NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2015, 6 (4), P. 479–488

Evolution of size and composition of a multicomponent gas bubble in liquid solution

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PACS 05.70.Fh, 05.70.Np, 47.55.db, 47.57.eb, 68.35.Rh

DOI 10.17586/2220-8054-2015-6-4-479-488

The equations describing the transient and steady stages of size and composition evolution for a gas bubble which grows or shrinks due to the diffusion of several gases dissolved in liquid solution have been derived. The diffusion fluxes for gases in the liquid mixture caused by the bubble growth or dissolution were assumed to be quasi-stationary and the mixture of the gases in the bubble was treated as ideal. The analytical solutions for the obtained evolution equations have been found for bubbles of any size with an arbitrary number of components in the case of equal products of diffusivities and solubilities of dissolved gases in the liquid solution, and for sufficiently large binary bubbles for which capillary effects can be neglected.

Keywords: bubble, multicomponent, supersaturated solution, diffusion, kinetics of growth.

Received: 20 June 2015

1. Introduction

This paper presents several analytical results in the theoretical description for the growth or dissolution kinetics of a single gas bubble in a supersaturated or undersaturated gas-liquid solution in the case when the bubble is comprised of several gases. Evidently, finding such description is a fundamental issue because it turns to be a necessary element for the general theory on the decay of multicomponent metastable liquid solutions [1, 2]. As discussed previously, the formation of gas bubbles is widely applied in technological processes for the creationc of new porous materials and nano- and microcontainers [3, 4]. The rapid, explosive growth of water vapor and accompanying dissolved gas bubbles in magmatic melt is one contributing mechanism for volcanic eruptions [5–7]. The control of gas bubble growth and dissolution in blood and biological tissues is a very important issue for those undergoing decompression [8]. These examples highlight how the problem of theoretically describing bubble growth or dissolution is still very relevant.

The diffusion growth of single-component bubbles in a supersaturated solution of gas in liquid has been previously considered under the assumption of a quasi-stationary state for the concentration fields of dissolved gas in the vicinity of bubble and with the use of a self-similar solution for the non-stationary diffusion equation [9–13]. Quasi-steady-state diffusion approaches to the description of multicomponent bubble evolution were considered by Ramos [14, 15] and Cable and Frade [16]. In particular, it was reported in [15, 16] that large enough bubbles reach a state of stationary growth with fixed composition and growth rate and may demonstrate a nonmonotonic change in their radius during the initial stage of their evolution. Formulation of the theory of non-stationary self-similar growth of a binary bubble with stationary composition in the case of high supersaturation had been done by Gor and Kuchma [17].

Recently, we have formulated and analyzed a set of equations for the size, composition and temperature of a droplet consisting of a binary or multicomponent solution, which non-isothermally condenses or evaporates under a quasi-steady-state diffusion regime in a multicomponent mixture of vapors and non-condensable carrier gas [18–22]. Some of these equations have been analytically solved. The problem of a multicomponent droplet growing or evaporating in the diffusion regime in the vapor-gas mixture is similar to the problem of growth or dissolution of a multicomponent bubble in the liquid solution with several dissolved gas. In this paper, we will extend the analytical approach which was useful for a small droplet to the case of a multicomponent bubble.

2. General relations

We consider a multicomponent spherical bubble of radius R which grows or shrinks due to the diffusion of several gases dissolved in liquid solution at fixed absolute temperature T and pressure P. Under mechanical equilibrium of the bubble and the solution, the total pressure $P_{tot}(R)$ in the bubble depends on its radius R and pressure P in the solution according to the formula:

$$P_{tot}(R) = P + \frac{2\sigma}{R}, \tag{1}$$

where σ is the surface tension at the bubble-liquid interface. Let n_i be the volume concentration of the molecules of i-th gas ($i = 1, 2, \dots, k$) within the bubble, then the total gas concentration n in the bubble equals:

