boundary conditions

$$u(0,t) = u_1(t),\ v(0,t) = v_1(t),\ \ 0 \leq t \leq T, \tag{3a}$$

$$u(1,t) = u_2(t),\ v(1,t) = v_2(t),\ \ 0 \leqslant t \leqslant T, \tag{3b}$$

where $\sigma_1,\ \sigma_2,\ \beta_1,\ \beta_2,\ p_1,\ p_2$ are real numbers which specified surrounding and front parameters.

The equations of the system characterize the migration of salt or dust taking into account humidity and change in humidity taking into account the migration of salt or dust, respectively.

We represent the solution of the system (1) in the following form

$$u\,(t,x) = (T+t)^{n_1}\,(t)\,w_1\,(\tau,x)\,, \tag{4}$$

$$v\,(t,x) = (T+t)^{n_2}\,(t)\,w_2\,(\tau,x)\,, \tag{5}$$

where $\tau(t)$ is function of time. Using equations (4),(5) we find n_1 and n_2:

$$n_1 = \frac{(p_1-2)(2p_2-p_2\sigma_2-2)-2\beta_1(p_2-2)}{(2p_1-p_1\sigma_1-2)(2p_2-p_2\sigma_2-2)-4\beta_1\beta_2}, \quad n_2 = \frac{(p_2-2)(2p_1-p_1\sigma_1-2)-2\beta_2(p_1-2)}{(2p_1-p_1\sigma_1-2)(2p_2-p_2\sigma_2-2)-4\beta_1\beta_2}. \tag{6}$$

We will assume that the functions $w_1(x,\tau)$ and $w_2(x,\tau)$ are representable in the form $w_1(x,\tau)=f_1(\xi)$, $w_2(x,\tau)=f_2(\xi)$, where $\xi = \dfrac{x}{\sqrt{\tau}}$.

Now system (1) can be represented as

$$\begin{cases} -\dfrac{\xi}{2}\dfrac{df_1}{d\xi} = \dfrac{d}{d\xi}({f_1}^{\sigma_1}\dfrac{df_1}{d\xi}) - (n_1\sigma_1+1)^{\frac{p_1-2}{2}}\cdot {f_2}^{\beta_1}\left|\dfrac{df_1}{d\xi}\right|^{p_1} - \dfrac{n_1}{n_1\sigma_1+1}\cdot f_1, \\ -\dfrac{\xi}{2}\dfrac{df_2}{d\xi} = \dfrac{d}{d\xi}({f_2}^{\sigma_2}\dfrac{df_2}{d\xi}) - (n_2\sigma_2+1)^{\frac{p_2-2}{2}}\cdot {f_1}^{\beta_2}\left|\dfrac{df_2}{d\xi}\right|^{p_2} - \dfrac{n_2}{n_2\sigma_2+1}\cdot f_2. \end{cases} \tag{7}$$

Where the functions f_1 and f_2 are chosen in the form

$$f_1 = (a+\xi)^{\gamma_1},\ f_2 = (a+\xi)^{\gamma_2}. \tag{8}$$

Then we find that the parameters of system (7) must satisfy the following conditions

$$\begin{cases} \gamma_1(\sigma_1\gamma_1+\gamma_1-1) = (n_1\sigma_1+1)^{\frac{p_1-2}{2}}|\gamma_1|^{p_1}, \\ \gamma_2(\sigma_2\gamma_2+\gamma_2-1) = (n_2\sigma_2+1)^{\frac{p_2-2}{2}}|\gamma_1|^{p_1}, \end{cases} \tag{9}$$

$$(\sigma_i\gamma_i+\gamma_i-1) > 0, \quad i=\overline{1,2}$$

Now, let us calculate the values γ_1 and γ_2. We obtain that

$$\gamma_1 = \frac{(\delta_2-p_2+1)(2-p_1)+\beta_1(2-p_2)}{(\delta_1-p_1+1)(\delta_2-p_2+1)-\beta_1\beta_2}; \quad \gamma_2 = \frac{(\delta_1-p_1+1)(2-p_2)+\beta_2(2-p_1)}{(\delta_1-p_1+1)(\delta_2-p_2+1)-\beta_1\beta_2}.$$

The following theorem holds for the upper bound of the solutions obtained.

