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ATTRACTIVE OR REPULSIVE CASIMIR EFFECT AND BOUNDARY CONDITION

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The Casimir force between two identical bodies, although highly dependent on their geometry and structure of boundaries, is always attractive. However, this force can become repulsive if the nature of the two boundaries is different. We analyze from a global perspective the analytic properties of the Casimir energy function in the space of the consistent boundary conditions $\mathcal{M}_F$ for a massless scalar field confined between two homogeneous parallel plates. The analysis allow us to completely characterize the boundary conditions which give rise to attractive and repulsive Casimir forces. In the interface between both regimes there is a very interesting family of boundary conditions which do not generate any type of Casimir force. We also find Casimirless boundary conditions which are invariant under the renormalization group flow. The conformal invariant boundary conditions which do not generate a Casimir force have not yet been exploited in string theory but open new interesting possibilities.

Keywords: Casimir effect, vacuum energy, boundary conditions, boundary renormalization group flow.

1. Introduction

The Casimir effect is a genuine quantum phenomenon induced by the vacuum fluctuations of quantum fields. The pressure of quantum fluctuations on impermeable solid bodies generates a force between these bodies [1]. The characteristics of this force depend essentially on the type of field theory, the geometry of the bodies and the physical properties of the bodies boundaries [2–8]. However, the Casimir force between two identical bodies is always attractive due to the Kenneth-Klich theorem [9]. Now, if the boundary conditions of the bodies are different the Casimir force can become repulsive. In fact, new repulsive regimes of the Casimir effect have been found between dielectric plates and ingenious combinations of them, which give rise to a repulsive effect [10]. In this paper we analyze from a global viewpoint the dependence of the Casimir phenomenon on the physical properties of the boundary, i. e. on the boundary conditions satisfied by the quantum fields. Based on a new technique which highly simplifies the calculation of the vacuum energy for arbitrary boundary conditions we analyze the nature of Casimir force between two homogeneous parallel plates induced by the fluctuations of massless free scalar field.

In this geometry the only variable elements are the distance between the plates and the boundary conditions at the plates. The set of physically admissible boundary conditions is a four dimensional manifold which includes Dirichlet, Neumann, Robin, Zaremba, periodic, quasi-periodic and anti-periodic [11, 12]. Thus, we analyze the dependence of the Casimir force on these five variables and in particular, we analyze in great detail the transition from the attractive Casimir regime to the repulsive Casimir regime. We also analyze the role of Casimir energy as a finite size effect of the conformal anomaly and the role of boundary renormalization group flow. In particular, we search for boundary conditions which being invariant under renormalization do not present finite size effects which generate Casimir energy. In the case of massless scalar fields we fully characterize the three parameter family of boundary conditions satisfying both requirements.

2. Quantum Fields in Bounded Domains

In quantum theories the unitarity principle imposes severe constraints on the boundary behavior of quantum states of systems confined in bounded domains [11]. The consistency of the quantum field theory imposes a much more stringent condition on the type of acceptable boundary conditions in order to prevent any type of pathological behavior. In relativistic field theories, causality imposes further requirements [13, 14]. The space of boundary conditions compatible with both constraints has interesting global geometric properties.

The dynamics of free complex scalar field φ is governed by the following Hamiltonian

$$H = \frac{1}{2}\int_{\Omega} d^3\mathbf{x}\left(|\pi(\mathbf{x})|^2 + \phi^*(\mathbf{x})(-\boldsymbol{\Delta} + m^2)\,\phi(\mathbf{x})\right) \tag{1}$$

in terms of the Laplacian operator $\boldsymbol{\Delta}$ and the canonical momenta π of the fields ϕ satisfying the standard canonical quantization rules

$$[\pi(\mathbf{x}), \phi(\mathbf{x}')] = -i\,\delta(\mathbf{x} - \mathbf{x}'), \tag{2}$$

where we using natural units ($c = \hbar = 1$). The Hamiltonian (1) describes the dynamics of an infinite number of decoupled harmonic oscillators given by the Fourier modes of the operator $-\boldsymbol{\Delta} + m^2$. When the fields are confined in a bounded domain $\Omega \subset \mathbb{R}^3$ with regular boundary $\partial\Omega$ the modes of the oscillators become discretized and highly dependent on the boundary conditions of the fields specially for the low energy modes. The consistency conditions require that the corresponding oscillating frequencies must be real and positive, which can be fulfilled for any value of the mass if and only if all eigenvalues of the Laplacian operator $-\Delta$ are real and nonnegative, i.e. $-\Delta$ is a selfadjoint non-negative operator. This requirement imposes severe constraints on the conditions the fields must satisfy at the boundary $\partial\Omega$ in order to have a consistent quantum field theory. The positivity condition can be relaxed for a fixed geometry and given mass, but the boundary conditions which are consistent for any size of the domain Ω require the non-positivity of the selfadjoint extension of the Laplacian operator Δ.