$$n = \sum_{i=1}^{k} n_i. \tag{2}$$

Assuming the gas mixture in the bubble is ideal, we have:

$$n = \frac{P_{tot}}{k_B T} = \frac{P}{k_B T}\left(1 + \frac{2\sigma}{PR}\right) = \bar{n}\left(1 + \frac{R_*}{R}\right), \tag{3}$$

where k_B is the Boltzmann constant, and we have introduced the new notation:

$$\bar{n} \equiv \frac{P}{k_B T}, \quad R_* \equiv \frac{2\sigma}{P}. \tag{4}$$

For characterization of the composition of the bubble, we will use the molecular fractions x_i of i-th component determined as:

$$x_i = \frac{n_i}{n} = \frac{N_i}{N} \quad (i = 1, 2, \dots k), \tag{5}$$

where N_i is the number of molecules of i-th gas and $N = \sum_{i=1}^{k} N_i$ is the total number of gas molecules in the bubble. Because $\sum_{i=1}^{k} x_i = 1$, only $k - 1$ molecular fractions x_i can be considered to be independent. In light of Eqs. (3) and (4), the total number N of molecules in the bubble can also be expressed as:

$$N = \frac{4\pi}{3} n R^3 = \frac{4\pi}{3}\bar{n} R^3\left(1 + \frac{R_*}{R}\right) = \frac{4\pi}{3}\bar{n} R^2 (R + R_*). \tag{6}$$

The emission and absorption of gas molecules by the bubble produce the diffusion profiles of each component around the bubble in a liquid solution. We denote the local volume concentration of i-th component at distance $r \geq R$ from the center of the bubble as $\rho_i(r,t)$. Correspondingly, the initial bulk volume concentration of the i-th component in liquid is $\rho_{i0} = \rho_i(r \to \infty, t)$. We will assume that the concentration $\rho_i(R,t)$ at the boundary of the bubble is determined by conditions of chemical equilibrium for the i-th component in the bubble and solution and is related in this way with the volume concentration n_i within the bubble. Henry's law gives:

$$\rho_i(R,t) = s_i n_i = s_i x_i n \quad (i = 1, 2, \ldots k), \tag{7}$$

where s_i is the solubility of i-th component of the gas mixture in the liquid solution.

3. The equations governing the evolution of the composition and size of the bubble

Differentiating both sides of definition (5) with respect to time gives:

$$\dot{x}_i = \frac{\dot{N}_i}{N} - x_i \frac{\dot{N}}{N} \quad (i = 1, 2, \ldots k), \tag{8}$$

where the dot over the quantity marks the time derivative or rate of the corresponding quantity. The expression for the rates $\dot{N}_i$ and $\dot{N}$ have the following form under assumption of stationary diffusion of the dissolved gases in solution:

$$\dot{N}_i = 4\pi R D_i \left[\rho_{i0} - \rho_i(R,t)\right] \quad (i = 1, 2, \ldots k), \tag{9}$$

$$\dot{N} = 4\pi R \sum_{i=1}^{k} D_i \left[\rho_{i0} - \rho_i(R,t)\right], \tag{10}$$

where D_i is the diffusivity of i-th component of the gas mixture in the liquid solution. Substitution of Eqs. (9) and (10) into the right-hand side of Eq. (8) and using Eqs. (4) and (6) yields:

$$\dot{x}_i = \frac{3}{\bar{n} R (R + R_*)} \left(D_i \left[\rho_{i0} - \rho_i(R,t)\right] - x_i \sum_{j=1}^{k} D_j \left[\rho_{j0} - \rho_j(R,t)\right] \right) \quad (i = 1, 2, \ldots k). \tag{11}$$

Let us now introduce the i-th gas supersaturation in solution as:

$$\zeta_i \equiv \frac{\rho_{i0}}{s_i \bar{n}} - 1 \quad (i = 1, 2, \ldots k). \tag{12}$$

