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Original article

Pinned gradient measures of SOS model associated with H_A-boundary laws on Cayley trees

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ABSTRACT This paper investigates pinned gradient measures for SOS (Solid-On-Solid) models associated with H_A-boundary laws of period two, a class that encompasses all 2-height periodic gradient Gibbs measures corresponding to a spatially homogeneous boundary law. While previous research has predominantly focused on a spatially homogeneous boundary law and corresponding GGMs on Cayley trees, this study extends the analysis by providing a comprehensive characterization of such measures. Specifically, it demonstrates the existence of pinned gradient measures on a set of G-admissible configurations and precisely quantifies their number under certain temperature conditions.

KEYWORDS SOS model, gradient configuration, G-admissible configuration, spin values, Cayley tree, gradient measure, gradient Gibbs measure, two periodic boundary law

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1. Introduction

Gradient Gibbs measures (GGMs) on trees, particularly on the Cayley tree, are an important class of models in statistical mechanics used to study interfaces and phase transitions. These measures arise in systems where the spin variables, i.e. "heights" are defined on the vertices of a tree and exhibit a gradient interaction between neighboring sites, meaning the energy of the system depends on the difference between spins at adjacent vertices. The main interest in GGMs on trees lies in their ability to capture complex behaviors such as long-range correlations, coexistence of multiple phases, and non-trivial periodic solutions, even in low-dimensional settings.

The Cayley tree, an infinite, connected, acyclic graph where each vertex has a fixed number of neighbors (called the order of the tree), serves as a natural setting for studying such measures. Unlike lattice systems, the tree structure introduces unique challenges due to the absence of loops, resulting in boundary effects that play a dominant role in the behavior of the system (e.g., [1–5]).

For GGMs on trees, the construction is typically based on boundary laws as solutions of recursive equations that describe the influence of the outer boundary on the system. This recursive structure facilitates the exploration of non-translation-invariant solutions, including periodic or quasi-periodic Gibbs measures. Notably, the work of Zachary [6] laid the foundation for describing Gibbs measures on trees using these boundary conditions. Models with denumerable (non-compact) set of spin values which potentials are invariant under a joint height-shift of all values of the spin-variables are notable in statistical mechanics, under the names interface models or gradient models. For lattice spin systems, a theory demonstrates the existence and uniqueness of gradient Gibbs measures with a fixed tilt, assuming uniform strictly convex potentials in dimensions $d = 2$ investigated by Funaki and Spohn [7]. (See however Remark 4.4 of [8] on existence for non-convex potentials.) This extends to random models [9, 10] in dimensions $d \geq 3$, while for $d = 2$ such random gradient states cannot exist ([11]) since they experience local destabilization due to the impact of quenched randomness. In [12], the establishment of gradient Gibbs measures on trees through boundary laws is provided. Also, in the paper, authors generalize the theory of Zachary [6, 13] for a non-normalizable boundary law (i.e. Zachary's theory can not be applied).

In the context of the SOS model, a classic example of an interface model, GGMs on the Cayley tree have been shown to exhibit rich behavior, including multiple periodic solutions and phase transitions depending on the parameters of the model. Such systems allow for the study of gradient Gibbs measures that are both translation-invariant and those that break translation symmetry, leading to periodic configurations (e.g. [14–20]).

In this paper, we build upon the research presented in the preceding papers by investigating the existence of 2-height periodic pinned gradient measures for the SOS model restricted to a set of G-admissible configurations on the Cayley tree of order $k \geq 2$. The notion of H_A-periodicity is typically defined (see [4]) for boundary conditions. If pinned gradient measures (GGMs) exist for such boundary conditions, they are referred to as H_A-periodic pinned gradient measures (GGMs). We define H_A-boundary laws by formula (3.14). Note that if there exists spatially translation invariant GGMs corresponding to a spatially homogeneous boundary law (i.e. $\{l_{xy}\} = \{l\}$), then it is possible to call these GGMs corresponding to H_A-boundary laws. In this case, we set $l^{(1)} = l^{(2)} := l$ for the family of vectors $\{l_{xy}\}_{\langle x,y\rangle \in \vec{L}} = \{l^{(1)}, l^{(2)}\}$.

The results in Theorem 1 show that all three H_A-boundary laws of period two define a spatially homogeneous boundary law resulting in three GGMs (see [17, 21, 22]). On the other hand, the results in Theorem 2 indicate that for specific ranges of the interaction parameter θ, there are exactly three 2-height periodic pinned gradient measures on a G_2-admissible configuration space: one of them is associated with a trivial boundary law and the other two are derived from spatially inhomogeneous (H_A) boundary laws. This reveals the presence of symmetry breaking in the model, where distinct periodic solutions emerge depending on the parameter θ.

