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Original article

Quantum coupling between radio modes in a single-atom maser with two resonators

Eugeniy N. Popov[1,a], Alexander I. Trifanov[1,b], Maria A. Moskalenko[1,c], Vladimir A. Reshetov[2,d]

[1]ITMO University, Saint Petersburg, Russia

[2]Togliatti State University, Tolyatti, Russia

[a]enp-tion@yandex.ru, [b]alextrifanov@gmail.com, [c]moskalenko.mary@gmail.com, [d]vareshetov@tltsu.ru

Corresponding author: Eugeniy N. Popov, enp-tion@yandex.ru

ABSTRACT In this work, we investigate the effect of quantum coupling between radio fields in a single-atom maser with two spatially separated resonators. Each atom in a beam, depending on its state, can emit one photon into the first resonator and absorb another from the second, thereby entangling the quantum states of two independent modes. Resulting from entanglement, we obtain a coherence between states of two-mode field with the same total number of photons in the both modes. To study the arising coupling, an analytical solution of the stationary master equation is found under conditions of a trapped field state and the dependence of the von Neumann entanglement entropy on the quality factor of the resonators. Numerical analysis reveals that the best conditions for the appearance of quantum coupling are the low quality factor of the first resonator and the high quality factor of the second one.

KEYWORDS single-atom maser, quantum coupling, trapped field state

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1. Introduction

A single-atom maser is widely accepted as one of the most efficient tools for observing the quantum effects of the atom-field interaction [1, 2]. An example of its application is the verification of the Jaynes-Cummings model [3]. Multiple publications containing a lot of experimental and theoretical work have also been published [4–14]. The most important research on the development of a single-atom masers was carried out by G. Walter at the end of the previous century. Thereby, the quantum properties of a single-atom masers with one resonator has already been described [15, 16]. Despite its success, the technical implementation of such a quantum single-atom maser circuit has proven to be a rather difficult and complex task. Therefore, further research continued only after the appearance of alternative schemes of controlled interaction between selected mode and atom. More recent research, most notably, includes the observation and prediction of new effects using a nanomaser on an ultra-fast driving LC circuit [17–20] as well as experiments on the study of interaction effects between single atoms and a series of coupled resonators [21–23]. Furthermore, active development is underway in the theory of single-atom masers [24]. However, in fact, we have not found any work on the formation of quantum connection between several independent modes using a common pumping system in a single-atom maser. Therefore, in this work, we build the initial theoretical foundation for this phenomenon with the ultimate goal of obtaining new physical results in subsequent studies.

Our article describes the initial theory of a single-atom masers that contains two resonators located in the path of the atomic beam, as shown in Fig. 1. With the use of focused pumping light, beam atoms are excited into a mixed populational state of lower and upper levels. Then the prepared atoms fly through a pair of resonators. The atomic beam is considered to be quite sparse, so that only one atom can possibly be located inside a pair of resonators at the same moment of time. During the propagation through each resonator, the atom periodically moves from one level to another, either emitting a photon into the mode or absorbing it. After the atom leaves the second resonator, the quantum state of the atom is destroyed due to the interaction with the environment. The atomic field wave function experiences collapse: the two-mode field will turn into a statistical ensemble of those states that correspond to the atom that was flew out at the lower or upper level. The process repeats for each atom in the beam.

Based on the described physical model, we propose a theoretical description for the formation of quantum coupling between the modes of spatially separated resonators. This coupling is explained by the preservation of the quantum state of the atom during its flight between the first and second resonators. During the interaction, the information about the field of the first resonator is recorded into a superposition of quantum states and then transferred to the second resonator by the atom. The information transfer process looks like this: an atom absorbs a photon from the first resonator and emits it