Eqs. (3) and (7) allows us to write:

$$\rho_i(R,t) = s_i x_i \bar{n} \left(1 + \frac{R_*}{R}\right) \quad (i = 1, 2, \ldots k). \tag{13}$$

Substituting Eqs. (12) and (13) in Eq. (11) yields:

$$\dot{x}_i = \frac{3}{R (R + R_*)} \left[D_i s_i \left(\zeta_i + 1 - x_i \left(1 + \frac{R_*}{R}\right)\right) - x_i \sum_{j=1}^{k} D_j s_j \left(\zeta_j + 1 - x_j \left(1 + \frac{R_*}{R}\right)\right)\right]$$
$$(i = 1, 2, \ldots k). \tag{14}$$

Analogously, differentiating both sides of definition (6) with respect to time and using Eq. (3) gives:

$$\dot{N} = 4\pi n R^2 \dot{R} + \frac{4\pi}{3}\dot{n}R^3 = 4\pi\bar{n}\left(1 + \frac{R_*}{R}\right)R^2\dot{R} - \frac{4\pi}{3}\bar{n}\frac{R_*}{R^2}\dot{R}R^3 = 4\pi\bar{n}R\dot{R}\left(R + \frac{2}{3}R_*\right), \quad (15)$$

while substituting Eqs. (7) and (12) in Eq. (10) yields:

$$\dot{N} = 4\pi R\bar{n}\sum_{i=1}^{k} D_i s_i\left[\zeta_i + 1 - x_i\left(1 + \frac{R_*}{R}\right)\right]. \quad (16)$$

After comparing Eqs. (15) and (16), we obtain:

$$\dot{R}\left(R + \frac{2}{3}R_*\right) = \sum_{i=1}^{k} D_i s_i\left(\zeta_i + 1 - x_i\left(1 + \frac{R_*}{R}\right)\right). \quad (17)$$

Finally, we have coupled evolution equations (14) and (17) for the bubble's composition and radius with initial conditions $x_i\,(t = 0) = x_{i0}$, $R\,(t = 0) = R_0$. Note that in the case of a single-component gas in solution and bubble ($x_1 = 1$, $\zeta_i + 1 = 0$ at $i \geq 2$), Eq. (17) is obviously reduced to the equation for the bubble radius in the known form:

$$R\dot{R}\frac{R + R_\sigma}{\zeta_1 R - R_*} = D_1 s_1, \quad (18)$$

where

$$R_\sigma \equiv \frac{2}{3}R_*. \quad (19)$$

For a supersaturated gas-liquid solution, when $\zeta_1 > 0$, Eq. (18) can be rewritten:

$$R\dot{R}\frac{R + R_\sigma}{R - R_c} = D_1 s_1 \zeta_1, \quad (20)$$

where $R_c = \dfrac{R_*}{\zeta_1} = \dfrac{2\sigma}{P\zeta_1}$ is the radius of the critical bubble (at $R_0 > R_c$ the size of the bubble increases monotonically with time, while at $R_0 < R_c$, the bubble dissolves irreversibly). The growth dynamics of a supercritical bubble in a highly supersaturated ($\zeta_1 >> 1$) gas-liquid solution, determined by Eq. (20), was studied in detail in [11]. If there is $\zeta_1 \leq 0$, a bubble of any initial size irreversibly dissolves.