Theorem. Let

1) $\sigma_1 \geq 0; \quad \sigma_2 \geq 0;$

2) $a^{\frac{\beta_1}{\sigma_2}+\frac{(1-\sigma_1)p_1-1}{\sigma_1}+\frac{p_1}{2}} \geq \dfrac{(\frac{n_1}{n_1\sigma_1+1}+\frac{1}{2})}{(n_1\sigma_1+1)^{\frac{p_1-1}{2}}}$ and $a^{\frac{\beta_2}{\sigma_1}+\frac{(1-\sigma_2)p_2-1}{\sigma_2}+\frac{p_2}{2}} \geq \dfrac{(\frac{n_2}{n_2\sigma_2+1}+\frac{1}{2})}{(n_2\sigma_2+1)^{\frac{p_2-1}{2}}}$;

3) $u(t,0) \leq u_+(t,0)$, $v(t,0) \leq v_+(t,0)$, $x \in \mathbb{R}$.

Then there is a global solution to problem (1)-(3) and the following conditions are valid:

$$u(x,t) \leq u_+(x,t) = (T+t)^{n_1} f_1(\xi),\ v(x,t) \leq v_+(x,t) = (T+t)^{n_2} f_2(\xi). \tag{10}$$

Proof: To prove the theorem, we use the comparison [1]. We choose the following function as a comparison function

$$u(x,t) = (T+t)^{n_1}\cdot f_1(\xi),\ v(x,t) = (T+t)^{n_2}\cdot f_2(\xi). \tag{11}$$

Substituting (11) into system (1), we obtain the following system

$$\begin{cases} \dfrac{\xi}{2}\dfrac{df_1}{d\xi} + \dfrac{d}{d\xi}({f_1}^{\sigma_1}\dfrac{df_1}{d\xi}) - (n_1\sigma_1+1)^{\frac{p_1-1}{2}}\cdot {f_2}^{\beta_1}\left|\dfrac{df_1}{d\xi}\right|^{p_1} - \dfrac{n_1}{n_1\sigma_1+1}\cdot f_1 \leqslant 0, \\ \dfrac{\xi}{2}\dfrac{df_2}{d\xi} + \dfrac{d}{d\xi}({f_2}^{\sigma_2}\dfrac{df_2}{d\xi}) - (n_2\sigma_2+1)^{\frac{p_2-1}{2}}\cdot {f_1}^{\beta_2}\left|\dfrac{df_2}{d\xi}\right|^{p_2} - \dfrac{n_2}{n_2\sigma_2+1}\cdot f_2 \leqslant 0, \end{cases} \tag{12}$$

$$\begin{cases} \dfrac{d}{d\xi}({f_1}^{\sigma_1}\dfrac{df_1}{d\xi}) + \dfrac{\xi}{2}\dfrac{df_1}{d\xi} + \dfrac{f_1}{2} - \dfrac{f_1}{2} - (n_1\sigma_1+1)^{\frac{p_1-1}{2}}\cdot {f_2}^{\beta_1}\left|\dfrac{df_1}{d\xi}\right|^{p_1} - \dfrac{n_1}{n_1\sigma_1+1}\cdot f_1 \leqslant 0, \\ \dfrac{d}{d\xi}({f_2}^{\sigma_2}\dfrac{df_2}{d\xi}) + \dfrac{\xi}{2}\dfrac{df_2}{d\xi} + \dfrac{f_2}{2} - \dfrac{f_2}{2} - (n_2\sigma_2+1)^{\frac{p_2-1}{2}}\cdot {f_1}^{\beta_2}\left|\dfrac{df_2}{d\xi}\right|^{p_2} - \dfrac{n_2}{n_2\sigma_2+1}\cdot f_2 \leqslant 0, \end{cases}$$

$$\frac{d}{d\xi}({f_1}^{\sigma_1}\frac{df_1}{d\xi}) = -\frac{d(\xi f_1)}{2d\xi}, \quad \frac{d}{d\xi}({f_2}^{\sigma_2}\frac{df_2}{d\xi}) = -\frac{d(\xi f_2)}{2d\xi}. \tag{13}$$

If equality (12) holds, then relation (13) holds.