2. Preliminaries

We would like to emphasize that the information below is based on the references [12, 18, 21–23]. Let us denote the Cayley tree of order k by $\Gamma^k = (V, L)$, where V and L is the set of vertices and the set of edges, respectively. An unoriented edge between two vertices $x, y \in V$ is denoted by $b = \{x, y\}$. For an oriented edge going from x to y, we write $\langle x, y\rangle$ and $\vec{L}$ is the set of all such edges. $d(x, y)$ denotes the number of edges along the unique smallest path from x to y. Let $\mathcal{N}$ be the collection of a finite subsets of V. The outer boundary set of $\Lambda \in \mathcal{N}$ is defined as

$$\partial\Lambda := \{x \notin \Lambda : d(x, y) = 1 \text{ for some } y \in \Lambda\}.$$

Let $\Omega := \mathbb{Z}^V = \{(\omega_x)_{x\in V} | \omega_x \in \mathbb{Z}\}$ denote the set of (integer-valued) height-configurations endowed with the product σ-algebra $\mathcal{F} = \mathcal{P}(\mathbb{Z})^V$ generated by the spin variables $\pi_x : \mathbb{Z}^V \to \mathbb{Z}$ is defined by $\pi_x(\omega) = \omega_x$ the projection onto the coordinate $x \in V$.

Let $\Lambda \subset V$ and $\pi_\Lambda : \Omega \to \mathbb{Z}^\Lambda$ be the projection onto the coordinates in Λ. We can write

$$\mathcal{F}_\Lambda = \sigma\left(\{\pi_y \mid y \in \Lambda\}\right) = \mathcal{P}(\mathbb{Z})^\Lambda$$

for the $\sigma-$algebra generated by the height-variables in the vertices $x \in \Lambda$.

Let ω_x be the state of the configuration ω at the vertex $x \in V$ and $b = \langle v, w\rangle \in \vec{L}$. The equation $\nabla\omega_b = \omega_w - \omega_v$ denotes the height difference of b. We define the gradient field of ω as

$$\nabla\omega := \{\nabla\omega_b | \, b \in \vec{L}\}.$$

The set of spin values $\eta_{\langle x,y\rangle} = \pi_y - \pi_x$ is called gradient spin variables for each $\langle x, y\rangle \in \vec{L}$. The state space of the gradient configurations is defined by $\Omega^\nabla = \mathbb{Z}^V/\mathbb{Z} = \mathbb{Z}^{\vec{L}}$. We will consider the standard σ-algebra on $\mathbb{Z}^{\vec{L}}$ which is defined as follows

$$\mathcal{F}^\nabla = \sigma\left(\left\{\eta_b \mid b \in \vec{L}\right\}\right) = \mathcal{P}(\mathbb{Z})^{\vec{L}}.$$

For each $b = \{x, y\} \in L$, *a symmetric nearest-neighbor gradient interaction potential* $U_b : \mathbb{Z} \to \mathbb{R}$ is given by $U_b(m) = U_b(-m)$ and the family of functions, i.e. *transfer operators* are defined by $Q_b(m) = \exp\left(-\beta U_b(m)\right)$ for all $m \in \mathbb{Z}$. Here β is interpreted as the inverse of a temperature. The following finite quantity is called [22] a *Hamiltonian* in the finite volume $\Lambda \in V$ is as follows

$$H^U_\Lambda(\omega) = \sum_{b\cap\Lambda\neq\emptyset} U_b\left(\nabla\omega_b\right), \ \Lambda \in \mathcal{N}.$$

In the SOS model on a Cayley tree, U_b is an unbounded symmetric nearest-neighbor gradient interaction potential defined by

$$U_b(\omega_x, \omega_y) = J_b|\omega_x - \omega_y|,$$

where $J_b \in \mathbb{R}$ is a coupling constant, which determines the energy cost of height differences.

In the article, it is assumed that $J_b = J > 0$, indicating the spatial homogeneity of the coupling constant. Furthermore, the system's energy increases as the height difference between adjacent sites increases. Thus, we can conclude that the parameter $\theta := e^{-J\beta}$ lies within the interval $0 < \theta < 1$.

The family of probability kernels [22] for the given Hamiltonian H^U_Λ, i.e., $(\gamma_\Lambda)_{\Lambda\in\mathcal{N}}$ from $(\Omega, \mathcal{F}_{\Lambda^c})$ to $(\Omega, \mathcal{F})$ is given by

$$\gamma_\Lambda(A|\tilde{\omega}) = Z^{-1}_\Lambda(\tilde{\omega}) \int_A \exp\left(-\sum_{b\subset\Lambda} U_b\left(\nabla\omega_b\right) - \sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}} U_{\{i,j\}}\left(\omega_i - \tilde{\omega}_j\right)\right) d\omega_\Lambda \tag{2.1}$$

for all $A \in \mathcal{F}$, where $Z_\Lambda(\tilde{\omega})$ denotes a normalization constant and $d\omega_\Lambda$ is the counting measure on $\Omega_\Lambda = \mathbb{Z}^\Lambda$.

