NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2013, 4 (4), P. 545–549

NEGATIVE EIGENVALUES OF THE Y-TYPE CHAIN OF WEAKLY COUPLED BALL RESONATORS

A. S. Anikevich

St. Petersburg National Research University of Information Technologies, Mechanics and Optics, St. Petersburg, Russia

shadowpoos@gmail.com

PACS 02.30.Tb

Spectral properties of a system are strongly associated with its geometry. The spectral problem for the Y-bent chain of weakly-coupled ball resonators is investigated. The Y-bent system can be described as a central ball linking three chains consisting of balls of the same radius. There is a δ -coupling condition with parameter α at every contact point. Specifically, it is assumed that the axis passing through the center of each ball lies in the same plane and the centers of balls that are the closest to the central ball form an equilateral triangle. The transfer-matrix approach and the theory of extensions are employed to solve the spectral problem for this system. It is shown that such system with a certain value of parameter α has at most one negative eigenvalue in the case of δ -coupling in contact points.

Keywords: negative eigenvalue, delta-coupling, operator extensions theory.

1. Introduction

The interrelation between the geometry of a system and its spectral properties is one of the most frequently asked questions that arises during research. For example, such a problem for quantum graphs is widely discussed, see, e.g., [1] – [5] etc. The problem is especially interesting when one deals with a so-called decorated graph (see, e.g., [4]). The subject of the present paper is close to this; namely, a system of coupled balls is considered. More precisely, a Y-type branching chain is studied. We assume that the Neumann Laplacian is defined inside the balls. The coupling is constructed by using of the theory of self-adjoint extensions of symmetric operators ([6] – [11]). The most intriguing question is about the existence of eigenvalues for the system. In the paper, we proved the existence of negative eigenvalues under certain conditions.

Let us describe briefly in general terms the problem under consideration. The geometry of the Y-type chain is shown at figure 1. The elementary cell of this system is a ball of unit radius. So, the system can be described as the "central"ball connected through the contact points with three semi-infinite direct chains that consist of similar balls.

To simplify considerations, we assume that there are no external fields. The system of units with $\hbar = 2m = 1$ is used. To show the method of solving such problem, a simple example of Y-geometry with additional assumptions is discussed:

1. for each branch, there is an axis passing through the centers of the balls forming this branch and all three axes lie in a single plane;
2. balls are numbered as shown at figure 2;
3. centers of the balls No (1,1); (2,1) and (3,1) are the vertices of an equilateral triangle.

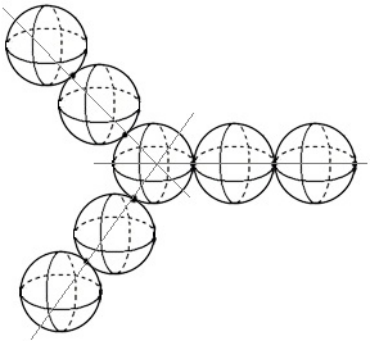

FIG. 1. Scheme of Y-type chain.

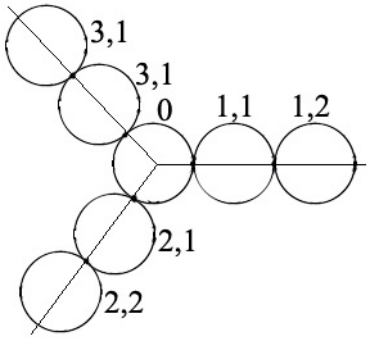

FIG. 2. Basic Y-type chain.

The described geometry of the system will subsequently be called "basic Y-type chain".

The weak coupling between resonators in our case should be considered as follows: all the interactions between elementary cells occur through pinholes (cf, [6] – [13]) at the contact points of resonators.

The wave function $\psi(x)$ of the stationary state of a spinless non-relativistic particle satisfies the stationary Schrödinger equation:

$$\hat{H}(x)\psi(x) = \lambda\psi(x).$$

And it is also considered that the Neumann boundary condition is satisfied at the boundary of the ball:

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0.$$

We use "restriction-extension"procedure, i.e., first, one constructs a symmetric restriction of the initial self-adjoint operator. Then, one considers its self-adjoint extensions, paying attention to those differing from the initial one. We restrict the operator $-\Delta$ on the set of all functions from $D(-\Delta)$ that are equal to zero near the contact points $\mathbf{x}_{i,j}$ (index i refers to chain's number, index j — to ball's number in a chain and the central ball has index 0). Thus, the symmetrical operator $-\Delta_0$ is obtained and its deficiency elements are Green's functions $G(\mathbf{x}, \mathbf{x}_j, \lambda)$ (cf, [14]). In our case, the Green's function written in spherical coordinates has the following form:

$$G\left(r,\theta,\varphi,r_j,\theta_j,\varphi_j,\lambda\right) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\Psi_{klm}\left(r,\theta,\varphi\right)\overline{\Psi_{klm}\left(r_j,\theta_j,\varphi_j\right)}}{\lambda_{lk}-\lambda},$$

where $\lambda_{lk} = x_{lk}^2$ are the eigenvalues, $\Psi_{klm}\left(r,\theta,\varphi\right) = N_{lk}j_l\left(x_{lk}r\right)Y_m^l\left(\theta,\varphi\right)$ are the eigenfunctions of the Neumann problem for the Laplace operator inside the ball, $Y_m^l\left(\theta,\varphi\right)$ are the spherical harmonics, $j_l\left(x\right)$ are the spherical Bessel's functions, x_{lk} is the k^{th} root of the following equation: $j_l'\left(x\right) = 0$, N_{lk} are the normalization coefficients.

As mentioned earlier, operator $-\Delta_0$ is a symmetrical operator, but it is not a self-adjoint operator. Keeping in mind that all self-adjoint extensions of $-\Delta_0$ are restrictions of the adjoint operator, we construct the operator $-\Delta_0^*$.

Any function U, $U \in D\left(-\Delta_0^*\right)$, has the following form:

$$U = U_0 + \frac{a}{|\mathbf{x}-\mathbf{x}_0|} + b,$$

where $U_0 \in D\left(-\Delta_0^*\right)$, $U_0\left(\mathbf{x}_0\right) = 0$; a, b are some coefficients. Henceforth, the following designations will be used: $a_{i,j}^+$, $a_{i,j}^-$ are the coefficients corresponding to pinholes in the ball

boundary with number i, j as you can see at the figure 3 (coefficients $b^+_{i,j}$, $b^-_{i,j}$ are denoted by a similar argument).

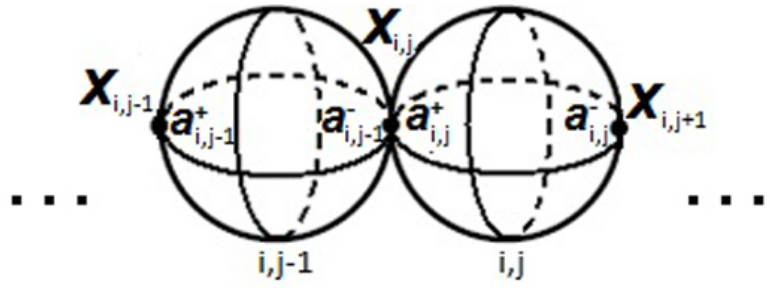

FIG. 3. Scheme of direct chain: coefficient's denotation

Using these notations one can describe a self-adjoint extension by the following relations between the coefficients:

$$\begin{cases} a^+_{i,j} = -a^-_{i,j-1}, \\ b^+_{i,j} = b^-_{i,j-1}. \end{cases} \tag{1}$$

Using these notations, one can describe a self-adjoint extension by the following relations between the coefficients:

$$\begin{cases} a^+_{i,j} = -a^-_{i,j-1}, \\ b^+_{i,j} = b^-_{i,j-1}. \end{cases} \tag{2}$$

We assume additionally that there are point-like potentials at the coupling points.

This letter discusses these results only. Full proofs will be published subsequently in a more detailed paper.

2. δ -coupling

The so-called δ -coupling imposes slightly different requirements for coefficients $a^\pm_{i,j}$, $b^\pm_{i,j}$. Resonators are supposed to be connected at the contact points by δ-coupling with the parameter α, so the following relations for the coefficients (instead of (2)) are considered:

$$\begin{cases} a^+_{i,j} = -a^-_{i,j-1}, \\ b^+_{i,j} - b^-_{i,j-1} = \alpha a^-_{i,j-1}, \end{cases} \tag{3}$$

where coefficients $b^+_{i,j}$, $b^-_{i,j-1}$ have the following form:

$$b^+_{i,j} = a^+_{i,j} \lim_{\mathbf{x}\to\mathbf{x}_{i,j}} \left(G\left(\mathbf{x}, \mathbf{x}_{i,j}, \lambda\right) - G\left(\mathbf{x}, \mathbf{x}_{i,j}, \lambda_0\right)\right) + a^-_{i,j} G\left(\mathbf{x}_{i,j+1}, \mathbf{x}_{i,j}, \lambda\right), \tag{4}$$

$$b^-_{i,j-1} = a^-_{i,j-1} \lim_{\mathbf{x}\to\mathbf{x}_{i,j}} \left(G\left(\mathbf{x}, \mathbf{x}_{i,j}, \lambda\right) - G\left(\mathbf{x}, \mathbf{x}_{i,j}, \lambda_0\right)\right) + a^+_{i,j-1} G\left(\mathbf{x}_{i,j-1}, \mathbf{x}_{i,j}, \lambda\right). \tag{5}$$