If the condition $\sigma_1 \geq 0; \quad \sigma_2 \geq 0;$ in the theorem holds, then, we will have the following system

$$\begin{cases} -(n_1\sigma_1+1)^{\frac{p_1-1}{2}}\cdot {f_2}^{\beta_1}\left|\dfrac{df_1}{d\xi}\right|^{p_1} - (\dfrac{n_1}{n_1\sigma_1+1}+\dfrac{1}{2})f_1 \leq 0, \\ -(n_2\sigma_2+1)^{\frac{p_2-1}{2}}\cdot {f_1}^{\beta_2}\left|\dfrac{df_2}{d\xi}\right|^{p_2} - (\dfrac{n_2}{n_2\sigma_2+1}+\dfrac{1}{2})f_2 \leq 0, \end{cases}$$

$$\begin{cases} -(n_1\sigma_1+1)^{\frac{p_1-1}{2}}\cdot(a-\frac{\sigma_2}{4}\xi^2)^{\frac{\beta_1}{\sigma_2}}\left|(a-\frac{\sigma_1}{4}\xi^2)^{\frac{1-\sigma_1}{\sigma_1}}\frac{\xi}{2}\right|^{p_1}-(a-\frac{\sigma_1}{4}\xi^2)^{\frac{1}{\sigma_1}}(\frac{n_1}{n_1\sigma_1+1}+\frac{1}{2})\leqslant 0,\\ -(n_2\sigma_2+1)^{\frac{p_2-1}{2}}\cdot(a-\frac{\sigma_1}{4}\xi^2)^{\frac{\beta_2}{\sigma_1}}\left|(a-\frac{\sigma_1}{4}\xi^2)^{\frac{1-\sigma_2}{\sigma_2}}\frac{\xi}{2}\right|^{p_2}-(a-\frac{\sigma_2}{4}\xi^2)^{\frac{1}{\sigma_2}}(\frac{n_2}{n_2\sigma_2+1}+\frac{1}{2})\leqslant 0.\end{cases}$$

Here $\xi=2\sqrt{\dfrac{a}{\sigma_1}}$ evaluate and expand the first expression

$$-(n_1\sigma_1+1)^{\frac{p_1-1}{2}}a^{\frac{\beta_1}{\sigma_2}}a^{\frac{(1-\sigma_1)p_1}{\sigma_1}}\left(\sqrt{\frac{a}{\sigma_1}}\right)^{p_1}-a^{\frac{1}{\sigma_1}}(\frac{n_1}{n_1\sigma_1+1}+\frac{1}{2})\leqslant 0,$$

$$-(n_2\sigma_2+1)^{\frac{p_2-1}{2}}a^{\frac{\beta_2}{\sigma_1}}a^{\frac{(1-\sigma_2)p_2}{\sigma_2}}\left(\sqrt{\frac{a}{\sigma_2}}\right)^{p_2}-a^{\frac{1}{\sigma_2}}(\frac{n_2}{n_2\sigma_2+1}+\frac{1}{2})\leqslant 0,$$

$$\frac{-(n_1\sigma_1+1)^{\frac{p_1-1}{2}}a^{\frac{\beta_1}{\sigma_2}+\frac{(1-\sigma_1)p_1-1}{\sigma_1}}a^{\frac{p_1}{2}}}{\sigma_1^{\frac{p_1}{2}}}\leqslant(\frac{n_1}{n_1\sigma_1+1}+\frac{1}{2}),$$

$$\frac{-(n_2\sigma_2+1)^{\frac{p_2-1}{2}}a^{\frac{\beta_2}{\sigma_1}+\frac{(1-\sigma_2)p_2-1}{\sigma_2}}a^{\frac{p_2}{2}}}{\sigma_1^{\frac{p_1}{2}}}\leqslant(\frac{n_2}{n_2\sigma_2+1}+\frac{1}{2}),$$

$$a^{\frac{\beta_1}{\sigma_2}+\frac{(1-\sigma_1)p_1-1}{\sigma_1}+\frac{p_1}{2}}\geqslant\frac{(\frac{n_1}{n_1\sigma_1+1}+\frac{1}{2})}{(n_1\sigma_1+1)^{\frac{p_1-1}{2}}}.$$