A transfer operator $Q_b = Q$ induces *local Gibbsian specification*

$$\gamma = \{\gamma_\Lambda : \mathcal{F} \times \Omega \to [0,1]\}_{\Lambda \in \mathcal{N}}$$

by the assignment which represent (2.1) in the form

$$\gamma_\Lambda(\sigma_\Lambda = \tilde{\omega}_\Lambda|\omega) = \frac{1}{Z_\Lambda(\omega_{\partial\Lambda})}\left(\prod_{\{x,y\}\subset\Lambda} Q(\tilde{\omega}_x - \tilde{\omega}_y)\right)\prod_{\substack{x\in\Lambda, y\in\Lambda^c\\ x\sim y}} Q(\tilde{\omega}_x - \omega_y)$$

for every $\Lambda \in \mathcal{N}$, $\tilde{\omega} \in \Omega_\Lambda$ and $\omega \in \Omega$. Here, the partition function Z_Λ gives for every $\omega \in \Omega$ the normalisation constant $Z_\Lambda(\omega) = Z_\Lambda(\omega_{\partial\Lambda})$ turning $\gamma_\Lambda(\cdot|\omega)$ into a probability measure on the height configuration space $(\Omega, \mathcal{F})$, ω_Λ and Ω_Λ denote the restrictions on $\Lambda \in V$.

The kernels γ_Λ can be projected to the *gradient Gibbs specification*

$$\gamma^\nabla = \{\gamma_\Lambda^\nabla : \mathcal{F}^\nabla \times \Omega^\nabla \to [0,1]\}_{\Lambda\in\mathcal{N}}.$$

The *outer gradient σ-algebra* [22] on Ω^∇ is defined by

$$\mathcal{T}_\Lambda^\nabla := \sigma((\eta_b)_{b\subset\Lambda^c}, [\eta]_{\partial\Lambda}) \subset \mathcal{F}^\nabla.$$

The kernels [21] are

$$\gamma_\Lambda^\nabla(\eta_\Lambda = \zeta_\Lambda|\zeta) := \gamma_\Lambda(\sigma_\Lambda = \omega_\Lambda|\omega)$$

for any $\omega \in \Omega$ such that $(\nabla\omega)_{\Lambda^c} = \zeta_{\Lambda^c}$ and $[\nabla\omega]_{\partial\Lambda} = [\zeta]_{\partial\Lambda}$.

Then a collection $\Sigma := \left(V, \mathcal{N}, \Omega^\nabla, \{\mathcal{T}_\Lambda^\nabla\}_{\Lambda\in\mathcal{N}}\right)$ can be considered as a *lattice system*. Let $\gamma = \{\gamma_\Lambda\}_{\Lambda\in\mathcal{N}}$ be a local specification on lattice systems. Then a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is called a *Gibbs measure with specification* γ if $\mu = \mu\gamma_\Lambda$ for each $\Lambda \in \mathcal{N}$. This definition of Gibbs measures originates from Dobrushin and Lanford and Ruelle (see [24–26]), and the last equations are called the *DLR-equations*. A Gibbs measure with the specification $\gamma^\nabla = \{\gamma_\Lambda^\nabla(\cdot|\zeta)|\zeta \in \Omega^\nabla, \Lambda \in \mathcal{N}\}$ is called a *gradient Gibbs measure* on the lattice system Σ.

3. Pinned gradient measures corresponding to two periodic boundary laws

It is known that the problem of expressing periodic Gibbs measures corresponding to various Hamiltonians typically reduces to solving systems of algebraic equations. Due to the lack of general formulas for solving such systems, many difficulties arise. Initially, we analyze the solutions of the following system of equations:

$$\begin{cases} x = \left(\dfrac{ay+b}{cy+a+b-c}\right)^k \\ y = \left(\dfrac{ax+b}{cx+a+b-c}\right)^k \end{cases}, \tag{3.1}$$

which is a generalization of systems of equations encountered in many papers [14, 15, 17, 27]. As an example for the case $b \neq c$, one can apply the result of the following proposition to the system of equations (4.3) analyzed in [27]. Applications of our proposition for the case $b = c$ will be explored later in Theorems 1 and 2.

Proposition 1. *Let $a, b, c > 0$ be real numbers satisfying the condition $a + b - c > 0$. The number of positive solutions (x, y) to the system (3.1) is determined by the value of $k \in \mathbb{N}$ and the relationship between a and c:*

(1) *If $a = c$ or $k = 1$, then the system has exactly one solution which is $(x, y) = (1, 1)$.*

(2) *If $a > c$ and $k > \dfrac{a+b}{a-c}$, then the system has exactly three distinct solutions which satisfy $x = y$.*

(3) *If $a < c$ and $k > \dfrac{a+b}{c-a}$, then the system also has exactly three distinct solutions one solution satisfies $x = y$ and the other two satisfy the condition $x \neq y$.*

Proof. Let's start with the following notation for simplicity

$$f(x) := \left(\frac{ax+b}{cx+a+b-c}\right)^k. \tag{3.2}$$

The case $a = c$ is indeed quite trivial, as in this scenario, the function simplifies to $f(x) = 1$ for all positive x. Consequently, there is only one pair of solutions, which is $(x, y) = (1, 1)$.