$$b^-_{i,0} = a^-_{i,0} \lim_{\mathbf{x}\to\mathbf{x}_{i,1}} \left(G\left(\mathbf{x}, \mathbf{x}_{i,1}, \lambda\right) - G\left(\mathbf{x}, \mathbf{x}_{i,1}, \lambda_0\right)\right) + \sum_{n\neq i} a^+_{n,0} G\left(\mathbf{x}_{n,1}, \mathbf{x}_{i,1}, \lambda\right), \qquad i, n = \overline{1,3}. \tag{6}$$

Matching of equations (3) and (4,5) for different points $\mathbf{x}_{i,j}$ allows one to obtain the discrete spectrum equation.

The relationship between coefficients $a^{\pm}_{i,j}$ and $a^{\pm}_{i,j-1}$ $\forall j \geq 2, i = \overline{1,3}$ can be obtained by substituting expressions (4,5) and the first equation of the system (3) into the second equation of this system:

$$\begin{pmatrix} a^{+}_{i,j} \\ a^{-}_{i,j} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{G(\mathbf{x}_{i,j-1},\mathbf{x}_{i,j},\lambda)}{G(\mathbf{x}_{i,j+1},\mathbf{x}_{i,j},\lambda)} & \frac{\alpha+2\left(G(\mathbf{x},\mathbf{x}_{i,j},\lambda)-G(\mathbf{x},\mathbf{x}_{i,j},\lambda_0)\right)|_{\mathbf{x}=\mathbf{x}_{i,j}}}{G(\mathbf{x}_{i,j+1},\mathbf{x}_{i,j},\lambda)} \end{pmatrix} \begin{pmatrix} a^{+}_{i,j-1} \\ a^{-}_{i,j-1} \end{pmatrix} = \mathbf{M_j} \begin{pmatrix} a^{+}_{i,j-1} \\ a^{-}_{i,j-1} \end{pmatrix},$$

where matrix $\mathbf{M}_j$ is a transfer matrix (cf [15]). It is simple to find its eigenvalues $\mu^{\pm}_j$ and the corresponding eigenvectors $\nu^{\pm}_j$. Let us choose unit first components of the eigenvectors. Then, it has the form:

$$\nu^{\pm}_j = \begin{pmatrix} 1 \\ -\mu^{\pm}_j \end{pmatrix}.$$

The relation between coefficients $a^{\pm}_{i,1}$ and $a^{\pm}_{i,0}$ where $i = \overline{1,3}$ can be also obtained by substituting of expressions (4,6) into system (3). Considering now equality (up to a multiplicative constant) of one of vectors (assume for certainty vector $\left(a^{+}_{1,1}, a^{-}_{1,1}\right)^T$) to eigenvector ν and using linear dependence, one obtains the following system:

$$\begin{cases} (\mu^{\pm}_2 - X)c + Aa^{-}_{2,0} + Ba^{-}_{3,0} = 0 \\ Ac + (\mu^{\pm}_2 - X)a^{-}_{2,0} - Ca^{-}_{3,0} = 0 \\ Bc - Ca^{-}_{2,0} + (\mu^{\pm}_2 - X)a^{-}_{3,0} = 0 \end{cases} \tag{7}$$

where c is a constant and the following notations were used:

$$X = \alpha + 2 \lim_{\mathbf{x}\to\mathbf{x}_{i,1}} \left(G\left(\mathbf{x}, \mathbf{x}_{i,1}, \lambda\right) - G\left(\mathbf{x}, \mathbf{x}_{i,1}, \lambda_0\right)\right),$$

$$A = G\left(\mathbf{x}_{2,1}, \mathbf{x}_{1,1}, \lambda\right), B = G\left(\mathbf{x}_{3,1}, \mathbf{x}_{1,1}, \lambda\right), C = G\left(\mathbf{x}_{3,1}, \mathbf{x}_{2,1}, \lambda\right).$$

It is obvious that system (7) has a nontrivial solution relative to c, $a^{-}_{2,0}$ and $a^{-}_{3,0}$ if its determinant equals zero:

$$\left(X - \mu^{\pm}_2 G\right)^3 - \left(X - \mu^{\pm}_2 G\right)\left(A^2 + B^2 + C^2\right) + 2ABC = 0. \tag{8}$$

Equation (8) is the main equation to investigate, but due to our additional assumptions, (namely, $A = B = C$) it can be rewritten in a simpler form:

$$\left(X - \mu^{\pm}_2 G\right)^3 - 3A^2\left(X - \mu^{\pm}_2 G\right) + 2A^3 = 0. \tag{9}$$

It should also be noted that equations (8,9) would be an equation on the discrete spectrum only if the corresponding eigenvalue μ satisfies the condition:

$$\left|\mu^{-}_2\right| < 1 \qquad \Leftrightarrow \qquad \frac{X}{2G} > 1, \tag{10}$$

$$\left|\mu^{+}_2\right| < 1 \qquad \Leftrightarrow \qquad \frac{X}{2G} < -1 \tag{11}$$

Investigating the solutions of equation (9) and keeping in mind conditions (10,11), we can obtain the following theorem (we omit the proof in this letter).