As for the second inequality of system (12), if the corresponding operations are carried out

$$a^{\frac{\beta_2}{\sigma_1}+\frac{(1-\sigma_2)p_2-1}{\sigma_2}+\frac{p_2}{2}}\geqslant\frac{(\frac{n_2}{n_2\sigma_2+1}+\frac{1}{2})}{(n_2\sigma_2+1)^{\frac{p_2-1}{2}}},$$

it follows that condition 2 in the theorem is valid.

Therefore, according to the hypotheses of the theorem and the principle of comparison, the following relations are valid: $u(t,x)\le u_+(t,x)$; $v(t,x)\le v_+(t,x)$. **The theorem is proven.**

Thus, it was found that the self-similar solution for system (1) can be represented in the form:

$$\begin{aligned} u(x,t)&=(T+t)^{n_1}\cdot f_1(\xi)=(T+t)^{n_1}\cdot(a-\xi)^{\gamma_1},\\ v(x,t)&=(T+t)^{n_2}\cdot f_2(\xi)=(T+t)^{n_2}\cdot(a-\xi)^{\gamma_2}.\end{aligned}\tag{14}$$

3. Calculation results

Using self-similar solutions (14) of system (1) - (3), the iteration or sweep method, numerical solutions were found, the graphs of which are given below. (The graphs presented have a horizontal axis representing the values of the variable x and a vertical axis intended to represent the values of the functions u and v.)

Graphs of solutions to the mutual diffusion problem: The results of studies of the processes of mutual diffusion for parameters $\sigma_1,\ \sigma_2,\ \beta_1,\ \beta_2,\ p_1,\ p_2,\ a$ are described and graphs are presented for analysing changes in moisture (u) and the content of salt and dust particles (v) under different conditions.

In Fig. 1, the results for the following values of the parameters are shown: $\sigma_1=4,\ \sigma_2=5,\ \beta_1=2.5,\ \beta_2=2.1,\ p_1=2.25,\ p_2=3.8,\ a=0.5$

In Fig. 2, the parameters indicated are as follows: $\sigma_1=4,\ \sigma_2=5,\ \beta_1=2.2,\ \beta_2=2.1,\ p_1=2.87,\ p_2=3.76,\ a=3.5$

The graphs of v-salt-dust migration for cases in steps are given in different colors.

Under the influence of certain parameters, the movement of moisture and salt (or dust) occurs “almost evenly” over a “short time interval”. This suggests that the diffusion under consideration is likely rapid. Rapid diffusion is characterized by concentration gradients becoming uniform over short intervals, which corresponds to the description of significant acceleration of diffusion processes and minimal concentration variations over time.

In Fig. 3, the diffusion process for the followimg parameter values: $\sigma_1=4,\ \sigma_2=5,\ \beta_1=1.7,\ \beta_2=2.1,\ p_1=3.1,\ p_2=3.2,\ a=0.5$, is shown, where changes in humidity occur almost identically to the migration of salt-dust particles.

In Fig. 3, the changes in moisture and salt-dust content are shown depending on environmental parameters. It can be seen that the initial phases of the diffusion processes differ, but later the changes become almost uniform.

In Fig. 4, the diffusion process for the parameter values is presented: $\sigma_1=4,\ \sigma_2=7,\ \beta_1=2.91,\ \beta_2=2.22,\ p_1=3.8,\ p_2=2.5,\ a=2.5$.

In Figs. 3 and 4, the results for various values of $\sigma_2,\ \beta_2$ are shown. It is evident that with an increase in σ_2 , the process accelerates, and the concentration gradients equalize more quickly.