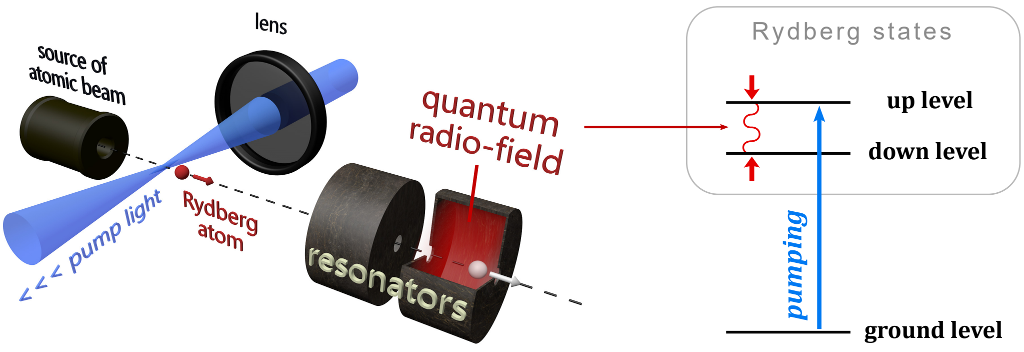

FIG. 1. The scheme of a single-atom maser with double resonator. Red and white spheres are atoms emitted from the source of the atomic beam. There is a scheme of atomic levels on right.

into the second, or vice versa; as a result, coherence is formed between field states with the same total number of photons in both modes. Strictly speaking, our task is to find an analytical form of the stationary state field modes under atomic pumping. The analysis of coupling between two cavities containing bosons is important in many theoretical and applied problems, so this observation may be useful in other studies [25–28].

2. Evolution of two-mode field

To model a single-atom maser with two resonators, we use the density matrix ρ_F, which describes the state of two coupled field modes. Each mode belongs to its own resonator. The first resonator, an atom flies through which, will be denoted as a, and the second as b:

$$\rho_F(t) = \sum_{m,n\,\mu,\nu} \mathcal{F}_{m,n|\mu,\nu} \left|m,n\right\rangle \left\langle \mu,\nu\right| , \qquad m,n,\mu,\nu \in \mathbb{N}_0; \tag{1}$$

$$\left|m,n\right\rangle = \left|m\right\rangle_a \otimes \left|n\right\rangle_b , \qquad \left\langle \mu,\nu\right| = \left\langle \mu\right|_a \otimes \left\langle \nu\right|_b . \tag{2}$$

Here $\left|k\right\rangle_\alpha$ is the Fock-state of the resonator mode α with occupation number k, and $\mathcal{F}_{m,n|\mu,\nu}$ are elements of a field density matrix.

In our case, the beam atoms inside the resonator can be in two orthogonal energy states: lower $|d\rangle$ with energy E_d, and upper $|u\rangle$ with energy E_u. These two states collectively constitute a two-dimensional subspace $\mathbb{C}_2$, within which all atomic transformations are described by the algebra of the following operators:

$$\sigma_z = |u\rangle\langle u| - |d\rangle\langle d| , \qquad \sigma_+ = |u\rangle \langle d| , \qquad \sigma_- = |d\rangle\langle u| . \tag{3}$$

When an atom in the beam propagates through one of the resonators with index $\alpha \in a, b$, then in accordance with the Jaynes-Cummings model, the evolution of such an atomic field system is described using the von Neumann equations in the rotating wave approximation [29]:

$$\frac{d\rho_{AF}}{dt} = i \left[V_\alpha, \rho_{AF}\right] , \tag{4}$$

$$V_\alpha = \left(\frac{\omega_\alpha}{2} - \frac{E_u - E_d}{2\hbar}\right) \cdot \sigma_z + g_\alpha \left(\alpha^\dagger \sigma_- + \alpha\sigma_+\right) , \tag{5}$$

where V_α is the interaction operator; density matrix ρ_{AF} is represented in the basis of atomic-field's states; α^+ and α are the operators of photon creation and annihilation inside the cavity mode α; ω_α is the frequency of the mode of the resonator α; real coefficient g_α characterizes the coupling between the mode and the passing atom.