The self-adjointness condition requires the cancellation of the probability flux across the boundary. This flux is given by

$$\Psi\left(\phi\right) = i\int_{\partial\Omega}\left[(\partial_n\phi^*)\phi - \phi^*\partial_n\phi\right]. \tag{3}$$

The set of boundary conditions which preserve the probability and have a null flux (3) across the boundary can be identified with the space of unitary operators U of $L^2(\partial\Omega)$, i.e. $U \in \mathcal{U}(L^2(\partial\Omega))$. This characterizationion is equivalent to that introduced by von Neumann, Krein [15–18] and by boundary triples approches [19–24]. Each selfadjoint extension is characterized by the restriction of the adjoint operator of $-\Delta^\dagger$ of the Laplacian operating only on smooth functions of compact support in Ω, to the domain [11, 12]

$$\mathcal{D}_U = \{\psi \in \mathcal{D}(\Delta^\dagger);\ (\varphi, \dot{\tilde{\varphi}}) \in H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega);\ \underline{\varphi} - i\underline{\dot{\varphi}} = U(\underline{\varphi} + i\underline{\dot{\varphi}})\}, \tag{4}$$

where φ denotes the boundary value of ψ and $\dot{\tilde{\varphi}}$ the boundary normal derivative of $(1-\Delta^\dagger)^{-1}(\psi - \Delta^\dagger\psi)$ and

$$\underline{\varphi} = \frac{1}{\sqrt{\delta}}\left(-\Delta^\dagger_{\partial\Omega} + \frac{1}{\delta^2}\mathbb{I}\right)^{-\frac{1}{4}}\varphi \quad \text{and} \quad \underline{\dot{\varphi}} = \sqrt{\delta}\left(-\Delta^\dagger_{\partial\Omega} + \frac{1}{\delta^2}\mathbb{I}\right)^{\frac{1}{4}}\dot{\tilde{\varphi}}, \tag{5}$$

δ being an arbitrary dimensionfull parameter. The Sobolev space $H^k(\partial\Omega)$ is defined as the closure of the subspace $C^\infty(\partial\Omega)$ of differentiable functions, with respect to the Sobolev norm of class k

$$\| \varphi \|_k^2 = \int_{\partial\Omega} d^2x \sqrt{g_{\partial\Omega}}\, \varphi(x)^\dagger(-\Delta_{\partial\Omega} + \mathbb{I})^k \varphi(x). \tag{6}$$

The boundary conditions (4) define selfadjoint extensions of the Laplacian but the consistency of the quantum field theory also requires the positivity of the corresponding selfadjoint operator and this condition imposes further constraints. In particular, since [11]

$$-(\psi, \Delta_U \psi) = |\vec{\nabla}\psi|^2 - (\varphi, \dot{\varphi}) = |\vec{\nabla}\psi|^2 - \left(\underline{\varphi}, \underline{\dot{\varphi}}\right) = |\vec{\nabla}\psi|^2 - \left(\underline{\varphi}, A\underline{\varphi}\right),$$

where

$$A = -i\frac{\mathbb{I} - U}{\mathbb{I} + U}, \tag{7}$$

is the selfadjoint Cayley transform of U. Whenever A is well defined it must be selfadjoint and non-positive. Since all eigenvalues of unitary operators U are of the form $\lambda(\alpha) = \mathrm{e}^{i\alpha}$ with $\alpha \in [0, 2\pi]$, the positivity condition of A translates into the following constraint on the eigenvalues of U

$$\tan\left(\frac{\alpha}{2}\right) \geqslant 0; \quad \text{i.e. } 0 \leqslant \alpha \leqslant \pi. \tag{8}$$

Thus, the space of boundary conditions $\mathcal{M}_F$ which lead to consistent quantum field theories is given by [13, 14]

$$\mathcal{M}_F \equiv \left\{U \in \mathcal{U}\left(L^2(\partial M)\right); \quad \lambda = \mathrm{e}^{i\alpha} \in \sigma(U),\ 0 \leqslant \alpha \leqslant \pi\right\}. \tag{9}$$