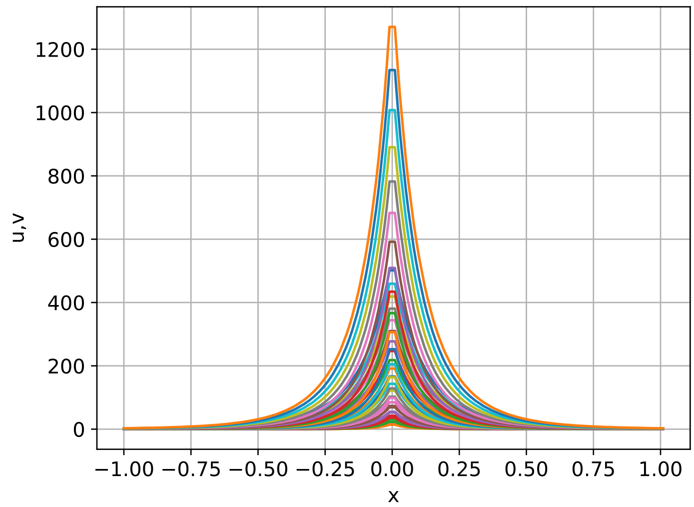

FIG. 1. Solutions of the system of equations representing the mutual diffusion process

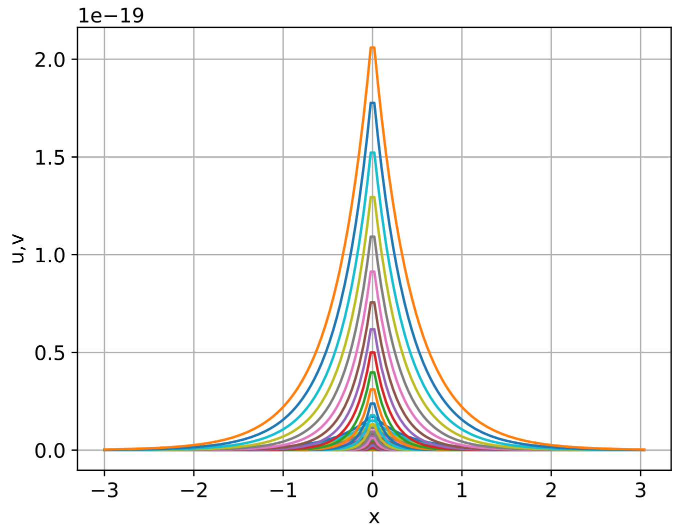

FIG. 2. Graphical representation of moisture and salt-dust diffusion process

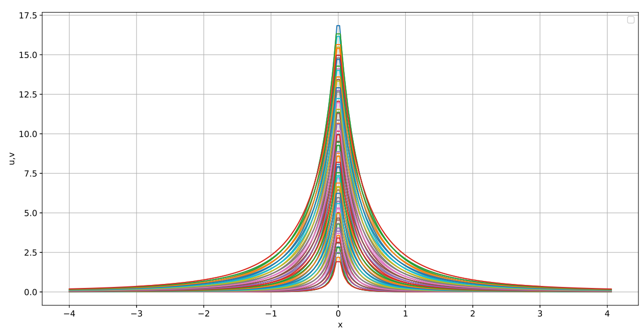

FIG. 3. Solutions of the diffusion problem corresponding to the given values of the parameters

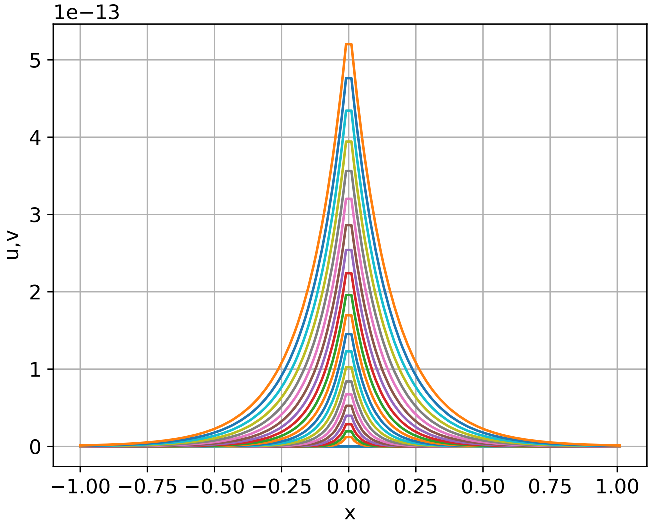

FIG. 4. Solutions of the diffusion problem corresponding to the given values of the parameters

