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Original article

Some conditions for the existence of 4-periodic solutions in non-homogeneous differential equations involving piecewise alternately advanced and retarded arguments

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ABSTRACT The manuscript introduces a method to characterize 4-periodic solutions in first-order non- homogeneous differential equations involving piecewise alternately advanced and retarded argument. It systematically delineates the prerequisites for these solutions to exist and furnishes precise methodologies for their determination. Additionally, the paper includes the illustrative example, including scenarios with infinitely many solutions, to demonstrate the effectiveness of the proposed approach.

KEYWORDS Piecewise alternately advanced and retarded argument, Periodic solution.

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1. Introduction

A differential equation with piecewise alternately advanced and retarded argument (DEPCA) is represented by the following equation:

$$x'(t) = f\left(t, x(t), x\left(2\left[\frac{t+1}{2}\right]\right)\right), \tag{1.1}$$

where $[\cdot]$ denotes the greatest integer function, and f is continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. DEPCAs are hybrid equations that amalgamate features of discrete equations and continuous systems. They bear significant relevance in applications pertaining to biomedical dynamics and physical phenomena (refer to [1,2] and the associated references [3–7]).

Recent publications [8–17] have investigated particular formulations of the DEPCA. Authors in [18–22] have streamlined the problem of n-periodic solvability to a set of n linear equations. Leveraging established properties of linear systems in algebra, they systematically delineated all conditions necessary for the existence of n-periodic solutions and furnished explicit formulas for solving these equations.

In 2024, M.I. Muminov and T.A. Radjabov [22] examined the existence conditions for 2-periodic solutions of first order differential equations with piecewise constant delay:

$$T'(t) = a(t)T(t) + b(t)T([t]) + f(t).$$

The authors developed a method for identifying 2-periodic solutions, thoroughly outlined the existence conditions, and presented explicit formulas for these solutions.

To the best of our knowledge, only one study has addressed the existence of infinitely many periodic solutions related to the DEPCA [22]. However, none of the available works have provided clear criteria for determining the existence of such solutions in differential equations with piecewise alternately advanced and retarded arguments.

In this paper, we examine a non-homogeneous differential equation involving piecewise alternately advanced and retarded arguments, given by

$$y'(t) = a(t)y(t) + b(t)y\left(2\left[\frac{t+1}{2}\right]\right) + g(t), \quad t \geq 1, \tag{1.2}$$

where the functions $a(t)$, $b(t)$, and $g(t)$ are continuous and nonzero on $[1, \infty)$. The general case of this problem was previously examined in [23, 24], where the authors established conditions for solution existence and demonstrated a Gronwall's type integral inequality as an application. This note is dedicated to elucidating the conditions necessary for the existence of 4-periodic solutions to this initial value problem. We present an example that illustrates an equation with an infinite number of 4-periodic solutions, thus offering additional insights to complement the uniqueness assertions made in prior studies focusing on homogeneous cases.

2. Alternately advanced and retarded differential equation

Let us define the solution for DEPCA (1.2). A function y is considered a solution of DEPCA (1.2) on $[1,\infty)$ if the following conditions are met: i) y is continuous on $[1,\infty)$. ii) The derivative $y'(t)$ exists at each point $t \in [1,\infty)$, except possibly at points $t = 2n-1$ for $n \in \mathbb{N}$, where one-sided derivatives exist. iii) The DEPCA (1.2) is satisfied by y on each interval $(2n-1, 2n+1)$ for $n \in \mathbb{N}$, and it holds for the right-hand derivative at the points $2n-1$ for $n \in \mathbb{N}$.

