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MODEL OF NON-AXISYMMETRIC FLOW IN NANOTUBE

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The asymmetric Stokes flow in a circular cylinder due to a rotlet is considered. This is a model for nanotube flow induced by a small rotating particle. The 3D Stokes and continuity equations are reduced to boundary problems for two scalar functions. Analytical solutions in terms of the Fourier transform is obtained.

Keywords: Stokes flow, rotlet.

1. Introduction

The Stokes flow description is a classical problem of fluid mechanics, having a long history. In recent years, it attracted new interest due to appearance of a new field, related with the development of nanotechnology. The flow through nanostructures is known to have many interesting unusual peculiarities [1]. Particularly, one observes a phenomenon analogous to superfluidity [2], dependence of the viscosity on the nanotube diameter [3] and other effects. The theory of nanoflow is not well-developed. There are only a few works suggesting theoretical explanation of these phenomena (see, e.g., [4], [5], [6]). It has been shown that hydrodynamic equations should be modified for nanoflows [7], but the Stokes approximation is appropriate due to the smallness of the Reynolds number [8].

In the present paper, we use the Stokes model for nanotube flow. Namely, we study the creeping flow inside the nanotube induced by a rotlet. From a physical point of view, a molecule rotating due to external magnetic field in the nanotube can play the role of the rotlet. A rotlet is the point source of vorticity, i.e. it is the solution of the Stokes equation with point singularity (see, e.g., [9], [10]). Correct mathematical description of such type of singular solutions was given in the framework of the theory of self-adjoint extensions of symmetric operators [11], [12], [13]. The advantage of the approach is that it allows one to obtain analytical solutions in many interesting cases. In the present paper we consider asymmetric Stokes flow in a cylinder caused by a rotlet having the axis orthogonal to the cylinder axis. Earlier the asymmetric Stokes flow was describe only for the domain between two parallel planes [14].

2. Problem formulation

Consider a cylinder of radius R_0 having OZ as the axis. We deal with the Stokes flow inside the cylinder caused by a rotlet at the origin having OX as the axis. Let $\mathbf{v}$ be the flow velocity, p be the pressure. Then the Stokes and continuity equations takes place:

$$\Delta \mathbf{v} = \nabla p, \qquad \nabla \cdot \mathbf{v} = 0. \tag{1}$$

We assume that the flow is induced by a rotlet, hence, we seek the solution in the form:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \qquad v_0 = \frac{\mathbf{i} \times \mathbf{r}}{|\mathbf{r}|},$$

$\mathbf{r}$ is radius-vector of a point, $\mathbf{i}$ is the unit vector, parallel to the OX axis.

We will use the cylindrical coordinates (ρ, φ, z). There is an interesting technique suggested in [15] which allows one to reduce the system (1) of the Stokes and continuity equation to two equations for two scalar functions ψ, χ. Namely, the solution of (1) can be represented in the form:

$$\begin{aligned} \mathbf{v} &= rot(rot(\frac{\psi}{\rho}\mathbf{k}\cos\varphi) + \frac{\chi}{\rho}\mathbf{k}\sin\varphi), \\ p &= \frac{1}{\rho}\frac{\partial}{\partial z}(L_{-1}\psi)\cos\varphi, \end{aligned}$$

where ψ, χ are scalar functions of two variables (ρ, z) satisfying the equations:

$$L_{-1}^2\psi = 0, \quad L_{-1}\chi = 0,$$

$$L_{-1} = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}.$$

The velocity components is related with these functions by the following manner:

$$v_z = -\frac{\partial}{\partial \rho}(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho})\cos\varphi, \tag{2}$$

$$v_\rho = (\frac{\partial}{\partial \rho}(\frac{1}{\rho}\frac{\partial \psi}{\partial z}) + \frac{\chi}{\rho^2})\cos\varphi, \tag{3}$$

$$v_\varphi = -(\frac{1}{\rho^2}\frac{\partial \psi}{\partial z} + \frac{\partial}{\partial \rho}(\frac{\chi}{\rho}))\sin\varphi. \tag{4}$$

The no-slip boundary condition (zero velocity at the boundary) is as follows:

$$\begin{aligned} (\frac{\partial}{\partial \rho}(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}))\Big|_{\rho=R_0} &= 0, \\ (\frac{\partial}{\partial \rho}(\rho\frac{\partial}{\partial \rho}(\frac{1}{\rho}\frac{\partial \psi}{\partial z})) - \frac{1}{\rho^2}\frac{\partial \psi}{\partial z})\Big|_{\rho=R_0} &= 0, \\ (\frac{\partial}{\partial \rho}(\rho\frac{\partial}{\partial \rho}(\frac{\chi}{\rho})) - \frac{\chi}{\rho^2})\Big|_{\rho=R_0} &= 0. \end{aligned}$$

Moreover, $\mathbf{v} \to 0$ if $z \to \infty$.