In the particular case of a domain bounded by two parallel homogeneous plates the translation invariance along the plates imply that U should be constant on each plate. In that case the most general boundary condition for the fields is given by a two dimensional unitary matrix $U \in U(2)$, which can be written in terms of Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$$\begin{aligned} U(\alpha, \beta, \mathbf{n}) &= \mathrm{e}^{i\alpha}\left(\cos(\beta)\mathbb{I} + i\sin(\beta)\,\mathbf{n}\cdot\boldsymbol{\sigma}\right) \\ &\quad \alpha \in [0, 2\pi],\ \beta \in [-\pi/2, \pi/2], \end{aligned} \tag{10}$$

and an unitary vector

$$\mathbf{n}(\theta, \chi) = (\sin(\theta)\cos(\chi), \sin(\theta)\sin(\chi), \cos(\theta)); \quad \theta \in [0, \pi],\ \chi \in [0, 2\pi]. \tag{11}$$

of the unit sphere S^2 satisfying the constraints $0 \leqslant \alpha \pm \beta \leqslant \pi$. The domain of the positive self-adjoint operator $-\Delta_U$ is given by

$$\begin{pmatrix} \varphi(0) - iL\dot{\varphi}(0) \\ \varphi(1) - iL\dot{\varphi}(1) \end{pmatrix} = \mathrm{e}^{i\alpha} \begin{pmatrix} \cos\beta + in_3\sin\beta & (in_1 + n_2)\sin\beta \\ (in_1 - n_2)\sin\beta & \cos\beta - in_3\sin\beta \end{pmatrix} \begin{pmatrix} \varphi(0) + iL\dot{\varphi}(0) \\ \varphi(1) + iL\dot{\varphi}(1) \end{pmatrix},$$

where L is the distance between the plates. This property guarantees the positivity of the operator $-A$ and in consequence that of the corresponding selfadjoint extension of the Laplacian. Therefore, the space of consistent boundary conditions $\mathcal{M}_F$ is [13, 14]

$$\mathcal{M}_F \equiv \{U(\alpha,\ \beta,\ \mathbf{n}) \in U(2)\,|\, 0 \leqslant \alpha \pm \beta \leqslant \pi\}. \tag{12}$$

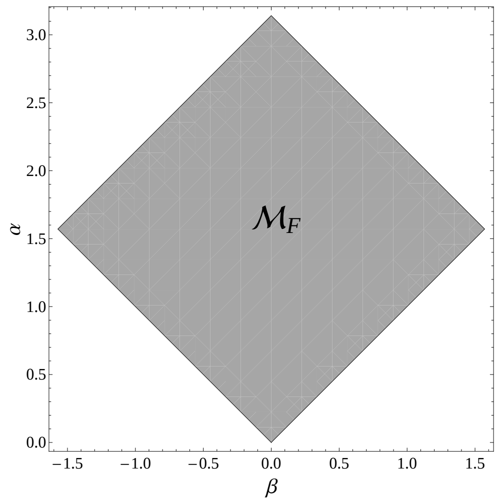

Fig. 1. Space of consistent boundary conditions for a scalar field theory confined between two homogeneous parallel plates

3. Boundary Renormalization Group Flow

Some symmetries of the classical theory can be broken upon quantization by quantum interactions. In the case of fields confined in bounded domains only the symmetries which leave the boundary invariant can be preserved for some boundary conditions. However, in the case of scale invariance ($x \to x/\Lambda$) the presence of the boundaries does not automatically imply the breaking of the symmetry at the quantum level because the rescaling involved in the Wilson renormalization group transformation restores the system back to the same boundary domain Ω. Now, the scale invariance in the massless quantum field theory can still be broken because not all boundary conditions preserve this symmetry. In fact, it has been shown in Refs. [25, 26] that the renormalization group acts on the space of boundary conditions according to the flow

$$\Lambda\, U_\Lambda^\dagger \partial_\Lambda U_\Lambda = \frac{1}{2}\left(U_\Lambda^\dagger - U_\Lambda\right) \tag{13}$$

or

$$U_t^\dagger \partial_t U_t = \frac{1}{2}\left(U_t^\dagger - U_t\right) \tag{14}$$

for $\Lambda = \Lambda_0\, \mathrm{e}^t$.

The only boundary conditions which preserve scale invariance are the fixed points of the renormalization group flow (14), i.e. boundary conditions whose unitary operators U are hermitian unitary matrices $U^\dagger = U = U^{-1}$ [26].