To determine the solution of DEPCA (1.2), following the approach outlined in [25], we integrate DEPCA (1.2) and obtain the following result:

$$y(t) = e^{\int_2^t a(s)ds} y(2) + \int_2^t b(s) y\left(2\left[\frac{s+1}{2}\right]\right) e^{\int_s^t a(r)dr} ds + \int_2^t g(s) e^{\int_s^t a(r)dr} ds, \quad t \in [1,3).$$

We define

$$\lambda(t,s) := e^{\int_s^t a(n)dn} + \int_s^t e^{\int_u^t a(n)dn} b(u)du,$$

$$\Psi(t,s) = \lambda\left(t, 2\left[\frac{t+1}{2}\right]\right) \prod_{j=\left[\frac{s+1}{2}\right]+1}^{\left[\frac{t+1}{2}\right]} \frac{\lambda(2j-1, 2j-2)}{\lambda(2j-1, 2j)}, \quad t \geq s,$$

$$G(t,s) = \int_s^t e^{\int_u^t a(\kappa)d\kappa} g(u)du,$$

where $t, s \in [1,\infty)$, $\lambda(2j-1, 2j) \neq 0$, $j \in \mathbb{N}$.

Theorem 2.1. *If $\lambda(2j-1,2j) \neq 0$ for $j \in \mathbb{N}$, then $y(t)$ represents the unique solution to DEPCA (1.2) for $t \geq 1$ if and only if $y(t)$ is expressed as*

$$\begin{aligned} y(t) = {} & \Psi(t,1)\, y(1) + \sum_{k=1}^{[(t+1)/2]} \int_{2k-1}^{2k} \Psi(t, 2k-1)\, G(2k-1, s) ds \\ & + \sum_{k=1}^{[(t+1)/2]-1} \int_{2k}^{2k+1} \Psi(t, 2k+1)\, G(2k+1, s) ds \\ & + G(t, 2[(t+1)/2]). \end{aligned} \tag{2.1}$$

The demonstration of the theorem closely resembles the proof provided for Theorem 1 in [25] and Theorem 2.1 in [23].

3. 4-periodic solutions

In this section, we present a methodology for identifying 4-periodic solutions of DEPCA (1.2) in scenarios where $a(\cdot)$, $b(\cdot)$, and $g(\cdot)$ are continuous functions defined on the interval $[1,\infty)$ and demonstrate a pattern of 4-periodic behavior.

Integrating DEPCA (1.2), we obtain:

$$y(t) = \lambda(t, 2n) y(2n) + G(t, 2n), \quad 2n-1 \leq t < 2n+1,$$

where $G(t, 2n) = \int_{2n}^t g(s) e^{\int_s^t a(r)dr} ds$ and $n \in \mathbb{N}$. Let $y(t)$ be 4-periodic on $[1,\infty)$. The function $y(t)$ on $[1,5)$ can be represented as:

$$y(t) = \begin{cases} \dfrac{\lambda(t,2)}{\lambda(1,2)} (y(1) - G(1,2)) + G(t,2), & t \in [1,3), \\ \dfrac{\lambda(t,4)}{\lambda(3,4)} (y(3) - G(3,4)) + G(t,4), & t \in [3,5). \end{cases} \tag{3.1}$$

This indicates that the expression on the right-hand side of (3.1) is solely dependent on the unknowns $y_1 = y(1)$ and $y_3 = y(3)$. Utilizing the continuity of $y(\cdot)$, we define y_1 and y_3 as $y(3) = \lim_{t\to 3^-} y(t)$ for $t \in [1,3)$ and $y_5 = y(5) = \lim_{t\to 5^-} y(t)$ for $t \in [3,5)$. Given the continuity and periodicity of $y(\cdot)$, it follows that $y(1) = y(5)$. To determine $y_1 = y_5$ from (3.1), we obtain the system of equations:

$$\begin{aligned} \frac{\lambda(3,2)}{\lambda(1,2)} y(1) - y(3) &= \frac{\lambda(3,2)}{\lambda(1,2)} G(1,2) - G(3,2), \\ y(1) - \frac{\lambda(5,4)}{\lambda(3,4)} y(3) &= -\frac{\lambda(5,4)}{\lambda(3,4)} G(3,4) + G(5,4). \end{aligned} \tag{3.2}$$

Let Δ denote the determinant of the matrix $\mathcal{M}$, where

$$\mathcal{M} = \begin{pmatrix} \frac{\lambda(3,2)}{\lambda(1,2)} & -1 \\ 1 & -\frac{\lambda(5,4)}{\lambda(3,4)} \end{pmatrix}.$$

We then obtain the following theorem for the existence of 4-periodic solutions to the DEPCA (1.2).