For the case $a \neq c$, the function $f(x)$ exhibits specific properties depending on the relationship between a and c. The derivative of $f(x)$ is given by

$$f'(x) = \frac{k(a+b)(a-c)f(x)}{(ax+b)(cx+a+b-c)}.$$

The function $f(x)$ is bounded, strictly increasing when $a > c$, and strictly decreasing when $a < c$. For $x > 0$, it holds that $f(0) = \left(\frac{b}{a+b-c}\right)^k > 0$, and as $x \to \infty$ we find that $\lim_{x\to\infty} f(x) = \left(\frac{a}{c}\right)^k$.

Now assume that $a < c$. In this case, we conclude that there exists a unique solution, given by $x = y = 1$ on the assumption that $x = y$.

Let $x \neq y$. Now we find the conditions for the existence of solutions with $x \neq y$ in the system (3.1). To do this, we will study the equation

$$f(f(x)) = x. \tag{3.3}$$

Since the function $f(x)$ is invertible for $x > 0$, we can rewrite the equation as $f(x) = f^{-1}(x) := g(x)$, where

$$g(x) = f^{-1}(x) = \frac{(a+b-c)\sqrt[k]{x} - b}{-c\sqrt[k]{x} + a}. \tag{3.4}$$

From $f(x) > 0$, it follows that $g(x) > 0$. Therefore, the domain of the function $g(x)$ is (x_1, x_2), where

$$\begin{cases} x_1 = \left(\frac{a}{c}\right)^k < x < \left(\frac{b}{a+b-c}\right)^k = x_2, & \text{if } a < c \\ x_1 = \left(\frac{b}{a+b-c}\right)^k < x < \left(\frac{a}{c}\right)^k = x_2, & \text{if } a > c \end{cases}. \tag{3.5}$$

Now let us consider the case $a < c$. Note that by solving the equation $h(x) = 0$ for the function $h(x) = \ln \frac{f(x)}{g(x)} = \ln f(x) - \ln g(x)$, we obtain the same solution set as for equation (3.3). Clearly, $x = 1$ is a solution to this equation, i.e., $h(1) = 0$. Using the derivatives

$$f'(x) = \frac{k(a+b)(a-c)f(x)}{(ax+b)(cx+a+b-c)}$$

and

$$g'(x) = \frac{(a+b)(a-c)g(x)}{k\sqrt[k]{x^{k-1}}\left[(a+b-c)\sqrt[k]{x} - b\right]\left(-c\sqrt[k]{x} + a\right)},$$

we have

$$\begin{aligned} h'(x) = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} &= \frac{(a+b)(a-c)}{k} \cdot \left(\frac{k^2}{(ax+b)(cx+a+b-c)}\right) \\ &- \frac{(a+b)(a-c)}{k} \cdot \frac{1}{\sqrt[k]{x^{k-1}}\left[(a+b-c)\sqrt[k]{x} - b\right]\left(-c\sqrt[k]{x} + a\right)}. \end{aligned}$$

Denoting $\sqrt[k]{x} = t$, we rewrite the derivative $h'(x)$ as

$$v(t) = \frac{(a+b)(c-a)p(t)}{kt^{k-1}(at^k+b)(ct^k+a+b-c)\left[(a+b-c)t - b\right](-ct+a)},$$

where

$$\begin{aligned} p(t) = &act^{2k} + k^2c(a+b-c)t^{k+1} - (k^2-1)(a^2+ab+bc-ac)t^k + \\ &+ k^2abt^{k-1} + b^2 + ab - bc. \end{aligned} \tag{3.6}$$

Let $k = 1$. Then the function $h'(x)$ gets always positive $(a < c)$ or negative $(a > c)$ value for any $x \in (x_1, x_2)$. Therefore, the only way for the function $h(x)$ to cross the x-axis is at the point $x = 1$.

Let $k \geq 2$. Then by Descartes' rule [28] of signs, the polynomial (3.6) has at most two positive roots. It is easy to verify that

$$\lim_{x\to x_1} h(x) = -\infty, \quad h(1) = 0, \quad \lim_{x\to x_2} h(x) = +\infty. \tag{3.7}$$

Hence, the equation $h(x) = 0$ has at least one solution x_0 for $x < 1$ and at least one solution x_0' for $x > 1$ if $h'(1) < 0$. From this condition,

$$h'(1) = \frac{(a+b)(a-c)}{k} \cdot \left(\frac{k^2}{(a+b)^2} - \frac{1}{(a-c)^2}\right) < 0, \tag{3.8}$$

we find that $k > \frac{a+b}{c-a}$ since $\frac{a+b}{c-a} > 0$. Moreover,

$$\lim_{x\to x_1} h'(x) = +\infty, \quad \lim_{x\to x_2} h'(x) = +\infty \tag{3.9}$$

for all $k > \dfrac{a+b}{c-a}$. From the condition $h'(1) < 0$, it follows that the function $h(x)$ has exactly two critical points ξ_1 and ξ_2 such that $x_1 < \xi_1 < 1$ and $1 < \xi_2 < x_2$ (see Figure 1).