In the parametrization given by (10) and using spherical coordinates (11) for the normal unit vector $\mathbf{n}(\theta, \varphi)$ the flow reads (see Fig. 2)

$$\alpha'(\Lambda) + \frac{1}{\Lambda}\sin(\alpha)\cos(\beta) \;=\; 0; \tag{15}$$

$$\beta'(\Lambda) + \frac{1}{\Lambda}\cos(\alpha)\sin(\beta) \;=\; 0; \tag{16}$$

$$\theta'(\Lambda) = \chi'(\Lambda) \;=\; 0, \tag{17}$$

which defines a vector field that can be extended to the whole group $U(2)$.

Thus all fixed points are located at the corners of the rhombus in figure 1 of $\mathcal{M}_F$. The upper and lower corners correspond to Dirichlet and Neumann ($U = \mp\mathbb{I}$) boundary conditions.

The other fixed points are located in the other two corners and correspond to a S^2 manifold given by

$$U = \mathbf{n} \cdot \boldsymbol{\sigma} \tag{18}$$

$\mathbf{n}$ being an arbitrary unit vector of $\mathbb{R}^3$, which includes pseudo-periodic and quasiperiodic boundary conditions.

For mixed boundary conditions the RG flows from Dirichlet boundary conditions (ultraviolet fixed point) toward Neumann boundary conditions (infrared fixed point) [25, 26].

$$U = \mathrm{e}^{2\mathrm{i}\arctan \mathrm{e}^{-t}}\mathbb{I}. \tag{19}$$

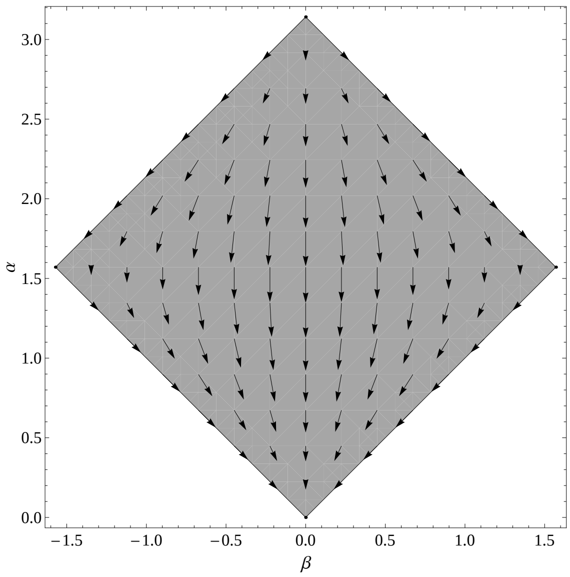

Fig. 2. Renormalization group flow in $\mathcal{M}_F$. Notice that fixed points are located at the corners of the rhombus. Neumann boundary conditions are at the lowest corner which is the only stable fixed point. This point is an attractor point of the whole renormalization group flow

4. Vacuum energy.

The Casimir effect in massless quantum theories is a consequence of the scale symmetry anomaly which arises in the form of finite size corrections to the vacuum energy. Within the global framework of boundary conditions formulated above it is possible to analyze with complete generality the characterization of attractive and repulsive regimes generated by this anomaly.

The scalar free field theory defined by a boundary condition U of $\mathcal{M}_F$ has a unique vacuum state which in the functional Schrödinger representation corresponds to the Gaussian state

$$\Psi(\phi) = \mathcal{N}\, \mathrm{e}^{-\frac{1}{2}(\phi,\, \sqrt{-\Delta_U}\, \phi)} \tag{20}$$

where $\mathcal{N}$ is a normalization constant and $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ product. The corresponding vacuum energy given by the sum of the eigenvalues of $\frac{1}{2}\sqrt{-\Delta_U}$ is ultraviolet divergent, but the Casimir effect is associated to some finite volume corrections to the vacuum energy which are

UV finite and universal. It is convenient to regularize the UV divergences involved in the sum of the trace by means of the heat kernel method

$$E_U = E_U^{(L,\epsilon)} = \frac{1}{2}\mathrm{tr}\sqrt{-\Delta_U}\mathrm{e}^{\epsilon\Delta_U} \tag{21}$$

The Casimir energy can be identified from the asymptotic expansion in powers of $\frac{\sqrt{\epsilon}}{L}$ of the vacuum energy per unit plate area [27],

$$\frac{E_U^{(L,\epsilon)}}{A} = \frac{c_0}{\epsilon^2}L + \frac{c_1}{\epsilon^{3/2}} + \frac{c_U}{L^3} + \mathcal{O}\left(\frac{\sqrt{\epsilon}}{L}\right). \tag{22}$$