The solution of equation (3) can be written using the operator exponent, since the interaction operators V_α corresponding to different moments of time commute:

$$\rho_{AF}(t^{'}) = e^{-iV_\alpha(t'-t)} \rho_{AF}(t) e^{iV_\alpha(t'-t)} . \tag{6}$$

Thus, evolution of the atom-field density matrix when an atom passes through a pair of resonators can be written as

$$\rho'_{AF} = U\rho^{(0)}_{AF}U^\dagger, \qquad U = e^{-iV_bT_b} \cdot e^{-iV_aT_a} . \tag{7}$$

where $\rho^{(0)}_{AF}$ is the atom-field density matrix at the moment the atom enter the resonator α; ρ'_{AF} is the density matrix at the moment the atom leaves the resonator b; T_α is the time of propagation of an atom through the resonator α. Before the interaction begins, the atom of beam and the field are independent, so we can write:

$$\rho^{(0)}_{AF} = \rho_A \otimes \rho_F(0), \tag{8}$$

where ρ_A is the mixed state of the atom after pumping, and $\rho_F(0)$ is the state of the field before the start of interaction with the atom.

For simplicity, we will assume that the frequency of the atomic transition coincides with the frequencies of the modes of both resonators: $\hbar\omega_\alpha = \hbar\omega_b = E_u - E_d$. Then the evolution operator U in equation (6) can be represented in the following form:

$$U = \sum_{f,i} U_{fi} \left|f\right\rangle\!\left\langle i\right|, \qquad f, i \in d, u; \tag{9}$$

$$U_{dd} = C_a \otimes C_b - S_a \otimes S_b', \qquad U_{uu} = C_a' \otimes C_b' - S_a' \otimes S_b, \tag{10}$$

$$U_{du} = -i\left(S_a' \otimes C_b + C_a' \otimes S_b'\right), \qquad U_{ud} = -i\left(S_a \otimes C_b' + C_a \otimes S_b\right). \tag{11}$$

The second term in equations (9) corresponds to the re-emission of a photon from one resonator to another through the passing atom. Operators C_α, $C_\alpha^{'}$, S_α, and $S_\alpha^{'}$ act in the subspace of mode states resonator α. Using the notation $\theta_k^\alpha = g_\alpha T_\alpha \sqrt{k}$, one can present their explicit forms:

$$C_\alpha \left|k\right\rangle_\alpha = \cos\theta_k^\alpha \left|k\right\rangle_\alpha, \qquad S_\alpha \left|k\right\rangle_\alpha = \sin\theta_{k+1}^\alpha \left|k-1\right\rangle_\alpha, \tag{12}$$

$$C_\alpha' \left|k\right\rangle_\alpha = \cos\theta_{k+1}^\alpha \left|k\right\rangle_\alpha, \qquad S_\alpha' \left|k\right\rangle_\alpha = \sin\theta_{k+1}^\alpha \left|k+1\right\rangle_\alpha. \tag{13}$$

We consider the case when the time of propagation of an atom through the resonators is short enough to neglect the relaxation processes for both the atom and the field. The relaxation and its effects will be taken into account later in the master equation, which describes the dynamics of the field over a long time interval, during which much more than one atom passes through the resonators.

3. Master equation

The dynamics of the micromaser field over a long time interval is described by the master equation [2, 9, 17]. We modify it for the case of a two-mode resonator, leaving the structure of the equation unchanged. That is, the first term on the right-hand side describes the transformation of the field due to interaction with the atoms of the beam, and the second term describes the decay of field modes:

$$\frac{\mathrm{d}\rho_F}{\mathrm{d}t} = \mathcal{I}\left(\langle U\left(\rho_F \otimes \rho_A\right)U^\dagger\rangle_{at} - \rho_F\right) - \sum_{\alpha=a,b}\frac{\gamma_\alpha}{2}\left(\alpha^\dagger\alpha\rho_F - 2\alpha\rho_F\alpha^\dagger + \rho_F\alpha^\dagger\alpha\right), \tag{14}$$

$$\rho_A = \xi_d \left|d\right\rangle\!\left\langle d\right| + \xi_u \left|u\right\rangle\!\left\langle u\right|. \tag{15}$$

The quantity $\mathcal{I}$ characterizes the intensity of the atomic beam, and the constants γ_α are equal to the inverse lifetimes of a photon inside the resonator α. Averaging over atomic states in the first term occurs due to collapse of the atomic wave function after it leaves the second resonator. Thus, coupling with the field is destroyed.