Theorem. *The basic Y-type chain of weakly coupled ball resonators has at most one negative eigenvalue if there is condition of δ -coupling in contact points with a coupling constant α (where $\alpha < -\frac{9}{2}\sum_{k=1,l=0}^{\infty} \frac{N_{lk}}{x^2_{lk}} - \frac{3}{2\pi\lambda_0}$).*

3. Conclusion

The operator extensions theory model for a base Y-type chain of weakly-coupled ball resonators was discussed. The main aim was to find negative eigenvalues for such a system with a certain type of coupling at contact points. For the system with δ -coupling it was proven that it has at most one negative eigenvalue (under certain conditions imposed on the coupling constant).

Acknowledgements

This work was supported by FTP "Scientific and Educational Human Resources for Innovation-Driven Russia"(contract 16.740.11.0030), grant 11-08-00267 of RFBR, FTP "Researches and Development in the Priority Directions Developments of a Scientific and Technological Complex of Russia 2007-2013"(state contract 07.514.11.4146) and grants of the President of Russia (state contract 14.124.13.2045-MK and grant MK-1493.2013.1)

References

[1] P. Duclos, P. Exner, O. Turek. On the spectrum of a bent chain graph. *J. Phys. A: Math. Theor.* **41**, P. 415206 (2008).

[2] N.I. Gerasimenko, B.S. Pavlov. Scattering problems on noncompact graphs. *Theoret. Math. Phys.* **74**, P. 230–240 (1988).

[3] P. Kuchment. Graph models for waves in thin structures. *Wave in random media*. **12** (4), R1-R24 (2002).

[4] E. Korotyaev, I. Lobanov. Schrödinger operators on zigzag graphs. *Ann. Henri Poincaré*. **8** (6), P. 1151–1176 (2007).

[5] I.Yu. Popov, P.I. Smirnov. Spectral problem for branching chain quantum graph. *Phys. Lett. A*. **377**, P. 439–442 (2013).

[6] I.Yu. Popov. The resonator with narrow slit and the model based on the operator extensions theory. *J. Math. Phys.* **33** (11), P. 3794–3801 (1992).

[7] R. Brown, P.D. Hislop, A. Martinez. Eigenvalues and resonances for domains with tubes: Neumann boundary conditions. *J. Diff. Equs*. **115** (2), P. 458–476 (1995).

[8] I.Yu. Popov. Model of point-like window for electromagnetic Helmholtz resonator. *Zeitschrift fur Analysis und ihre Anwendungen*. **32** (2), P. 155–162 (2013).

[9] R.R. Gadylshin. On the vibration amplitude for the Helmholtz resonator. *DAN SSSR*. **310** (5), P. 1094–1097 (1990).

[10] I.Yu. Popov. Operator extensions theory model for electromagnetic fieldelectron interaction. *Journal of Mathematical Physics*. **53** (6), P. 063505/1-8 (2012).

[11] J. Brüning, P. Exner, V. Geyler. Large gaps in point-coupled periodic systems of manifolds. *J. Phys. A, Math. Gen.* **36** (17), P. 4875–4890 (2003).

[12] J. Brüning, G. Martin, B. Pavlov. Calculation of the Kirchhoff coefficients for the Helmholtz resonator. *Russian J. Math. Phys*. **16** (2), P. 188–207 (2009).

[13] A. Laurain, S.A. Nazarov, J. Sokolowski. Singular perturbations of curved boundaries in three dimensions. The spectrum of the Neumann Laplacian. *Z. Anal. Anwend.* **30** (2), P. 145–180 (2011).

[14] N.I. Akhiezer, I.M. Glazman. *Theory of Linear Operators in Hilbert Space.* Vol. I. Translated from the third Russian edition by E. R. Daqson. Translation edited by W. N. Everitt. Monographs and Stidies in Mathematics, 9. London: Pitman (Advanced Publishing Program), 1981. 312 pp.

[15] N. Bagraev, G. Martin, B.S. Pavlov, A. Yafyasov. Landau-Zener effect for a quasi-2D periodic sandwich. *Nanosystems: Physics, Chemistry, Mathyematics*, **2** (4), P. 32–50 (2011).