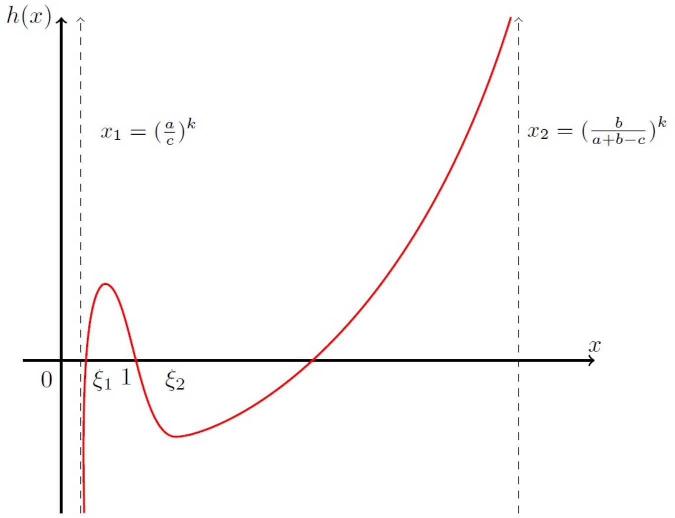

FIG. 1. The graph illustrates the number of possible solutions to the equation (3.3) for $a < c$ and $k > \dfrac{a+b}{c-a}$

This indicates that $h(x)$ is increasing on the intervals $x_1 < x < \xi_1$ and $\xi_2 < x < x_2$ and decreasing on the interval $\xi_1 < x < \xi_2$. Therefore, the equation $h(x) = 0$ has exactly two solutions except for 1, denoted as $x_0 < 1 < x_0'$.

Finally, since the function $f(x)$ is strictly decreasing for $a < c$, from the second equation of the system (3.1), we have $f(x_0) := y_0 > f(1) = 1 > f(x_0') := y_0'$. We conclude that $x_0 \neq y_0$ $(x_0' \neq y_0')$ for the pairs of solutions (x_0, y_0) and (x_0', y_0') corresponding to x_0 and x_0' respectively. Thus, the system of equations (3.1) has exactly three distinct solutions under the condition $k > \dfrac{a+b}{c-a}$.

The case $a > c$ is analogous to the case $a < c$, so we will provide a brief proof. In this case the function f is a strictly increasing function for $x > 0$. Assume that $y < x$ and (x, y) is a solution to (3.1). This would imply $f(x) < f(y)$ but due to the fact that f is strictly increasing, we would have $x < y$ which contradicts our assumption. The case $x < y$ proceeds analogously and consequently there can not be a solution $x \neq y$ if $a > c$.

Therefore, it suffices to consider the case of $x = y$ as the solutions to the system (3.1).

After denoting $\sqrt[k]{x} := z$, the system of equations (3.1) becomes the following equation

$$z = \frac{az^k + b}{cz^k + a + b - c}. \tag{3.10}$$

Alternatively, using the function (3.4), the equation (3.10) can be rewritten in the following form

$$z^k = g(z^k) = \frac{(a+b-c)z - b}{-cz + a}. \tag{3.11}$$

In this case, for the values $\sqrt[k]{x_1} := z_1$ and $\sqrt[k]{x_2} := z_2$ in the domain (z_1, z_2) of the function $g(z^k)$ the equations in (3.7) take the form

$$\lim_{z \to z_1} h(z) = +\infty, \quad h(1) = 0, \quad \lim_{z \to z_2} h(z) = -\infty.$$

The inequality corresponding to (3.8) becomes

$$h'(1) = \frac{(a+b)(a-c)}{k} \cdot \left(\frac{k^2}{(a+b)^2} - \frac{1}{(a-c)^2} \right) > 0, \tag{3.12}$$

and the derivative condition in (3.9) transforms to

$$\lim_{z \to z_1} h'(z) = -\infty, \quad \lim_{z \to z_2} h'(z) = -\infty,$$

for $k > \dfrac{a+b}{a-c}$. Under this condition, the system (3.1) admits exactly two solutions, apart from the trivial case $z = y = 1$ (see Figure 2).