4. Analytical solution of evolution equations in the case of equal products of diffusivities and solubilities of dissolved gases

Let us now consider the conditions on the system parameters which allows us to find analytical solutions for Eqs. (14) and (17). In the particular case when $D_i s_i \equiv Ds$, for any gas component, (i.e. when the diffusivity and solubility products for dissolved gases in liquid solution are equal) Eqs. (14) and (17) take the form:

$$\dot{x}_i = \frac{3Ds}{R\,(R + R_*)}\left[\zeta_i + 1 - x_i\left(\bar{\zeta} + 1\right)\right] \quad (i = 1, 2, \ldots k), \quad (21)$$

$$R\dot{R}\frac{R + R_\sigma}{R - R_*/\bar{\zeta}} = Ds\bar{\zeta}, \quad (22)$$

where we have introduced new notation:

$$\bar{\zeta} \equiv \sum_{i=1}^{k} (\zeta_i + 1) - 1. \tag{23}$$

As is seen from Eqs. (20) and (22), changing the bubble size with time does not depend in this particular case on the bubble's composition and is described by the same equation as in the case of single-component bubble. The role of supersaturation is played now by the quantity $\bar{\zeta}$, and we have $R_*/\bar{\zeta}$ instead of R_c. Proceeding by analogy with [11], taking into account the initial value $R\,(t=0) = R_0$ and Eq. (19), we find for the bubble radius at $\bar{\zeta} \neq 0$:

$$\frac{R^2 - R_0^2}{2} + \left(1 + \frac{2\bar{\zeta}}{3}\right) \frac{R_*}{\bar{\zeta}} \left(R - R_0 + \frac{R_*}{\bar{\zeta}} \ln \left| \frac{\bar{\zeta} R - R_*}{\bar{\zeta} R_0 - R_*} \right| \right) = D s \bar{\zeta} t. \tag{24}$$

In the case $\bar{\zeta} = 0$, it follows from Eq. (22) that the bubble radius satisfies the equation:

$$R\dot{R} \left(\frac{R}{R_*} + \frac{2}{3}\right) = -Ds. \tag{25}$$

It is clear from Eq. (25) that $\dot{R} < 0$ and the bubble irreversibly dissolves in this case. Using the initial condition for the bubble radius, we find the solution of Eq. (25) in the form:

$$\frac{1}{3}\left(\frac{R^3 - R_0^3}{R_*} + R^2 - R_0^2\right) = -Dst. \tag{26}$$

Setting $R\,(t_d) = 0$, we can find the time t_d of complete dissolution of the bubble of initial radius R_0 at $\bar{\zeta} = 0$ as:

$$t_d\,(R_0) = \frac{R_0^2}{3Ds} \frac{R_0 + R_*}{R_*}. \tag{27}$$

There is a special case when $\zeta_i + 1 = 0$ for all gas components, and, correspondingly, $\bar{\zeta} + 1 = 0$. This case refers to the dissolution of a gas bubble in pure solvent. As follows from Eq. (21), we have $\dot{x}_i = 0$ for each gas component within the bubble in this case, and the composition of the bubble in the dissolution process remains unchanged. Such situation is fully equivalent to the description of the single-component bubble dissolution in the pure solvent. Substituting $\bar{\zeta} = -1$ into Eq. (24), we find:

$$\frac{R^2 - R_0^2}{2} - \frac{1}{3} R_* \left[R - R_0 - R_* \ln \left(\frac{R + R_*}{R_0 + R_*}\right)\right] = -Dst. \tag{28}$$

As follows from Eq. (28), the time $\bar{t}_d$ of complete dissolution of the bubble of initial radius R_0 at $\bar{\zeta} = -1$ is determined as:

$$\bar{t}_d\,(R_0) = \frac{1}{Ds}\left[\frac{R_0^2}{2} - \frac{1}{3} R_* \left(R_0 - R_* \ln\left(\frac{R_0}{R_*} + 1\right)\right)\right]. \tag{29}$$

In the case $R_0 << R_*$, this time approximately equals:

$$\bar{t}_d\,(R_0) \approx \frac{R_0^2}{3Ds}\left(1 + \frac{R_0}{3R_*}\right). \tag{30}$$