The eigenvalues $\lambda_n = (k^1)^2 + (k^2)^2 + k_n^2$ of the Laplacian operator $-\Delta_U$ are given in terms of the zeros k_n of the spectral function [14,28]

$$\begin{aligned} h_U(k) &= 4k\det U\cos kL - 2i(1+k^2)\det U\sin kL + 4k(U_{21}+U_{12}) \\ &\quad -2i(1+k^2)\sin kL - 4k\cos kL + 2i(1-k^2)\mathrm{tr}U\sin kL \\ &= 4ke^{2i\alpha}\cos kL - 2i(1+k^2)e^{2i\alpha}\sin kL + 8in_1ke^{i\alpha}\sin\beta \\ &\quad -2i(1+k^2)\sin kL - 4k\cos kL + 4i(1-k^2)e^{i\alpha}\cos\beta\sin kL. \end{aligned} \tag{23}$$

and two arbitrary real parallel components k^1, k^2. The spectral function $h_U(k)$ is obtained from the determinant of the coefficients of the eigenvalue equation of $-\Delta_U$ for plane waves with momenta $(0,0,k)$. Notice that the spectral function is not only dependent on the invariants of the boundary unitary matrix $\det U$ and $\mathrm{tr}\, U$ but also in the entries U_{21} and U_{12}, which implies that the spectrum of the quantum theory may be different for matrices with the same spectrum if they are non-equivalent as matrices. The vacuum energy can be formally given in terms of the spectral function h_U [28] (see [29,30]) for an historical review) by

$$E_0 = \frac{1}{24\pi^2 i}\oint dz\, z^3\,\partial_z \log h_U(z) \tag{24}$$

or equivalently

$$E_0 = -\frac{1}{12\pi^2}\int_0^\infty dk\, k^3\,\frac{d}{dx}\log h_U(ik). \tag{25}$$

It is straightforward to extract the leading terms of the asymptotic expansion in $1/L$ by substracting the divergent values for a fixed reference value $L_0 << L$ of the distance between the plates [13,14]

$$c_U = \frac{-L_0^3}{12\pi^2(L^3-L_0^3)}\int_0^\infty dk\, k^3\left[L - L_0 - \frac{d}{dk}\log\left(\frac{h_U^{(L)}(ik)}{h_U^{(L_0)}(ik)}\right)\right]. \tag{26}$$

From this expression it is possible to compute in a very efficient way the Casimir energy for arbitrary boundary conditions $U \in \mathcal{M}_F$.

In some cases the Casimir energy can be computed analytically [13,14,28,31–34]:

(1) In the case of pseudo-periodic boundary conditions

$$U_{\mathrm{pp}} = \cos\xi\sigma_1 - \sin\xi\sigma_2; \quad \varphi(L) = e^{i\xi}\xi(0) \tag{27}$$

we have that

$$h_{\mathrm{pp}} = -8k(\cos kL - \cos\xi) \tag{28}$$

and

$$E_{\mathrm{pp}}(\xi) = \frac{A}{L^3}\left(-\frac{\pi^2}{90} + \frac{\xi^2}{12} - \frac{\xi^3}{12\pi} + \frac{\xi^4}{48\pi^2}\right); \quad \xi \in [0,\, 2\pi]. \tag{29}$$

They contain two special cases, when $\xi = 0$ we have periodic boundary conditions,

$$E_{\mathrm{p}} = -\frac{\pi^2 A}{90 L^3}, \tag{30}$$

where the Casimir force is attractive, and when $\xi = \pi$ anti-periodic boundary conditions with repulsive Casimir force

$$E_{\mathrm{ap}} = \frac{7\pi^2 A}{720 L^3}. \tag{31}$$

(2) In the case of Dirichlet boundary conditions

$$U_D = -\mathbb{I} \tag{32}$$

$$h_D = -8i \sin kL \tag{33}$$

the Casimir energy per unit area

$$E_{\mathrm{D}} = -\frac{\pi^2 A}{1440 L^3} = E_{\mathrm{N}} \tag{34}$$

is the same as for Neumann boundary conditions

$$U_N = \mathbb{I} \tag{35}$$

with

$$h_N = -8ik^2 \sin kL. \tag{36}$$

In both cases the Casimir force is attractive.