□

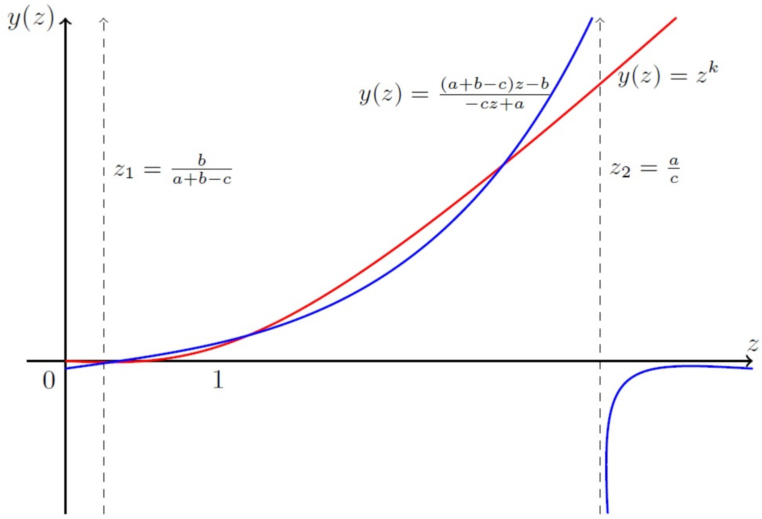

FIG. 2. The graph depicts the number of possible solutions to the equation (3.11) in the case $a > c$ and $k > \dfrac{a+b}{c-a}$

Definition 1. *[22] A family of vectors $\{l_{xy}\}_{\langle x,y\rangle \in \vec{L}}$, where $l_{xy} \in (0,\infty)^{\mathbb{Z}}$, is called a* boundary law for the transfer operators $\{Q_b\}_{b\in L}$ *if for each $\langle x,y\rangle \in \vec{L}$, there exists a constant $c_{xy} > 0$ such that the consistency equation*

$$l_{xy}(\omega_x) = c_{xy} \prod_{z\in\partial x\setminus\{y\}} \sum_{\psi_z\in\mathbb{Z}} Q_{zx}(\omega_x - \psi_z) l_{zx}(\psi_z) \tag{3.13}$$

holds for every $\omega_x \in \mathbb{Z}$. A boundary law is called q-periodic *if $l_{xy}(\omega_x + q) = l_{xy}(\omega_x)$ for every oriented edge $\langle x,y\rangle \in \vec{L}$ and each $\omega_x \in \mathbb{Z}$.*

Gradient measures and gradient Gibbs measures are constructed using q-periodic boundary laws on the space of gradient configurations (see Chapters 3 and 4 in [22]). Theorem 3.1 establishes that for a vertex $\Lambda \in \mathcal{N}$ and class label $s \in \mathbb{Z}_q$, any q-periodic boundary law $\{l_{xy}\}_{\langle x,y\rangle\in\vec{L}}$ for $\{Q_b\}_{b\in L}$ defines a consistent family of probability measures (pinned gradient measures) on Ω^{∇}. Chapter 4 discusses a spatially homogeneous boundary law, with the gradient Gibbs measure given by equation (4.3).

Let G_k be the free product of $k+1$ cyclic groups of order two, with generators $a_1, a_2, \dots, a_{k+1}$. It is known that there is a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k(see Proposition 1.1 in [4]).

Any element $x \in G_k$ has the following form

$$x = a_{i_1} a_{i_2} ... a_{i_n}, \quad \text{where} \quad 1 \le i_m \le k+1,\ m = 1, ..., n.$$

The number n is called the length of the word and the number of letters $a_i, i = 1, ..., k+1$, that enter the non contractible representation of the word x is denoted by $\omega_x(a_i)$. Let $N_k = \{1, \dots, k+1\}$, and define the set

$$H_A = \left\{ x \in G_k \mid \sum_{i\in A} \omega_x(a_i) \text{ is even} \right\}.$$

By Proposition 1.2 in [4], for any $\emptyset \ne A \subseteq N_k$, the set $H_A \subset G_k$ is a normal subgroup of index two.

Now, we define a spatially inhomogeneous boundary law associated with H_A (a H_A-boundary law), i.e., $\{l_{xy}\}_{\langle x,y\rangle\in\vec{L}} = \{l^{(1)}, l^{(2)}\}$ assuming $A = N_k$ as follows

$$l_{xy} = \begin{cases} l^{(1)}, & \text{if } x \in H_A \text{ and } y \in G_k \setminus H_A \\ l^{(2)}, & \text{if } y \in H_A \text{ and } x \in G_k \setminus H_A \end{cases}. \tag{3.14}$$

It is essential to observe that when $l^{(1)} = l^{(2)}$, the boundary conditions are spatially homogeneous [17, 21, 22]. Conversely, when $l^{(1)} \ne l^{(2)}$, the boundary conditions become spatially inhomogeneous, a phenomenon that is further investigated in this paper.

Now we define the vectors $z = (...z_{-2}, z_{-1}, 1, z_1, z_2, ...)$ and $t = (...t_{-2}, t_{-1}, 1, t_1, t_2, ...)$ for simplicity under the assumption $l_{xy}(0) \ne 0$ in the following way

$$\frac{l_{xy}(i)}{l_{xy}(0)} = \begin{cases} z_i, & \text{if } x \in H_A \text{ and } y \in G_k \setminus H_A \\ t_i, & \text{if } y \in H_A \text{ and } x \in G_k \setminus H_A \end{cases},$$

where $i \in \mathbb{Z}$.