Let us now consider a relation between the bubble radius and composition. If $\bar{\zeta} + 1 > 0$, then we have from Eq. (21):

$$\dot{x}_i = -\frac{3Ds\left(\bar{\zeta} + 1\right)}{R\,(R + R_*)}\,(x_i - x_{is})\,, \quad (i = 1, 2, \ldots k), \tag{31}$$

where:

$$x_{is} \equiv \frac{\zeta_i + 1}{\bar{\zeta} + 1}, \tag{32}$$

is the stationary value of molecular fraction of i-th component of the gas mixture in the bubble. With the help of Eq. (22), we can find from Eq. (31) the relation between differentials dx_i and dR:

$$\frac{dx_i}{x_i - x_{is}} = -\frac{3\left(\bar{\zeta}+1\right)\left(R+R_\sigma\right)dR}{\left(R+R_*\right)\left(\bar{\zeta}R - R_*\right)} \quad (i = 1, 2, \dots k). \tag{33}$$

The solution of Eq. (33) at $\bar{\zeta} \neq 0$ and $R\,(t=0) = R_0$, $x_i\,(t=0) = x_{i0}$ $(i = 1, 2, \dots k)$ has the form:

$$\left|\frac{x_i - x_{is}}{x_{i0} - x_{is}}\right| = \frac{R_0 + R_*}{R + R_*}\left|\frac{\bar{\zeta}R_0 - R_*}{\bar{\zeta}R - R_*}\right|^{\dfrac{2\bar{\zeta}+3}{\bar{\zeta}}}. \tag{34}$$

Equation (34) shows that molecular fractions of all gas components in the bubble relax to their steady-state values according to one and the same power law. If $R >> R_*/\left|\bar{\zeta}\right|$ and $R_0 >> R_*/\left|\bar{\zeta}\right|$, Eq. (34) simplifies as:

$$\left|\frac{x_i - x_{is}}{x_{i0} - x_{is}}\right| = \left|\frac{R_0}{R}\right|^{3\dfrac{\bar{\zeta}+1}{\bar{\zeta}}} \quad (i = 1, 2, \dots k), \tag{35}$$

or

$$R^2 = R_0^2\left(\frac{x_{i0} - x_{is}}{x_i - x_{is}}\right)^{\dfrac{2}{3}\dfrac{\bar{\zeta}}{\bar{\zeta}+1}}. \tag{36}$$

In the case when $\bar{\zeta} = 0$, the dependence $x_i\,(R)$ $(i = 1, 2, \dots k)$ can be easily found by two ways. First we can set $\bar{\zeta} = 0$ just in Eq. (33) for x_i and solve it to obtain:

$$\left|\frac{x_i - x_{is}}{x_{i0} - x_{is}}\right| = \frac{R_0 + R_*}{R + R_*}\exp\left(-3\frac{R_0 - R}{R_*}\right) \quad (i = 1, 2, \dots k). \tag{37}$$

The same result can also be obtained by taking the limit $\bar{\zeta} \to 0$ in the solution (34) and using the relation:

$$\left|\frac{\bar{\zeta}R_0 - R_*}{\bar{\zeta}R - R_*}\right|^{\dfrac{2\bar{\zeta}+3}{\bar{\zeta}}} = \left|\frac{1 - \bar{\zeta}R_0/R_*}{1 - \bar{\zeta}R/R_*}\right|^{\dfrac{2\bar{\zeta}+3}{\bar{\zeta}}} \approx \left(1 - \bar{\zeta}\frac{R_0 - R}{R_*}\right)^{\dfrac{3}{\bar{\zeta}}} \xrightarrow[\bar{\zeta}\to 0]{} \exp\left(-3\frac{R_0 - R}{R_*}\right).$$