It can be shown that in the master equation (14) there are no terms inducing the excitation of phase coherence between field states with different total numbers of photons in both modes. Physically, this can be explained by the fact that the emission or absorption of one photon while preserving the field phase occurs only with a change of atomic energy. In essence, coherence can arise only for a pair of atomic-field states in which the atom populates different levels, and corresponding terms in the master equation (14) are neglected during the trace over atomic variables. As a result, the field of a single-cavity single-atom maser in a stationary state is always an incoherent mixture of the Fock field states with a certain number of photons.

In the case of two resonators, the field is not a mix of the Fock states, since an atom can absorb a photon from the first resonator and emit it into the second, or vice versa. The atom enters and leaves the pair of resonators in the same state. Consequently, such a photon exchange process will induce the coherence between field states with the same total number of photons in the system. Formally, this will lead to non-zero off-diagonal elements of the density matrix $\mathcal{F}_{m,k-m|_\mu,k-\mu}$, where k is the total number of photons in a pair of resonators.

The above reasoning allows us to split the field density matrix into non-zero blocks that describe excited field modes with a certain total number of photons k:

$$\rho_F = \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus ..., \tag{16}$$

$$\rho_k = \sum_{m,\mu=0}^{k} \mathcal{F}_{m,k-m|\mu,k-\mu} \left|m, k-m\right\rangle \left\langle \mu, k-\mu\right|. \tag{17}$$

In new notation, master equation (14) takes the form of a linear dynamic system for matrices ρ_k:

$$\frac{1}{\mathcal{I}}\frac{d\rho_k}{dt} = \mathcal{L}^{\uparrow}\rho_{k-1} + \mathcal{L}^{o}\rho_k + \mathcal{L}^{\downarrow}\rho_{k+1}, \quad \gamma_\alpha = \frac{\Gamma_\alpha}{\mathcal{I}}, \tag{18}$$

$$\mathcal{L}^{\uparrow}\rho_k = \xi_u U_{du}\rho_k U_{ud}^{\dagger}, \tag{19}$$

$$\mathcal{L}^{o}\rho_k = \xi_u U_{uu}\rho_k U_{uu}^{\dagger} + \xi_d U_{dd}\rho_k U_{dd}^{\dagger} - \rho_k - \sum_{\alpha=a,b}\frac{\gamma_\alpha}{2}[\alpha^{\dagger}\alpha, \rho_k], \tag{20}$$

$$\mathcal{L}^{\downarrow}\rho_k = \xi_d U_{ud}\rho_k U_{du}^{\dagger} + \sum_{\alpha=a,b}\gamma_\alpha \alpha\rho_k\alpha^{\dagger}, \tag{21}$$

where for any k linear maps $\mathcal{L}^{\uparrow}$ act from the Hermitian matrix space ρ_k to the Hermitian matrix space ρ_{k+1}; $\mathcal{L}^{0}$ acts inside the Hermitian matrix space ρ_k; $\mathcal{L}^{\downarrow}$ acts from the Hermitian matrix space ρ_k to the Hermitian matrix space ρ_{k-1}.

We are interested in the case of stationary dynamics, in which the process of photons leaving the cavity is completely compensated by atomic pumping. However, the search for a stationary solution of equation (18) through the recurrence relation leads to the increasing number of unknown variables due to rank of matrices ρ_k. This problem can be solved for trapped states.

At a certain moment in time, while atoms propagate through the resonator, the number of photons reaches a maximum value and stops increasing. This effect occurs due to the integer value of the Rabi oscillations. That is, the atom has time to emit a photon into the mode and immediately absorb it while propagating through the resonator. In the case of two micromaser resonators, the condition for appearance of such trapped states must be written for each resonator α separately; it is expressed through solution (6):

$$g_\alpha T_\alpha \sqrt{N_\alpha + 1} = \pi k_\alpha, \qquad k_\alpha \in \mathcal{N}, \tag{22}$$

where N_α is the maximum number of photons in the resonator α, after which the atoms in the upper level stop emitting a photon into the mode; k_α is the number of integer Rabi oscillations.