Theorem 3.1. *Let $a(\cdot)$, $b(\cdot)$, and $g(\cdot)$ be 4-periodic continuous functions.*

(a) *If $\Delta \neq 0$, then DEPCA* (1.2) *possesses a unique 4-periodic solution presented in* (3.1)*, where (y_1, y_3) denotes the exclusive solution of the system* (3.2)*.*

(b) *If $\Delta = 0$ and $G(1,2) = G(3,2) = G(3,4) = G(5,4) = 0$, then DEPCA* (1.2) *possesses an infinite number of 4-periodic solutions as detailed below:*

$$y(t) = \begin{cases} \alpha \frac{\lambda(t,2)}{\lambda(1,2)} (y(1) - G(1,2)) + G(t,2), & t \in [1,3), \\ \alpha \frac{\lambda(t,4)}{\lambda(3,4)} (y(3) - G(3,4)) + G(t,4), & t \in [3,5). \end{cases}$$

Here, (y_1, y_3) denotes an eigenvector of $\mathcal{M}$ associated with the eigenvalue 0*, while α represents a real number.*

(c) *If $\Delta = 0$ and the rank $\mathcal{M}$ is less than the rank of $(\mathcal{M}|b)$, where*

$$b = \left(\frac{\lambda(3,2)}{\lambda(1,2)} G(1,2) - G(3,2)\ ,\ -\frac{\lambda(5,4)}{\lambda(3,4)} G(3,4) + G(5,4) \right)^T,$$

then the DEPCA (1.2) *does not admit a 4-periodic solution.*

Proof. (a) Suppose $y(t)$ represents a 4-periodic solution to DEPCA (1.2). This solution can be described using (3.1), where (y_1, y_3) is the solution of (3.2). The solvability of the linear system (3.2) depends on the condition $\Delta \neq 0$. Therefore, it must hold that $\Delta \neq 0$. Conversely, if $\Delta \neq 0$, DEPCA (3.2) admits a unique solution (y_1, y_3). It can be shown that the function $y(\cdot)$, expressed as in (3.1), constitutes the periodic solution to DEPCA (1.2).

(b) The values of G at points $(1,2)$, $(3,2)$, $(3,4)$, and $(5,4)$ are all zero. Consequently, Equation (3.2) simplifies to a homogeneous form. A non-trivial solution to this equation exists if and only if Δ equals zero. The pair of non-zero solutions (y_1, y_3) represents an eigenvector of $\mathcal{M}$ associated with the eigenvalue 0. Thus, $(\alpha y_1, \alpha y_3)$ constitutes a non-trivial solution to Equation (3.2), where α denotes any non-zero scalar. Consequently, the 4-periodic function

$$y(t) = \begin{cases} \alpha \frac{\lambda(t,2)}{\lambda(1,2)} (y(1) - G(1,2)) + G(t,2), & t \in [1,3), \\ \alpha \frac{\lambda(t,4)}{\lambda(3,4)} (y(3) - G(3,4)) + G(t,4), & t \in [3,5), \end{cases}$$

satisfies DEPCA (1.2), where α can take any value.

(c) If $\Delta = 0$ and the rank $\mathcal{M}$ is less than the rank of $(\mathcal{M}\,|\,b)$, where

$$b = \left(\frac{\lambda(3,2)}{\lambda(1,2)} G(1,2) - G(3,2)\ ,\ -\frac{\lambda(5,4)}{\lambda(3,4)} G(3,4) + G(5,4) \right)^T,$$

then Eq. (3.2) has no solution. Consequently, DEPCA (1.2) does not possess a 4-periodic solution. This concludes the proof.