5. Analytical solution of evolution equations in the case of neglecting the capillary contributions

For sufficiently large bubbles, when $R_0 >> R_*$ and $R >> R_*$, we can neglect the capillary contributions to the evolution equations (14) and (17) and write:

$$\dot{x}_i = \frac{3}{R^2}\left(D_i s_i\left(\zeta_i + 1 - x_i\right) - x_i\sum_{j=1}^{k} D_j s_j\left(\zeta_j + 1 - x_j\right)\right), \tag{38}$$

$$R\dot{R} = \sum_{i=1}^{k} D_i s_i\left(\zeta_i + 1 - x_i\right). \tag{39}$$

An analytical solution for this set of equations is possible for $k = 2$ (for a binary gas bubble). In this case, Eqs. (38) and (39) are reduced to the form:

$$\dot{x}_1 = \frac{3}{R^2} f(x_1), \tag{40}$$

$$f(x_1) = -D_1 s_1 \left[(\gamma - 1) x_1^2 + (\zeta_1 + \gamma \zeta_2 + 2) x_1 - \zeta_1 - 1 \right], \tag{41}$$

$$R\dot{R} = g(x_1), \tag{42}$$

$$g(x_1) = D_1 s_1 \left[\zeta_1 + 1 + \gamma \zeta_2 + x_1 (\gamma - 1) \right], \tag{43}$$

where:

$$\gamma \equiv \frac{D_2 s_2}{D_1 s_1}. \tag{44}$$

This set of equations is similar to the set of equations describing the evolution of a two component droplet in the gas mixture, which has been studied in [18].

Recognizing that the case of $\gamma = 1$ (when $D_2 s_2 = D_1 s_1 = Ds$) was completely described in the previous section for multicomponent bubbles of any size, we consider further the situation with $\gamma \neq 1$, assuming for definiteness $\gamma > 1$. It is convenient to rewrite Eq. (41) in the form:

$$f(x_1) = -D_1 s_1 (\gamma - 1)(x_1 - x_{1s})(x_1 - \tilde{x}_{1s}), \tag{45}$$

where x_{1s} and $\tilde{x}_{1s}$ are the roots of function $f(x_1)$:

$$x_{1s} = \frac{-(\zeta_1 + 2 + \gamma \zeta_2) + \sqrt{[\zeta_1 + 2 + \gamma \zeta_2]^2 + 4(\gamma - 1)(\zeta_1 + 1)}}{2(\gamma - 1)}, \tag{46}$$

$$\tilde{x}_{1s} = \frac{-(\zeta_1 + 2 + \gamma \zeta_2) - \sqrt{[\zeta_1 + 2 + \gamma \zeta_2]^2 + 4(\gamma - 1)(\zeta_1 + 1)}}{2(\gamma - 1)}. \tag{47}$$

As follows from Eq. (47), at $\gamma > 1$, the root $\tilde{x}_{1s}$ is negative, $\tilde{x}_{1s} < 0$, and has no physical meaning. The root x_{1s} lies in the interval $0 < x_1 < 1$ and corresponds to the stationary composition of the bubble. As follows from Eqs. (42) and (43), the stationary rate $\left(R\dot{R}\right)_s$ of the changing the bubble size is determined as:

$$\left(R\dot{R}\right)_s = g(x_{1s}) = D_1 s_1 \left[\zeta_1 + 1 + \gamma \zeta_2 + x_{1s} (\gamma - 1) \right], \tag{48}$$

or, in view of Eq. (46), as:

$$\left(R\dot{R}\right)_s = \frac{\gamma}{1 + (\gamma - 1) x_{1s}} D_1 s_1 (\zeta_1 + \zeta_2 + 1). \tag{49}$$

As one can see from Eq. (49), the size of the bubble increases monotonically in the stationary case at $\zeta_1 + \zeta_2 + 1 > 0$. If the opposite inequality holds, the bubble dissolves.

