

Nonlinear optimal control problem in a two-point boundary regime for a pseudoparabolic equation with Samarskii–Ionkin type conditions

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ABSTRACT This paper is devoted to study a optimal movable point control problem for a pseudoparabolic equation with nonlinear control function in a two-point nonlinear boundary condition. The equation is studied with Samarskii–Ionkin type boundary conditions on spatial variable x . Spectral problem is studied and eigenvalues, eigenfunctions and optimality conditions are found. Loaded nonlinear functional equations are obtained with respect to control function. We prove the existence and uniqueness of the control function by the method of compressing mapping. The state function is determined. Convergence of the Fourier series for the state function is proved.

KEYWORDS Nonlinear loaded functional equation, pseudoparabolic equation, two-point boundary condition, Samarskii–Ionkin type conditions, eigenvalues, eigenfunctions, Fourier series, existence and uniqueness theorems

FOR CITATION Yuldashev T.K., Kadirkulov B.J., Ramazanova A.T., Shermamatov Zh.Zh. Nonlinear optimal control problem in a two-point boundary regime for a pseudoparabolic equation with Samarskii–Ionkin type conditions. *Nanosystems: Phys. Chem. Math.*, 2025, **16** (5), 563–576.

1. Formulation of the problem statement

Differential equations of mathematical physics have direct applications in the theory of nanosystems (see, for example [1–13] and [14]). Partial differential and integro-differential equations of parabolic and pseudoparabolic types with initial and boundary conditions were investigated widely by large number of scientists and have different applications in sciences and technology (see, for example [15–28]). The spectral problems for finding eigenvalues and eigenfunctions play an important role in solving mixed and boundary problems for differential equations of mathematical physics [29–37].

Optimal control theory is one of the most relevant branches of mathematical science. Many applied problems are reduced to finding the optimal control function and the corresponding state function. A large number of analytical and numerical methods for solving optimal control problems have been developed and are effectively used in solving various problems of optimization process in science and technology (see, for example, [38–50]). In the works [51–54], the time-optimal control problems for partial differential equations are studied.

It is well known that differential equations of parabolic type are associated with heat and diffusion processes. Neutron diffusion plays a significant role in the operation of nuclear reactors. The diffusion equation makes it possible to calculate the neutron density inside the core of a nuclear reactor, the neutron flux from the moderator surface, and the reflection and transmission of neutrons by biological protection structures. We note that the processes describing by parabolic equations can be studied more exactly by pseudoparabolic equations. Moreover, studying some problems for parabolic equations is more difficult than studying this problem for pseudoparabolic equations. When investigating the parabolic equation $u_t - u_{xx} = f(t, x)$ on $[0, T] \times [0, l]$ with final time condition $u(T, x) = \varphi(x)$ and the Dirichlet boundary conditions $u(t, 0) = u(t, l) = 0$, we obtain the function $\exp\{\lambda_n T\}$, which goes to infinity as $\lambda_n \rightarrow \infty$, where $\lambda_n > 0$ is eigenvalues of the spectral problem $\vartheta''(x) + \lambda \vartheta(x) = 0$, $\vartheta(0) = \vartheta(l) = 0$. Consequently, the present paper is devoted to study an optimal control problem for a pseudoparabolic equation with nonlinear control function in a two-point boundary condition. Control of the function in two-point boundary condition is important in metallurgy, partially, in aluminum production. The equation is studied with Samarskii–Ionkin type boundary value conditions on spatial variable x . Samarskii–Ionkin type boundary value conditions are nonlocal. It is used in the study of processes related to the gas lift oil production.

The eigenvalues and eigenfunctions problem is studied. So, in the domain $\Omega \equiv (0, T) \times (0, 1)$, we consider the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial^5}{\partial t \partial x^4} + \frac{\partial^4}{\partial x^4} \right) U(t, x) = f(t, x) \quad (1)$$

with boundary value conditions

$$\alpha(t)U(0, x) + \omega\beta(t)U(T, x) = \delta(x - \eta(t)) \varphi(t, x, p(t)), \quad (2)$$

$$U(t, 1) = 0, \quad U_{xx}(t, 0) = 0, \quad (3)$$

$$U_x(t, 0) = U_x(t, 1), \quad U_{xxx}(t, 0) = U_{xxx}(t, 1), \quad 0 \leq t \leq T, \quad (4)$$

where $\varphi(t, x, p(t))$ is nonlinear function depending from the control function $p(t)$, $f(t, x)$ is given function, $\alpha(t) \neq 0$, $\beta(t) \neq 0$ are given real functions, ω is real nonzero parameter.