(3) A different family of boundary conditions is provided by quasi-periodic boundary conditions

$$\begin{aligned} &U_q = \cos\theta\sigma_3 + \sin\theta\sigma_1 \\ &\varphi(L) = \tan\frac{\theta}{2}\,\varphi(0); \quad \partial_n\varphi(0) = \left(L\tan\frac{\theta}{2}\right)^{-1}\varphi(0), \end{aligned} \tag{37}$$

where

$$h_q = -8k(\cos kL - \sin\theta). \tag{38}$$

The Casimir energy per unit area in this case is

$$E_q = \frac{A}{L^3}\left(\frac{127\pi^2}{11520} - \frac{3\pi\theta}{32} - \frac{11\theta^2}{96} - \frac{4\theta^3 + |\pi - 2\theta|^3}{96\pi} + \frac{\theta^4}{48\pi^2}\right); \qquad \theta \in [0,\, \pi]\,. \tag{39}$$

Here there are three special cases, $\theta = \frac{\pi}{2}$ which corresponds to periodic boundary conditions, with attractive Casimir force, $\theta = -\frac{\pi}{2}$ which corresponds to anti-periodic boundary conditions, with repulsive Casimir force and $\theta = 0$ to Zaremba boundary conditions also with repulsive behavior

$$E_{\mathrm{z}} = \frac{7\pi^2 A}{11520 L^3}. \tag{40}$$

In summary, many of the conditions (e.g. Dirichlet, Neumann, Periodic) give rise to attractive forces between the plates, others (e.g. antiperiodic, Zaremba) induce repulsive forces, and between these two types of boundary conditions there exist a family of boundary conditions with no Casimir force [28]. In the case of quasi-periodic boundary conditions there are two values of the parameter $\theta_{\rm cp}$

$$\theta_{\rm cp}^{\pm} = \pi\left(\frac{1}{2} \pm \left(1 - \sqrt{1 - 2\sqrt{\frac{2}{15}}}\right)\right). \tag{41}$$

where the Casimir energy vanishes which signals the boundary between attractive and repulsive regimes of the Casimir effect. Indeed, when $0 \leqslant \theta < \theta_{\rm cp}^{-}$ and $\theta_{\rm cp}^{+} \leqslant \theta < \pi$ the Casimir energy is positive, and hence the Casimir force between plates has a repulsive character. On the other hand, when $\theta_{\rm cp}^{-} \leqslant \theta < \theta_{\rm cp}^{+}$, the Casimir force between plates becomes attractive, which corresponds to a negative Casimir energy. Notice that for boundary conditions which correspond to identical plates, ($\beta = 0$) the Casimir energy is always negative which agrees with the Kenneth-Klich theorem [9].

Something similar occurs in the case of pseudo-periodic boundary conditions where there are two values of ξ with null Casimir energy, and therefore there is no force between plates

$$\xi_{\rm pp}^{\pm} = \pi\left(1 \pm \sqrt{1 - 2\sqrt{\frac{2}{15}}}\right). \tag{42}$$

In this case, for $\xi_{\rm pp}^{-} < \xi < \xi_{\rm pp}^{+}$ the Casimir energy is negative, which leads an attractive Casimir force between plates, and for $-\pi < \xi < \xi_{\rm pp}^{-}$ or $\xi_{\rm pp}^{+} < \xi < \pi$, the Casimir energy is positive, and hence the force between plates is repulsive.

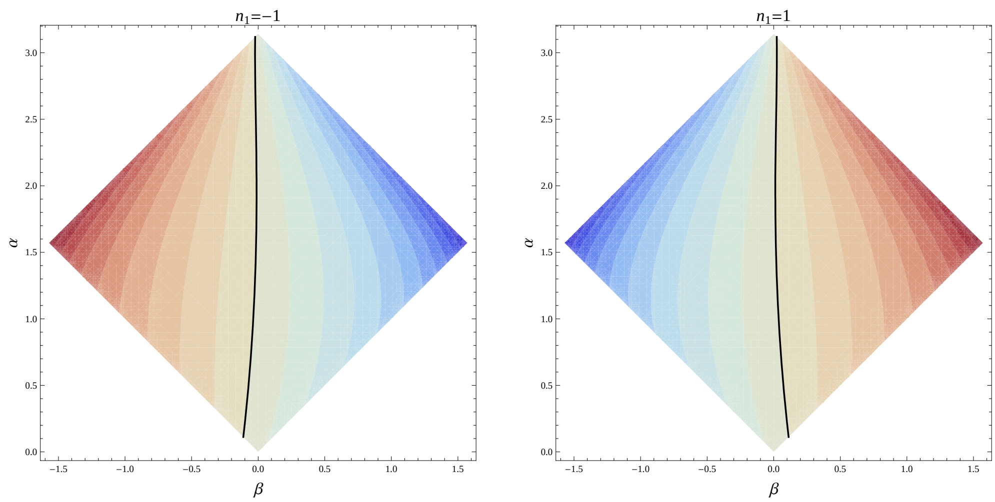

Fig. 3. [Color online] Behavior of the Casimir energy E_U in the consistency region $|\beta| < \alpha < \pi - |\beta|$, for $n_1 = \pm 1$. The Casimirless components are marked as thick lines. Blue colored regions correspond to negative values, and red colored to positive values