If condition (22) is satisfied for both resonators, then the number of photons k in a pair of resonators does not exceed the maximum value $N_a + N_b$. The matrix ρ_k rank in the case of trapping is determined by the number of all states $|m, k-m\rangle$ in which the sum of photons is equal to k and the occupation number of each mode does not exceed the maximum value N_α:

$$\text{rank}[\rho_k] = \min(N_\alpha, k) + 1 - k. \tag{23}$$

The number of non-zero matrices ρ_k becomes finite and equal to $N = N_a + N_b + 1$, and the first and last matrices have rank 1. The constraint of the sequence ρ_k allows us to solve the stationary master equation through the recurrence relation:

$$\frac{d\rho_k}{dt} = \mathcal{L}^{\uparrow}\rho_{k-1} + \mathcal{L}^{o}\rho_k + \mathcal{L}^{\downarrow}\rho_{k+1} = 0, \tag{24}$$

$$\rho_k = r_k + \mathcal{A}_k\rho_{k-1}, \qquad \rho_0 = r_0, \tag{25}$$

$$\mathcal{A}_k = -(\mathcal{L}^{o} + \mathcal{L}^{\downarrow}\mathcal{A}_{k+1})^{-1}\mathcal{L}^{\uparrow}, \qquad \mathcal{A}_N = 0, \tag{26}$$

$$r_k = -(\mathcal{L}^{\wr} + \mathcal{L}^{\downarrow}\mathcal{A}_{k+1})^{-1}\mathcal{L}^{\downarrow}r_{k+1}, \qquad r_N = \rho_N, \tag{27}$$

where $\rho_N = \mathcal{F}_{N_a,N_b|N_a,N_b}$ is firstly equal to unity and then calculated using renormalization ρ_F.

In the next chapter, solution (25) of the stationary governing equation (24) is used to observe the entanglement between modes for different single-atom maser parameters.

4. Entanglement of resonator modes

By varying the parameters γ_a, γ_b, ξ_u, N_a, k_a, N_b, k_b, it is possible to distinguish four stationary regimes of a single-atom maser, which allow one to observe its main qualitative properties:

(1) Both resonators are high-Q, and the beam consists of atoms at the upper level only;
(2) Both resonators are high-Q, and the beam consists of atoms at the upper and lower levels;
(3) The first resonator is high-Q, the second is low-Q, and the beam consists mainly of atoms at the upper level;
(4) The first resonator is low-Q, the second is high-Q, and the beam consists mainly of atoms at the upper level.

Let us assume that each mode can be excited only up to the one-photon state $N_a = N_b = 1$. In this case, the density matrix contains three elements: $\rho_0[1 \times 1]$, $\rho_1[2 \times 2]$ and $\rho_2[1 \times 1]$. Besides, let $k_a = k_b = 1$, that is, the atom has time to emit and absorb a photon in each mode only once. Explicit form of the density matrices are follow:

$$\rho_0 = \begin{pmatrix} \mathcal{F}_{00|00} \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} \mathcal{F}_{01|01} & \mathcal{F}_{01|10} \\ \mathcal{F}_{10|01} & \mathcal{F}_{10|10} \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \mathcal{F}_{11|11} \end{pmatrix}. \tag{28}$$

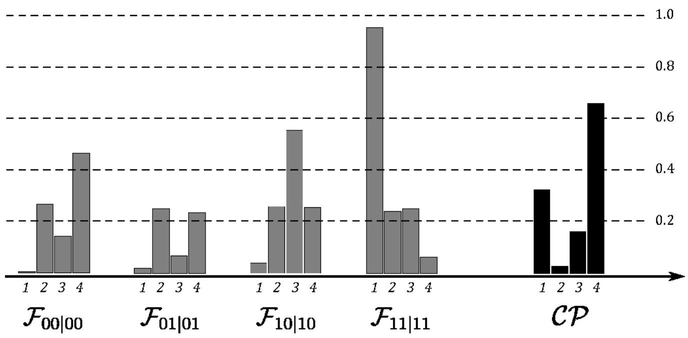

FIG. 2. Population diagram of the field density matrix. The gray bars show the value of the diagonal elements $\mathcal{F}_{m,n|\mu\nu}$, the numbers under the bars indicate the regime from the list. The black bars show the coherence parameters. The parameters for which a stationary solution was evaluated are given in the table above the graph.