Let G be a given graph. We specify the set of G-admissible configurations as follows.

Definition 2. *[14] A configuration ω is called a G-admissible configuration on the Cayley tree if $\{\omega_x, \omega_y\}$ is the edge of the graph G for any pair of nearest neighbors x, y in V.*

Let Ω_G denote the set of G-admissible configurations, Ω_G^{∇} indicate the set of G-admissible gradient configuration space and $L(G)$ be the set of edges of a graph G. We let $A \equiv A^G = (a_{ij})_{i,j\in\mathbb{Z}}$ denote the adjacency matrix of the graph G, i.e.,

$$a_{ij} = a_{ij}^G = \begin{cases} 1 & \text{if} \quad \{i,j\} \in L(G) \\ 0 & \text{if} \quad \{i,j\} \notin L(G) \end{cases}.$$

Applying the matrix A to the system of boundary law equations (3.13) for the SOS model, restricted to the set of G-admissible configurations, results in

$$\begin{cases} z_i = \left(\dfrac{a_{i0}\theta^{|i|} + \sum_{j\in\mathbb{Z}_0} a_{ij}\theta^{|i-j|}t_j}{a_{00} + \sum_{j\in\mathbb{Z}_0} a_{0j}\theta^{|j|}t_j} \right)^k \\ t_i = \left(\dfrac{a_{i0}\theta^{|i|} + \sum_{j\in\mathbb{Z}_0} a_{ij}\theta^{|i-j|}z_j}{a_{0,0} + \sum_{j\in\mathbb{Z}_0} a_{0j}\theta^{|j|}z_j} \right)^k \end{cases}, \tag{3.15}$$

where $i \in \mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$.

It should be noted that for any graph with the vertex set $\mathbb{Z}$, the system of equations (3.15) simplifies to the form (3.1). It is easily demonstrable that by altering the graph, one can derive parameter values b and c such that $b \neq c$. Specifically, the scenario where $b = c$ is examined for two selected graphs throughout the paper.

Let G_1 be the complete graph with vertex set $\mathbb{Z}$, where each vertex has a loop, i.e., $a_{ij} = 1$ for all $i, j \in \mathbb{Z}$. Using the transfer operator defined in the preliminaries and the parameter $\theta = e^{-J\beta}$ (with $0 < \theta < 1$), the system of equations (3.15) for our model becomes

$$\begin{cases} z_i = \left(\dfrac{\theta^{|i|} + \sum_{j\in\mathbb{Z}_0} \theta^{|i-j|}t_j}{1 + \sum_{j\in\mathbb{Z}_0} \theta^{|j|}t_j} \right)^k \\ t_i = \left(\dfrac{\theta^{|i|} + \sum_{j\in\mathbb{Z}_0} \theta^{|i-j|}z_j}{1 + \sum_{j\in\mathbb{Z}_0} \theta^{|j|}z_j} \right)^k \end{cases}. \tag{3.16}$$

We study the 2-periodic solutions of (3.16), assuming $u_i = \sqrt[k]{z_i}$ and $v_i = \sqrt[k]{t_i}$. In the 2-periodic case, the sequences are given by

$$l^{(1)} \sim (..., u_1, 1, u_1, 1, u_1, ...),$$
$$l^{(2)} \sim (..., v_1, 1, v_1, 1, v_1, ...).$$

By denoting $u_1 := x$ and $v_1 := y$ we obtain the following system of equations

$$\begin{cases} x = \dfrac{(\theta^2+1)y^k + 2\theta}{2\theta y^k + \theta^2 + 1} \\ y = \dfrac{(\theta^2+1)x^k + 2\theta}{2\theta x^k + \theta^2 + 1} \end{cases}. \tag{3.17}$$

Theorem 1. *Let $\theta_{cr} = \dfrac{\sqrt{k}-1}{\sqrt{k}+1}$ with $k \geq 2$. Then 2-height periodic boundary law of the type (3.14) determines 2-height periodic spatially homogeneous boundary law. Consequently, for the SOS model on Cayley tree of order k with the parameter $\theta \in (0, \theta_{cr})$ there exist precisely three 2-height periodic GGMs on $\Omega_{G_1}^{\nabla}$.*

Proof. We apply Proposition 1 with parameters $a = \theta^2 + 1$, $b = 2\theta$, and $c = 2\theta$. It's clear that we are in the regime where $a > c$ and consequently we obtain three different GGMs corresponding to the spatially homogeneous boundary law, i.e. $l^{(1)} = l^{(2)}$.