The function $\eta(t) \in C[0, T]$ describes the change in the position of a moving point source in the range from zero to 1. It is defined as the solution to the following Cauchy problem

$$\eta'(t) = \omega(t, \eta(t)), \quad \eta(0) = \eta_0 = \text{const},$$

where $\omega(t, \eta) \in C^{0,1}(\Omega)$.

We consider the control function $p(t) \in \{p : |p(t)| \leq M^*, t \in [0, T], 0 < M^* = \text{const}\}$ and the following functional of quality

$$J[p] = \int_0^1 [U(T, y) - \xi(y)]^2 dy + \gamma_1 \int_0^T p^2(t) dt + \gamma_2 \int_0^T \eta^2(t) dt, \quad (5)$$

where $0 < \gamma_\kappa = \text{const}$, $\kappa = 1, 2$ and $\xi(x)$ is a given continuous function.

Problem. We find a pair of functions $\{U(t, x); p(t) \in \{p : |p(t)| \leq M^*\}\}$, first of which satisfies the differential equation (1), the two-point boundary condition (2), the Samarskii-Ionkin type boundary conditions (3), (4), belongs to the class of functions

$$U \in C_{t,x}^{1,3}(\overline{\Omega}), \quad U_{txxxx} \in C(\Omega), \quad U_{xxxx} \in C(\Omega), \quad (6)$$

and the second of which deliver a minimum to the functional (5).

2. Some auxiliary materials about Riesz bases

Let $\{\vartheta_n(x)\}$ and $\{\sigma_n(x)\}$ be two complete systems of functions from $L_2(a, b)$. Let us denote by $(\vartheta, \sigma)_0$ the scalar product of functions $\vartheta(x)$ and $\sigma(x)$ in $L_2(a, b)$:

$$(\vartheta, \sigma)_0 = (\vartheta, \sigma)_{L_2(a,b)} = \int_a^b \vartheta(x)\sigma(x)dx.$$

We use some known facts from the book [55].

1. Let the systems $\{\vartheta_n(x)\}$ and $\{\sigma_n(x)\}$ form a biorthonormal system in some interval $[a, b]$:

$$(\vartheta_n, \sigma_k)_0 = \int_a^b \vartheta_n \sigma_k dx = \delta_{nk} = \begin{cases} 0, & n \neq k, \\ 1, & n = k. \end{cases}$$

Then the system $\{\sigma_n(x)\}$ is called biorthogonally adjoint to the system $\{\vartheta_n(x)\}$ in the interval $[a, b]$.

2. Let no element of the system $\{\vartheta_n(x)\}$ belongs to the closure of the linear span of the other elements of this system. Then the system $\{\vartheta_n(x)\}$ is called minimal in $L_2(a, b)$.

The minimality of the system ensures the existence of a biorthogonally conjugate system.

3. The biorthogonal expansion of a function $f \in L_2(a, b)$ in a system $\{\vartheta_n(x)\}$ is a series

$$f(x) \sim \sum_{n=1}^{\infty} c_n \vartheta_n(x),$$

where $c_n = (f, \sigma_n)_0$.

4. Let for any function $f \in L_2(a, b)$ there hold

$$\sum_{n=1}^{\infty} |(f, \sigma_n)_0|^2 < \infty,$$

where $\{\sigma_n\}$ is biorthogonally conjugate system to $\{\vartheta_n(x)\}$. Then a complete and minimal system of functions $\{\vartheta_n(x)\}$ is called Besselian.

5. A complete and minimal system of functions $\{\vartheta_n(x)\}$ is called a Hilbert system, if for any sequence $\{c_n\}$ with

$\sum_{k=1}^{\infty} c_n^2 < \infty$ there is one and only one function $f \in L_2(a, b)$, for which $\{c_n\} = (f, \sigma_n)_0$ are coefficients of its biorthogonal expansion in $\{\vartheta_n(x)\}$.