TABLE 1. The parameters of four regimes for which a stationary solution (25) was evaluated

regime	ξ_d	ξ_u	γ_a	γ_b
(1)	0	1	10^{-3}	10^{-3}
(2)	0.5	0.5	10^{-3}	10^{-3}
(3)	0.2	0.8	10^{-3}	1
(4)	0.2	0.8	1	10^{-3}

Figure 2 shows the results from which we can make a conclusion about the conditions for the emergence of a coherence between the states $|0,1\rangle$ and $|1,0\rangle$ in the matrix ρ_1. No other cases are generated in the system, since the rank of the matrices ρ_0 and ρ_2 is equal to one. To estimate the coherence, we consider the product $\mathcal{CP}$:

$$\mathcal{C} = \frac{|\lambda_2 - \lambda_1|}{\lambda_2 + \lambda_1}, \tag{29}$$

where λ_1 and λ_2 are the eigenvalues of the matrix ρ_1.

$$\mathcal{P} = 1 - \frac{|\mathcal{F}_{0,1|0,1} - \mathcal{F}_{1,0|1,0}|}{\mathcal{F}_{0,1|0,1} + \mathcal{F}_{1,0|1,0}}. \tag{30}$$

The parameter $\mathcal{C}$ tends toward unity if ρ_2 can be represented as the pure state density matrix, while the parameter $\mathcal{P}$ characterizes the degree of mutual excitation of both modes of a maser and tends to zero if only single of the states $|0,1\rangle$ and $|1,0\rangle$ is populated.

Let us focus to main results. Firstly, in the second regime, the coherence is approximately zero (black bars in the Fig. 2), despite the high quality of the resonators. We explain this effect by the fact that coherence is formed by atoms at the lower level and atoms at the upper level with the opposite phases. If all states of the field are populated equally, as in this regime, then the total contribution to coherence turns out to be practically empty. If the beam consists of atoms only at the upper level, as in the first regime, then coherence between the states $|0,1\rangle$ and $|1,0\rangle$ arises, but these states are practically unpopulated compared to the state $|1,1\rangle$. Secondly, in the fourth regime, the coherence is significant increased (black bars), and the levels $|0,1\rangle$ and $|1,0\rangle$ are populated equally (gray bars). It is worth noting that the condition for the coherence occurrence is the low quality factor of the first resonator into which the atom flies, while the lifetime of photons in the second resonator should be long. We explain this by the fact that the atoms at the upper level compensate for the decay process of the field in the first resonator, emit photons into it, and enter the second resonator already in a state of superposition of two levels.

In addition to observing coherence within a subspace of states with the same number of photons, we looked for conditions under which two field modes approach to the maximum entanglement. To estimate this one, we used the entanglement entropy [29] and denoted it by $\mathcal{S}$. Micromaser generates a mix of the field quantum states, therefore we should calculate entropy for each one and then sum it using the probabilities. Moreover, since the density matrix is a direct sum (16), the entropy can be calculated for each of the terms ρ_k independently.

$$\mathcal{S} = \sum_{k=0}^{N_a+N_b} \left(\sum_{j=1}^{\mathrm{rank}[\rho_k]} p_{k,j} \mathcal{S}_{k,j} \right), \qquad \mathcal{S}_{k,j} = - \sum_{q=1}^{\mathrm{rank}[\rho_k]} \lambda_{k,j,q} \ln \lambda_{k,j,q}, \tag{31}$$