For more general boundary conditions it is possible to numerically evaluate the Casimir energy. In this way we find the complete set of boundary conditions which give rise to attactive Casimir forces and those which give rise to repulsive forces. The interface between both regimes correspond to boundary conditions which do not produce any Casimir force between the plates (*Casimirless boundary conditions*). The total set of Casimirless conditions is a connected subspace but its restriction for fixed values of n_1 can have one or two connected components.

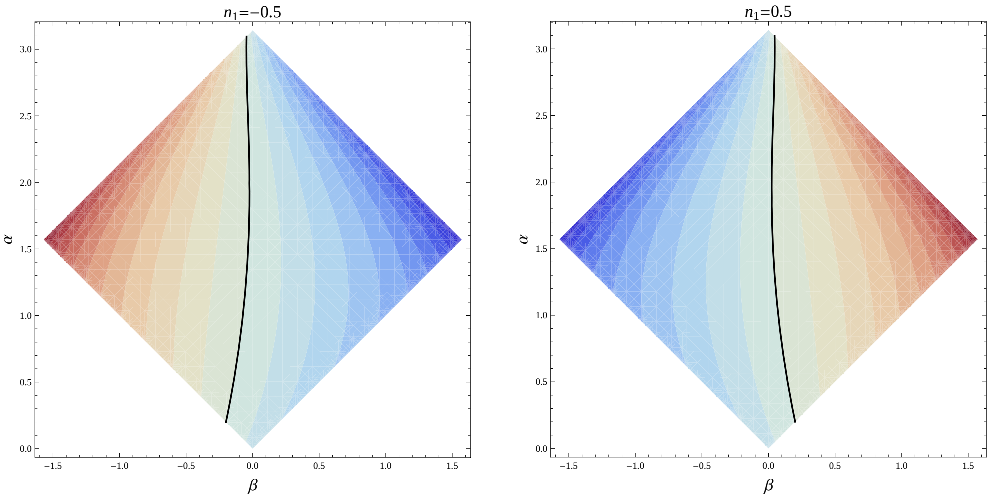

Fig. 4. [Color online] Behavior of the Casimir energy E_U in the consistency region $|\beta| < \alpha < \pi - |\beta|$, for $n_1 = \pm 0.5$. The Casimirless component appears as thick lines. Blue colored regions correspond to negative values, and red colored to positive values

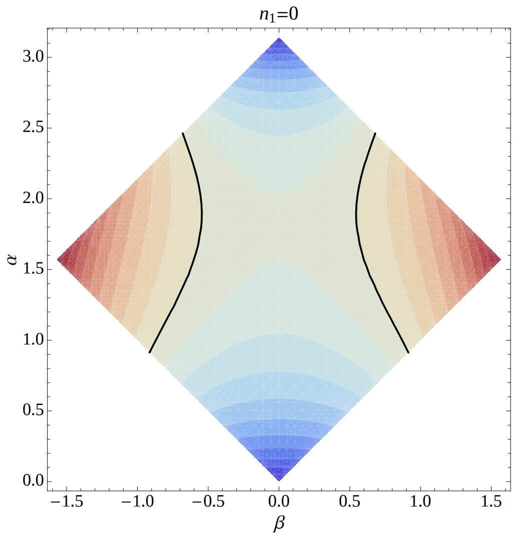

Fig. 5. [Color online] Behavior of the Casimir energy E_U in the consistency region $|\beta| < \alpha < \pi - |\beta|$, for $n_1 = 0$. The Casimirless components are marked with thick lines. Blue colored regions correspond to negative values, and red colored to positive values

Figures 3 and 4 show the behavior of the Casimir energy for the values $n_1 = \pm 1, \pm 0.5$, where there is only one connected component of Casimirless conditions.