□

4. Illustrative example

In this section, we present a relevant example to illustrate the practical applicability of our theory. We examine the following scalar equations with piecewise alternately advanced and retarded argument.

$$y'(t) = b(t)y\left(2\left[\frac{t+1}{2}\right]\right) + \sin(2\pi t) + \cos(2\pi t), \quad t \geq 1, \tag{4.1}$$

where

$$b(t) = \begin{cases} b_1(t), & t \in [4k+1, 4k+2), \\ b_2(t), & t \in [4k+2, 4k+3), \\ -\cos\left(\frac{\pi}{2}t\right) + \frac{\pi^2-4}{\pi^2-2\pi}, & t \in [4k+3, 4k+4), \\ \frac{2-3\pi}{2}\cos\left(\frac{\pi}{2}t\right) + 1 - \frac{2}{\pi}, & t \in [4k+4, 4k+5), \end{cases}$$

where $b_1(t) = \frac{2\pi^4+8\pi^3-48\pi^2+32\pi+32}{-2\pi^4+16\pi^3-40\pi^2+32\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{4\pi^4-12\pi^3+24\pi^2-80\pi+96}{2\pi^4-16\pi^3+40\pi^2-32\pi}$,

$b_2(t) = \frac{\pi^4-12\pi^3+16\pi^2+16\pi-16}{2\pi^3-12\pi^2+16\pi}\cos\left(\frac{\pi}{2}t\right) + \frac{\pi^2-4}{\pi^2-2\pi}$ and $k \in \mathbb{N}\cup\{0\}$.

DEPCA (4.1) is a specific instance of DEPCA (1.2) with $a = 0$ and $g(t) = \sin(2\pi t) + \cos(2\pi t)$. It is straightforward to verify that $G(1,2) = G(3,2) = G(3,4) = G(5,4) = 0$, and the matrix corresponding to the linear system of equations involving the variables y_1 and y_3 is as follows:

$$\mathcal{M} = \begin{pmatrix} \frac{\lambda(3,2)}{\lambda(1,2)} & -1 \\ 1 & -\frac{\lambda(5,4)}{\lambda(3,4)} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The determinant $\mathcal{M}$ is zero and $(1,1)$ is an eigenvector of $\mathcal{M}$ associated with the eigenvalue 0. By Theorem 3.1(b), the solution of DEPCA

$$y_\alpha(t) = \begin{cases} \alpha\lambda(t,2)y(1) + \frac{1+\sin(2\pi t)-\cos(2\pi t)}{2\pi}, & t \in [1,3), \\ \alpha\lambda(t,4)y(3) + \frac{1+\sin(2\pi t)-\cos(2\pi t)}{2\pi}, & t \in [3,5), \end{cases}$$

is a 4-periodic solution for any value of α.

The graphs of $y_\alpha(t)$ for $\alpha = 2$ and $\alpha = -3$ are presented in Figs. 1 and 2, respectively.