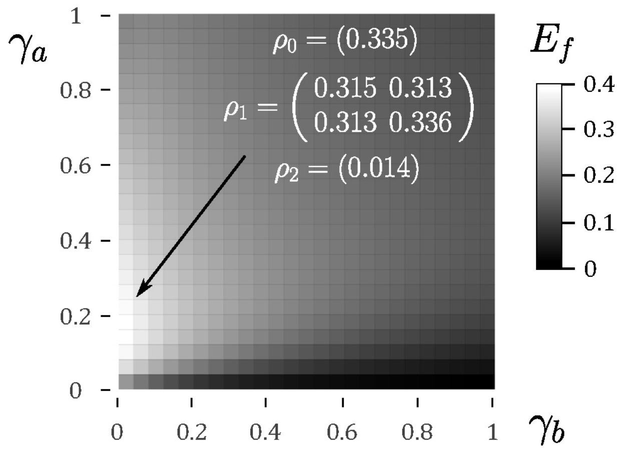

FIG. 3. The dependence of the parameter $\mathcal{S}$ defined by equation (30) of the maser's double-mode field on the quality factor of the resonators. The calculation was made with the following values: $\xi_u = 1, \xi_d = 0, k_a = k_b = 3$, $N_a = N_b = 1$. The figure contains the density matrix when the maximum entanglement $\mathcal{S} \approx 0.4$ is reached.

where $\mathcal{S}_{k,j}$ is the bipartite von Neumann entropy for j-th quantum state $|\Psi_{k,j}\rangle$ in the mix described by the density matrix ρ_k, symbol $\lambda_{k,j,q}$ denotes the q-th eigenvalue of the matrix $\rho_{k,j}^{\mathrm{Tr}}$.

$$\rho_k = \sum_{j=1}^{\mathrm{rank}[\rho_k]} p_{k,j}\rho_{k,j}, \qquad \rho_{k,j} = |\Psi_{k,j}\rangle\langle\Psi_{k,j}|\,, \tag{32}$$

$$\rho_{k,j}^{\mathrm{Tr}} = \sum_m \langle m|_a \, \rho_{k,j} \, |m\rangle_a \,. \tag{33}$$

Figure 3 shows a graph of the entanglement enthropy dependence on the relaxation constants of the resonators. When plotting the graph (as in the study of coherence), the case of trapped states with $N_a = N_b = 1$ is considered, the beam atoms are at the upper level, and the number of Rabi oscillations is set to new: $k_a = k_b = 3$. With these parameters, entanglement appears especially clearly.

Note that entanglement occurs when the quality factor of the first resonator is much less than the quality factor of the second resonator. The peak of entanglement at a certain value of the field relaxation rate in the first resonator γ_a seems to be curious, as it might show that there is a certain balance between the pumping and relaxation processes, in which the modes interact with the beam atoms as a single quantum system.

5. Conclusion

In this work, we have proposed the fundamental theory of a single-atom maser with two resonators, the field of which is pumped by a sparse atomic beam. In the case of trapped field states, a simple form of recurrence relation for the stationary solution of the master equation of the single-atom maser is obtained. The properties of the stationary solution indicate the existence of coherence between the states of a pair of modes with the same total number of photons. This effect was absent in a single-cavity single-atom maser, in which the stationary field is represented as an incoherent mixture of the Fock field states. The predicted coherence in our case leads to the possibility of generating entangled states, which was demonstrated by evaluating the entanglement enthropy. In this work, we consider only one special case of trapped states of a pair of resonators, which is the simplest for qualitative research of the properties of the proposed quantum system. Other trapped states may exhibit quantum properties that were not discovered in this work, which opens the prospect for further research.

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Information about the authors:

Eugeniy N. Popov – Laboratory of Quantum Processes and Measurements, Center of Mathematics, ITMO University, Kroverkskiy, 49, St. Petersburg, 197101, Russia; ORCID 0000-0003-0223-097X; enp-tion@yandex.ru

Alexander I. Trifanov – Center of Mathematics, ITMO University, Kroverkskiy, 49, St. Petersburg, 197101, Russia; ORCID 0000-0002-5109-2086; alextrifanov@gmail.com

Maria A. Moskalenko – Center of Mathematics, ITMO University, Kroverkskiy, 49, St. Petersburg, 197101, Russia; moskalenko.mary@gmail.com

Vladimir A. Reshetov – Togliatti State University, Belorusskaya str., 14, Togliatti, 445667, Russia; ORCID: 0009-0000-6418-6121; vareshetov@tltsu.ru

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