3. Problem solution

Note that the functions ψ, χ corresponding to the rotlet in free space are as follows

$$\psi_0 = -\sqrt{z^2 + \rho^2}, \quad \chi_0 = -\frac{z}{\sqrt{z^2 + \rho^2}}.$$

We seek the solution in the form $\psi = \psi_0 + \psi_1$, $\chi = \chi_0 + \chi_1$. At first, let us consider the problem for χ_1 :

$$\frac{\partial^2 \chi_1}{\partial z^2} + \frac{\partial^2 \chi_1}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \chi_1}{\partial \rho} = 0,$$

$$(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{\chi_1}{\rho}))-\frac{\chi_1}{\rho^2})\Big|_{\rho=R_0}=-(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{\chi_0}{\rho}))-\frac{\chi_0}{\rho^2})\Big|_{\rho=R_0}.$$

Making the Fourier transform F in respect to z, one gets

$$-k^2X+X''_{\rho\rho}-\frac{1}{\rho}X'_\rho=0,$$

$$(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{X}{\rho}))-\frac{X}{\rho^2})\Big|_{\rho=R_0}=-(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{X_0}{\rho}))-\frac{X_0}{\rho^2})\Big|_{\rho=R_0}.$$

Here $X(k,\rho)=F(\chi_1(z,\rho))$, $X_0(k,\rho)=F(\chi_0(z,\rho))$. Note that X,X_0 should be considered as distributions. One can solve the problem and find X and, correspondingly, χ :

$$\chi=\chi_0+\chi_1, \tag{5}$$

$$\chi_1=F^{-1}(X)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}AJ_1(ik\rho)e^{ikz}dk,$$

$$A=-\frac{(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{X_0}{\rho}))-\frac{X_0}{\rho^2})\Big|_{\rho=R_0}}{(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{J_1(ik\rho)}{\rho}))-\frac{J_1(ik\rho)}{\rho^2})\Big|_{\rho=R_0}},$$

where J_1 is the Bessel function.

Consider the problem for ψ. Making the Fourier transform in respect to z, one obtains the equation

$$(-k^2+\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho})^2\Psi=0,$$

where $\Psi(k,\rho)=F(\psi(z,\rho))$. To construct the solution to the boundary problem, we need two linearly independent solutions having no singularity at zero. One of them is, evidently, $J_1(ik\rho)$. The second solution is sought in the form $f(\rho)J_1(ik\rho)$. Routine procedure gives us the expression for $f(\rho)$:

$$f(\rho)=\int_0^\rho\frac{\rho_1d\rho_1}{J_1(ik\rho_1)}(\int_0^{\rho_1}\frac{J_1^2(ik\rho_2)}{\rho_2}d\rho_2).$$

The solution is linear combination of these two functions with coefficients determined by the boundary conditions. As a result, one has:

$$\psi=\psi_0+\psi_1, \tag{6}$$

$$\psi_1=F^{-1}(\Psi_1)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_1(k,\rho)e^{ikz}dk,$$

$$\Psi_1(k,\rho)=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{21}a_{12}}J_1(ik\rho)+\frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{21}a_{12}}f(\rho)J_1(ik\rho),$$

$$a_{11}=(\frac{\partial}{\partial\rho}(\frac{1}{\rho}\frac{\partial J_1(ik\rho)}{\partial\rho}))\Big|_{\rho=R_0},\quad a_{12}=(\frac{\partial}{\partial\rho}(\frac{1}{\rho}\frac{\partial(f(\rho)J_1(ik\rho))}{\partial\rho}))\Big|_{\rho=R_0},$$

$$a_{21}=(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{J_1(ik\rho)}{\rho}))-\frac{J_1(ik\rho)}{\rho^2})\Big|_{\rho=R_0},$$

$$a_{22}=(\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}(\frac{f(\rho)J_1(ik\rho)}{\rho}))-\frac{f(\rho)J_1(ik\rho)}{\rho^2})\Big|_{\rho=R_0},$$

$$b_1 = -\left(\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \Psi_0(\rho)}{\partial \rho}\right)\right)\Big|_{\rho=R_0},$$

$$b_2 = -\left(\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\left(\frac{\Psi_0(\rho)}{\rho}\right)\right) + \frac{\Psi_0(\rho)}{\rho^2}\right)\Big|_{\rho=R_0}.$$

Inserting of expressions (5), (6) for χ and ψ into (2), allows one to obtain the velocity and the pressure fields.

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