Since function $f(x_1)$ monotonically decreases over the $0 < x_1 < 1$ interval, it then follows from Eq. (40) that molecular fraction x_1 tends monotonically with time to its stationary value x_{1s}. This allows us to use the current value $x_1(t)$ as the independent variable for solving the system of equations for the evolution of the composition and size of the bubble. Accordingly, we find from Eqs. (40) and (42) the following differential equation:

$$\frac{dR^2}{R^2} = \frac{2}{3} \frac{g(x_1)}{f(x_1)} dx_1, \tag{50}$$

the solution of which at $R\left(t=0\right)=R_0$, $x_1\left(t=0\right)=x_{10}$ has the form:

$$R^2\left(x_1\right)=R_0^2\exp\left(\frac{2}{3}\int\limits_{x_{10}}^{x_1}\frac{g\left(y\right)}{f\left(y\right)}dy\right). \tag{51}$$

Substituting Eq. (51) in Eq. (40) yields another differential equation:

$$\frac{dx_1}{dt}=\frac{3}{R^2}f\left(x_1\right)=\frac{3}{R_0^2\exp\left(\frac{2}{3}\int_{x_{10}}^{x_1}\frac{g\left(y\right)}{f\left(y\right)}dy\right)}f\left(x_1\right), \tag{52}$$

the solution of which can be written as:

$$t\left(x_1\right)=\frac{R_0^2}{3}\int\limits_{x_{10}}^{x_1}\frac{dy}{f\left(y\right)}\exp\left(\frac{2}{3}\int\limits_{x_{10}}^{y}\frac{g\left(z\right)}{f\left(z\right)}dz\right). \tag{53}$$

It is convenient to transform expression (43) for function $g\left(x_1\right)$ as:

$$g\left(x_1\right)=D_1s_1\left(\gamma-1\right)\left(x_1-x_{1*}\right), \tag{54}$$

where:

$$x_{1*}=\frac{\zeta_1+\gamma\zeta_2+1}{1-\gamma}. \tag{55}$$

Using Eqs. (45) and (54) allows us to represent the ratio $g(x_1)/f(x_1)$ in the form:

$$\frac{g(x_1)}{f(x_1)}=-\frac{x_1-x_{1*}}{\left(x_1-x_{1s}\right)\left(x_1-\tilde{x}_{1s}\right)}=\frac{1}{x_{1s}-\tilde{x}_{1s}}\left(\frac{\tilde{x}_{1s}-x_{1*}}{x_1-\tilde{x}_{1s}}-\frac{x_{1s}-x_{1*}}{x_1-x_{1s}}\right), \tag{56}$$

and to perform integration:

$$\exp\left(\frac{2}{3}\int\limits_{x_{10}}^{x}\frac{g(y)}{f(y)}dy\right)=\left(\frac{x_{10}-x_{1s}}{x_1-x_{1s}}\right)^{\frac{2}{3}\frac{x_{1s}-x_{1*}}{x_{1s}-\tilde{x}_{1s}}}\left(\frac{x_1-\tilde{x}_{1s}}{x_{10}-\tilde{x}_{1s}}\right)^{\frac{2}{3}\frac{\tilde{x}_{1s}-x_{1*}}{x_{1s}-\tilde{x}_{1s}}}. \tag{57}$$

Using Eq. (57) in Eqs. (51) and (53), we finally obtain for the square of the radius, $R^2(x)$ and time, $t(x)$ the following expressions:

$$R^2(x_1)=R_0^2\left(\frac{x_{10}-x_{1s}}{x_1-x_{1s}}\right)^{\frac{2}{3}\frac{x_{1s}-x_{1*}}{x_{1s}-\tilde{x}_{1s}}}\left(\frac{x_1-\tilde{x}_{1s}}{x_{10}-\tilde{x}_{1s}}\right)^{\frac{2}{3}\frac{\tilde{x}_{1s}-x_{1*}}{x_{1s}-\tilde{x}_{1s}}}, \tag{58}$$

$$t(x_1)=\frac{R_0^2}{3f\left(x_{10}\right)}\int\limits_{x_{10}}^{x_1}dy\left(\frac{x_{10}-x_{1s}}{y-x_{1s}}\right)^{\frac{2}{3}\frac{x_{1s}-x_{1*}}{x_{1s}-\tilde{x}_{1s}}+1}\left(\frac{y-\tilde{x}_{1s}}{x_{10}-\tilde{x}_{1s}}\right)^{\frac{2}{3}\frac{\tilde{x}_{1s}-x_{1*}}{x_{1s}-\tilde{x}_{1s}}-1}. \tag{59}$$