The values of n_1 for which there are two connected components of Casimirless conditions are values close to $n_1 = 0$. In figure 5 can be seen in the case $n_1 = 0$, which has two Casimirless connected components. As long as n_1 goes towards to $n_1 = 0$ the subset of boundary conditions with zero Casimir energy changes its topology from one to two connected components. It is just at the transition point

$$n_1 = \cos \pi \left(1 \pm \sqrt{1 - 2\sqrt{\frac{2}{15}}}\right), \tag{43}$$

where the change occurs.

One particular case of interest is the case of fixed points of the renormalization group which are saddle points and are located at left and right corners of the rhombus of $\mathcal{M}_F$, i.e. boundary conditions corresponding to the points on the unit sphere S^2 for values $\alpha = \pm\beta = \frac{\pi}{2}$. This includes periodic, anti-periodic, quasi-periodic, and pseudo-periodic boundary conditions. The Casimir energy per unit area

$$E_{\rm sp}(n_1) = \frac{1}{L^3}\left(-\frac{\pi^2}{90} + \frac{(\arccos n_1)^2}{12} - \frac{(\arccos n_1)^3}{12\pi} + \frac{(\arccos n_1)^4}{48\pi^2}\right); \quad \arccos n_1 \in [0,\, 2\pi]. \tag{44}$$

has two attractive and repulsive regimes separated by a one dimensional circle of Casimirless boundary conditions given by

$$\alpha = \beta = \frac{\pi}{2}; \quad n_1 = \cos \pi \left(1 \pm \sqrt{1 - 2\sqrt{\frac{2}{15}}}\right). \tag{45}$$

We remark that in any case the submanifold of Casimirless boundary conditions only intersects the manifold of fixed points at the S^2 sphere of saddle fixed points of the renormalization group flow. Obviously, Dirichlet and Neumann boundary conditions have always a non-vanishing atractive Casimir energy. On the other hand, boundary conditions for identical plates correspond to $\beta = 0$ and from the numerical calculations it is shown that all these boundary conditions are always in the attractive regime as shown in figures 3 4 and 5, which in agreement with the Kenneth-Klich theorem [9].

In summary, the powerful method based on the use of the spectral function for the calculation of the Casimir effect permits to analyze from a global perspective the properties of the Casimir energy as a function over the space of consistent boundary conditions $\mathcal{M}_F$. Some of these global properties have a great physical interest. In particular, it is possible to find the extremal values of the Casimir energy $E_c(U)$ over the space $\mathcal{M}_F$. The continuity of $E_c(U)$ in $\mathcal{M}_F$ and the compactness of $\mathcal{M}_F$ imply the existence of at least two types of consistent boundary conditions where the Casimir energy attains its maximal and minimal values. In our case the minimum of Casimir energy corresponding to periodic boundary conditions whereas the maximum value of the Casimir energy corresponds to anti-periodic boundary conditions.

The fact that the maximum and the minimum of the Casimir energy are at the boundary of the space of consistent boundary conditions $\partial\mathcal{M}_F$ can be inferred from the analytic properties of the spectral function $h_U(k)$ which provides a bound for the Casimir energy $E_c(\alpha,\, \beta,\, n_1)$ in the interior of $\mathcal{M}_F$. The restriction of $h_U(k)$ to $\mathcal{M}_F \cap SU(2)$ is an harmonic function for any k and, thus, by the maximum principle all its local extrema must be at the boundary $\partial\mathcal{M}_F$. This behavior can be translated to the Casimir energy. The same argument implies the existence of extreme boundary conditions in the attractive and repulsive regimes of Casimir effect. In both cases the extreme values of E_c are reached in $\partial\mathcal{M}_F$, the minimum value corresponds to periodic boundary conditions which provide strongest attractive force between plates, and the maximum value is attained for anti-periodic boundary conditions which represents the strongest repulsive force between plates.

The global analysis of the dependence of infrared properties of field theories on the nature of boundary conditions also unveils many interesting physical effects. However, the characteristics of boundary conditions which encode the attractive or repulsive nature of the Casimir energy are still unknown, although the algorithm found in the previous section provides the simplest mechanism to determine such a character. On the other hand it will be very interesting to understand the special role of the Casimirless boundary conditions which are also fixed points of the renormalization group (45). Even if these Casimirless conformal invariant conditions are physically unstable under renormalization group flow they provide a new set of conformally invariant boundary conditions which are anomaly free (45). The existence of similar conditions in 1+1 dimensions opens a new approach for the study of string theory in non-critical dimensions. The role of such conformal boundary conditions in the corresponding string theory deserves further study.

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