Changes in parameters lead to an acceleration or deceleration of the mutual diffusion processes. At certain values, the concentration of moisture and salt-dust particles equalizes more quickly, reducing fluctuations. Changes in moisture (u) and salt-dust content (v) indicate that, depending on environmental parameter values, in some cases, changes in moisture occur almost identically to the migration of salt-dust particles. In system (1), representing cross-diffusion processes, under conditions of rapid diffusion, the movement of moisture and salt (or dust) changes almost uniformly over a short time interval. With the proper selection of parameter values, these changes asymptotically approach an infinitely small value. This means that due to significant acceleration of diffusion processes, the concentration gradients of moisture and salt become almost uniform throughout the system, leading to minimal concentration changes over time.

The figures show that, depending on the medium and front parameter values, the diffusion process begins differently and then changes almost uniformly.

u-moisture change, V-salt-dust content change, in which depending on the change of parameter values, in some cases it can be seen that the moisture changes almost identically with the salt-dust migration.

In the system (1) representing cross-diffusion processes, under conditions of rapid diffusion, the movement of moisture and salt (or dust) changes almost uniformly over a short time interval. As the parameter values are chosen appropriately, these changes asymptotically approach an infinitesimally small amount. This implies that due to the significant acceleration of the diffusion processes, the concentration gradients of moisture and salt become nearly uniform across the entire system, leading to minimal variations in concentration over time.

4. Conclusion

The method for developing a self-similar solution for a system of nonlinear differential equations that explain mutual diffusion processes in a one-dimensional spatial approximation is investigated. The results have clear practical implications when considering the Cauchy problem for equations with variable coefficients. An approach to approximation of the solution of a second-order nonlinear problem is described and supported. Based on the impacts of the finiteness of the disturbances' propagation speed and spatial localization, asymptotes of regular, finite, and unbounded solutions are derived. The observed results allow us to build an iterative procedure for the numerical solution of the mutual diffusion issue to acquire an initial approximation. These mathematical models can be used to solve many practical problems, including biological population issues, epidemic spread, and numerical modeling of various diffusion processes.

References

[1] Samarsky A.A., Mikhailov A.P. *Mathematical Modeling*, Fizmatlib, Moscow, 2001, 320 p.

[2] Aripov M.M. *Methods of Reference Equations for Solving Nonlinear Boundary Value Problems.*, Fan, Tashkent, 1988, 136 p.

[3] Aripov M.M., Sadullaeva Sh.A. *Computer Modeling of Nonlinear Diffusion Processes.* University, Tashkent, 2020, 656 p.

[4] Aripov M.M., Matyakubov A.S., Imomnazarov B.Kh. The cauchy problem for a nonlinear degenerate parabolic system in non-divergence form. *Mathematical Notes of NEFU*, 2020, **27**(3), P. 27–38.

[5] Matyakubov A.S., Raupov D. On some properties of the blow-up solutions of a nonlinear parabolic system non-divergent form with cross-diffusion. *Lec. Notes in Civil Engineering*, 2022, **180**, P. 289–301.

[6] Aripov M.M., Matyakubov A.S., Xasanov J.O. To the qualitative properties of self-similar solutions of a cross-diffusion parabolic system not in divergence form with a source. *AIP Conf. Proc.*, 2023, **2781**.

[7] Aripov M.M., Matyakubov A.S., Khasanov J.O., Bobokandov M.M. Mathematical modeling of double nonlinear problem of reaction diffusion in not divergent form with a source and variable density. *J. Phys. : Conf. Ser.*, 2021, **2131**(3).

[8] Muhamediyeva D.K. Properties of self similar solutions of reaction-diffusion systems of quasilinear equations. *Int. J. of Mech. and Prod. Eng. Research and Development*, 2018, **36**(8), P. 555–566.