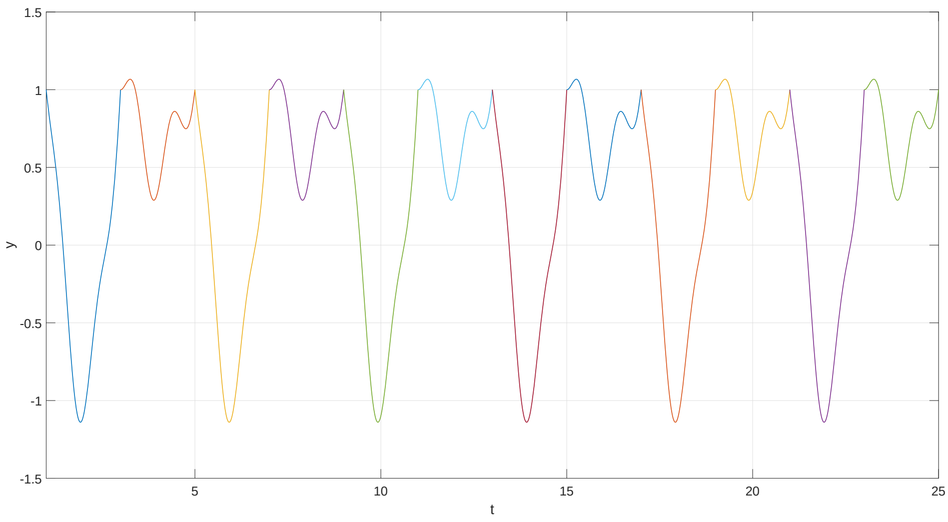

Fig. 1. 4-periodic solution of DEPCA (4.1) when $\alpha = 2$.

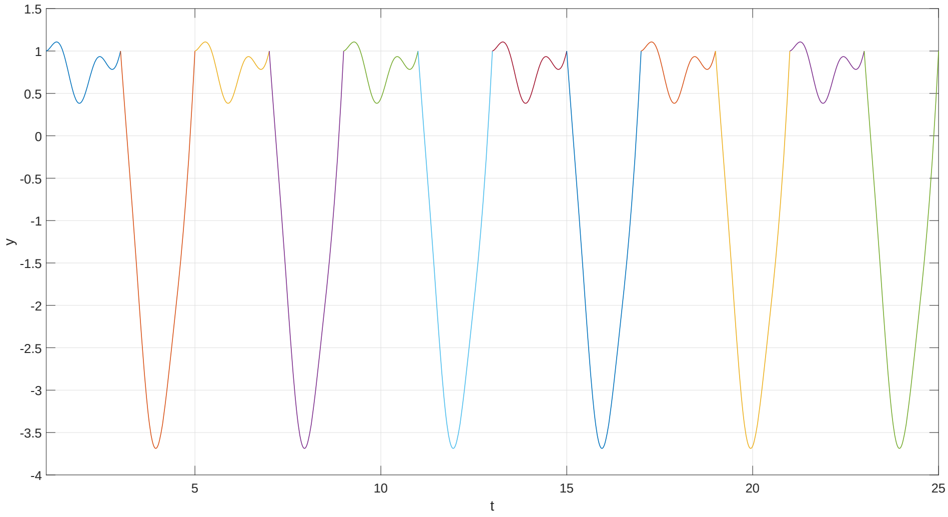

Fig. 2. 4-periodic solution of DEPCA (4.1) when $\alpha = -3$.

It is noteworthy that the parameters of the equation in this example adhere to the conditions delineated in the primary findings of papers [9]. Example 4.1 improves upon the findings of Theorem 4.4 in [9], which assert the uniqueness of the solution to DEPCA (1.2).

5. Conclusion and perspectives

This article investigates the existence of infinitely many periodic solutions for first-order differential equations involving piecewise alternately advanced and retarded arguments. We have established several theorems that ensure both the existence and uniqueness of solutions for the DEPCA with these types of arguments. By employing methods inspired by [22], we have derived sufficient conditions that guarantee the existence of infinitely many periodic solutions under suitable assumptions. Furthermore, we have provided a variety of numerical examples and simulations to illustrate the practical applicability of our findings.

Looking forward, several open problems warrant further investigation:

1. Examination of the existence of infinitely many periodic solutions for generalized types of DEPCA with piecewise alternately advanced and retarded arguments.
2. In-depth study of the asymptotic behavior and the existence of infinitely many periodic solutions for second-order DEPCA.
3. Comparative analysis of the asymptotic and periodic properties of first-order neutral differential equations with piecewise alternately advanced and retarded arguments, along with a comparison to equations incorporating linearly transformed arguments.

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