To determine the conditions for the existence of three gradient Gibbs measures corresponding to the subgroup H_A, we solve the following inequality as stated in part (2) of Proposition 1

$$k > \frac{(\theta+1)^2}{(\theta-1)^2}.$$

Through straightforward calculations, we establish the interval $0 < \theta < \theta_{cr}$ within which the system of equations (3.17) possesses exactly three solutions. Here, θ_{cr} is defined as $\theta_{cr} = \frac{\sqrt{k}-1}{\sqrt{k}+1}$ for $k \geq 2$. $\square$

Now we consider the graph G_2 containing $\mathbb{Z}$ as the vertices, i.e. one-dimensional lattice graph where additionally each vertex is connected to itself, considered in [14] (see Figure 3) with its adjacency matrix

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or } |i-j| = 1, \quad i,j \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}.$$

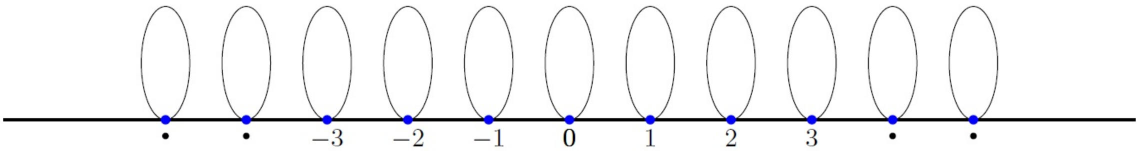

FIG. 3. The graph G_2 with the set $\mathbb{Z}$ of vertices

Then the system of equations (3.15) on the space Ω_{G_2} for the 2−periodic case becomes

$$\begin{cases} x = \left(\frac{y+2\theta}{2\theta y+1}\right)^k \\ y = \left(\frac{x+2\theta}{2\theta x+1}\right)^k \end{cases}. \tag{3.18}$$

Theorem 2. *Let $\theta_{cr}^- = \frac{k-1}{2k+2}$ for $k \geq 2$ and $\theta_{cr}^+ = \frac{k+1}{2k-2}$ for $k \geq 4$. Then for the SOS model restricted to a set of G_2-admissible configurations on the Cayley tree of order k the following assertions hold*

- *The boundary law (3.14) associated with H_A coincides with the spatially homogenous boundary law for $\theta \in (0, \theta_{cr}^-)$ which provides exactly three 2-height periodic GGMs on $\Omega_{G_2}^{\nabla}$.*
- *The boundary law (3.14) associated with H_A becomes spatially inhomogeneous for $\theta \in (\theta_{cr}^+, 1)$ resulting in three 2-height periodic pinned gradient measures on $\Omega_{G_2}^{\nabla}$.*

Proof. For the graph G_2 in Figure 3 we derive the parameters $a = 1$, $b = 2\theta$, and $c = 2\theta$ to apply Proposition 1 once more.

Case 1. Let $0 < \theta < \frac{1}{2}$. Then, it is evident that $a > c$, leading to three distinct GGMs corresponding to the spatially homogeneous boundary law, i.e., $l^{(1)} = l^{(2)}$. In this case, we use the inequality $k > \frac{a+b}{a-c}$ stated in part (2) of Proposition 1 in the form

$$k > \frac{2\theta+1}{1-2\theta}.$$

By solving last inequality, we obtain $0 < \theta < \theta_{cr}^-$, where $\theta_{cr}^- = \frac{k-1}{2k+2}$.

Case 2. Let $\frac{1}{2} < \theta < 1$. Then, it is evident that $a < c$, resulting in spatially inhomogeneous, i.e., $l^{(1)} \neq l^{(2)}$, boundary laws which always defines gradient measures by the equation (3.4) in [22]. In this case, the inequality $k > \frac{a+b}{c-a}$ stated in part (3) of Proposition 1 becomes

$$k > \frac{2\theta+1}{2\theta-1}.$$

Thus, it follows that the system of equations (3.18) has exactly three solutions, provided that $\theta_{cr}^+ < \theta < 1$, where $\theta_{cr}^+ = \frac{k+1}{2k-2}$. It is important to note that this condition on θ is valid when the order of the Cayley tree is strictly greater than 3. $\square$

Remark 1. *Are the two pinned gradient measures identified in Theorem 2 classified as gradient Gibbs measures (GGMs)? This question remains open.*

4. Conclusion

Our main goal is to quantify the number of pinned gradient measures for the SOS model on the Cayley tree of order $k \geq 2$ by analyzing boundary law equations (3.13) under certain temperature conditions. This work distinguishes itself from previous studies, which have focused on spatially homogeneous q-periodic boundary laws and their corresponding GGMs (see [14–19, 22]). The paper is organized as follows: we first prove Proposition 1, then use it to solve an infinite system of equations (3.13), i.e., to find 2-periodic boundary laws. In Theorem 1, we demonstrate the existence of three GGMs on the Cayley tree of order $k \geq 2$ for certain values of θ using different methods (see [17, 21, 22]). We also determine the critical temperature condition, i.e., $\theta \in (\theta_{cr}^{+}, 1)$, where spatially inhomogeneous boundary laws of period two defines pinned gradient measures for the SOS model restricted to the G_2-admissible configuration space.

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