[9] Muhamediyeva D.K. Qualitative properties of wave solutions of the equation of reaction-diffusion of a biological population. Conference proceedings of "2020 International Conference on Information Science and Communications Technologies (ICISCT)", Tashkent, Uzbekistan, 04-06 November 2020.

[10] Muhamediyeva D.K., Nurumova A.Y, Muminov S.Y. Fuzzy evaluation of cotton varieties in the natural climatic. In IOP Conf. Ser.: Earth and Environmental Science, 2022, **1076**, P. 012043.

[11] Muhamediyeva D.K., Madrakhimov A.Kh., Kodirov Z.Z. Construction of a system of differential equations taking into account convective transfer. Proceedings of "Computer applications for management and sustainable development of production and industry (CMSD2022)". Dushanbe, Tajikistan, 21-23 December 2022.

[12] Muhamediyeva D.K., Muminov S.Y., Shaazizova M.E., Hidirova Ch., and Bahromova Yu. Limited different schemes for mutual diffusion problems. *E3S Web of Conf*, 2023, **401**, P. 05057.

[13] Muhamediyeva D.K., Nurumova A.Y, Muminov S.Y. Numerical modeling of cross-diffusion processes. *E3S Web of Conf*, 2023, **401**, P. 05060.

[14] Muhamediyeva D.K., Nurumova A.Y., Muminov S.Y. Cauchy problem and boundary-value problems for multicomponent cross-diffusion systems. Proceedings of "Int. Conf. on Inf. Sci. and Comm. Tech.". Tashkent 2021. 01-05.

[15] Muminov S.Y. Construction of self-similar solutions of the system of nonlinear differential equations of cross-diffusion. Proceedings of "The int. Sci. and Prac. Conf.", China 2023. 57–60.

[16] Rakhmonov Z.R., Alimov A.A. Properties of solutions for a nonlinear diffusion problem with a gradient nonlinearity. *Int. J. App. Math.*, 2023, **36**(3), P. 1–20.

[17] Yang Liu, Yanwei Du, Hong Li, Jichun Li, and Siriguleng He. A two-grid mixed finite element method for a nonlinear fourth-order reaction-diffusion problem with time-fractional derivative. *Comp. and Math. with Appl.*, 2015, **70**(10), P. 2474–2492.

[18] Farina A., Gianni R. Self-similar solutions for the heat equation with a positive non-Lipschitz continuous, semilinear source term. *Nonl. Anal.: Real World Appl.*, 2024, **79**.

[19] Aripov M.M., Raimbekov J.R. The critical curves of a doubly nonlinear parabolic equation in non-divergent form with a source and nonlinear boundary flux. *J. Sib. Fed. Univ. - Math. and Phys.*, 2019, **12**(1), P. 112–124.

[20] Topaev T.N., Popov A.I., Popov I.Yu. On Keller-Rubinow model for Liesegang structure formation. *Nanosystems: physics, chemistry, mathematics*, 2022, **13**(4), P. 365–371.

[21] Fedorov E.G. Properties of an oriented ring of neurons with the Fitzhugh-Nagumo model. *Nanosystems: physics, chemistry, mathematics*, 2021, **12**(5), P. 553–562.

Submitted 7 August 2024; revised 28 October 2024, 12 November 2024; accepted 13 November 2024

Information about the authors:

Sokhibjan Muminov – Mamun university, Bolkhovuz street 2, Khiva 220900, Uzbekistan; Urgench state university, Khamid Alimjan street 14, Urgench 220100, Uzbekistan; ORCID 0000-0003-2471-4836; sokhibjan.muminov@mamunedu.uz

Praveen Agarwal – Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE; Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India; International Center for Basic and Applied Sciences, Jaipur-302029, India; ORCID 0000-0001-7556-8942; praveen.agarwal@anandice.ac.in

Dildora Muhamediyeva – Department of Software of Information Technologies, Tashkent University of information technologies named after Muhammad Al-Khwarizmi, Amir Temur Avenue 108, Tashkent 100084, Uzbekistan; ORCID 0000-0003-2154-4157; matematichka@inbox.ru

Conflict of interest: the authors declare no conflict of interest.