The dependence $x_1\left(t\right)$ can be found by reversing the dependence $t(x_1)$ obtained with the help of Eq. (59). Substituting the reversal function $x_1\left(t\right)$ in Eq. (58) determines the dependence $R^2\left(t\right)$.

Let us now consider the possibility of nonmonotonic behavior for rate $R\dot{R}$ as a function of system parameters and an initial bubble composition. As is clear from Eqs. (42) and (54), a nonmonotonic growth or dissolution of the bubble becomes possible when the parameter x_{1*}, determined by Eq. (55), appears within the $0<x_{1*}<1$ interval, and the concentration $x_1\left(t\right)$ passes during the bubble evolution through the value $x_1=x_{1*}$. In such a case, we have $\dot{R}>0$ at $x_1<x_{1*}$ and $\dot{R}<0$ at $x_1>x_{1*}$. According to Eq. (55), the condition $0<x_{1*}<1$ is reached if the double inequality $1-\gamma < \zeta_1 + \gamma\zeta_2 + 1 < 0$ is fulfilled. Moreover, if we have $0 < x_{1*} < x_{1s} < 1$ and $x_{10} < x_{1*}$, then the bubble radius will diminish at the initial stage of its evolution until the concentration $x_1 = x_{1*}$ is reached and then grow monotonically after that. Note that, as follows from Eqs. (46) and (55), inequality $x_{1*} < x_{1s}$ fulfils at $\zeta_1 + \zeta_2 + 1 > 0$.

To control our results, we will check the case when $\gamma \to 1$ in Eq. (58). Setting $\gamma \to 1$ in Eqs. (46), (47) and (55) and recognizing with the help of Eq. (23) that $\bar{\zeta} \equiv \zeta_1 + \zeta_2 + 1$ in the case of two dissolved gases, we find:

$$x_{1s} \to \frac{\zeta_1 + 1}{\bar{\zeta} + 1}, \quad \tilde{x}_{1s} \sim -\frac{\bar{\zeta} + 1}{\gamma - 1} \to \infty, \quad x_{1*} \sim -\frac{\bar{\zeta}}{\gamma - 1} \to \infty, \tag{60}$$

$$\frac{x_{1s} - x_{1*}}{x_{1s} - \tilde{x}_{1s}} \to \frac{\bar{\zeta}}{\bar{\zeta} + 1}, \quad \left(\frac{x_1 - \tilde{x}_{1s}}{x_{10} - \tilde{x}_{1s}}\right)^{\frac{2}{3}\frac{\tilde{x}_{1s} - x_{1*}}{x_{1s} - \tilde{x}_{1s}}} \to 1. \tag{61}$$

Using Eqs. (60) and (61) in Eq. (58), we obtain $R^2(x_1) \xrightarrow[\gamma \to 1]{} R_0^2 \left(\frac{x_{10} - x_{1s}}{x_1 - x_{1s}}\right)^{\frac{2}{3}\frac{\bar{\zeta}}{\bar{\zeta}+1}}$. As expected, this result coincides with Eq. (36) for the case when $D_1 s_1 = D_2 s_2 \equiv Ds$ and $R >> R_*/\left|\bar{\zeta}\right|$, $R_0 >> R_*/\left|\bar{\zeta}\right|$ which was considered in section 3.

Acknowledgements

This work was supported by the Russian Foundation for Basic Research (project No. 13-03-01049-a) and the Program of Development of St. Petersburg State University (project No. 11.37.183.2014).

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