6. A complete and minimal system is called a Riesz basis, if it is both Bessel and Hilbert basis.

Theorem 1 ([56]). The following statements are equivalent:

- 1). The sequence of functions $\{\sigma_j(x)\}_1^{\infty}$ forms a basis, equivalent to the orthonormal one, in the space R ;
- 2). The sequence of functions $\{\sigma_j(x)\}_1^{\infty}$ will be an orthonormal basis of the space R with the appropriate replacement of the scalar product (f, g) by some new one $(f, g)_1$, topologically equivalent to the original one;
- 3). The sequence of functions $\{\sigma_j(x)\}_1^{\infty}$ is complete in R and there exist positive constants a_1, a_2 , such that for any natural n and any complex numbers $\gamma_1, \gamma_2, \dots, \gamma_n$ there holds

$$a_2 \sum_{j=1}^n |\gamma_j|^2 \leq \sum_{j=1}^n |\gamma_j \sigma_j|^2 \leq a_1 \sum_{j=1}^n |\gamma_j|^2;$$

- 4). The sequence of functions $\{\sigma_j(x)\}_1^{\infty}$ is complete in R and its Gram matrix $(\sigma_j(x), \sigma_k(x))_1^{\infty}$ generates a bounded invertible operator in space ℓ_2 ;
- 5). The sequence of functions $\{\sigma_j(x)\}_1^{\infty}$ is complete in R , corresponds to complete biorthogonal sequence of functions $\{\chi_j(x)\}_1^{\infty}$ and for any $f(x) \in R$ it is true that

$$\sum_{j=1}^n |(f, \sigma_j)|^2 < \infty, \quad \sum_{j=1}^n |(f, \chi_j)|^2 < \infty.$$

Lemma 1 ([57]). Let be $f(x) \in L_2(0, 1)$ and $a_k = \int_0^1 f(x) e^{-\lambda k x} dx$, $b_k = \int_0^1 f(x) e^{\lambda k(x-1)} dx$, where λ is an

arbitrary complex number $\lambda = \alpha + i\beta$ with a positive real part $\alpha > 0$. Then the series $\sum_{k=1}^{\infty} |a_k|^2$, $\sum_{k=1}^{\infty} |b_k|^2$ converges.

3. Eigenvalues and eigenfunctions

First, we consider the following homogeneous differential equation

$$\frac{\partial U(t, x)}{\partial t} + \frac{\partial^5 U(t, x)}{\partial t \partial x^4} + \frac{\partial^4 U(t, x)}{\partial x^4} = 0. \quad (7)$$

We will look for a non-trivial particular solution of the equation in the form $U(t, x) = u(t) \cdot \vartheta(x)$. Substituting this product of functions, depending on different variables, into equation (7), we obtain

$$-\frac{u'(t)}{u'(t) + u(t)} = \frac{\vartheta^{(IV)}(x)}{\vartheta(x)}.$$

Hence, equating second fraction into λ , we obtain

$$\vartheta^{(IV)}(x) - \lambda \vartheta(x) = 0, \quad \lambda \geq 0. \quad (8)$$

Using conditions (3) and (4), from the product of two functions, we obtain conditions for the eigenvalues λ and eigenfunctions $\vartheta(x)$:

$$\vartheta(1) = 0, \quad \vartheta''(0) = 0, \quad \vartheta'(0) = \vartheta'(1), \quad \vartheta'''(0) = \vartheta'''(1). \quad (9)$$

Solving the spectral problem (8), (9), we derive the eigenvalues

$$\lambda_n = (2\pi n)^4, \quad n = 0, 1, 2, \dots \quad (10)$$

Eigenfunctions, corresponding to the eigenvalues (10), have the forms

$$\vartheta_0(x) = 2(1-x), \quad \vartheta_{1n}(x) = -2 \sin 2\pi n x, \quad \vartheta_{2n}(x) = \frac{e^{2\pi n x} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi n x. \quad (11)$$

The spectral problem (8), (9) is not self-adjoint and it is easy to see that the following problem is adjoint to it

$$\sigma^{(IV)}(x) - \lambda \sigma(x) = 0, \quad 0 < x < 1, \quad (12)$$

$$\sigma(0) = \sigma(1), \quad \sigma'(1) = 0, \quad \sigma''(0) = \sigma''(1), \quad \sigma'''(0) = 0. \quad (13)$$

We also consider adjoint to it problem (12), (13). Solving this problem, it is not difficult to see that the eigenfunctions, corresponding to eigenvalues (10), have the form

$$\sigma_0(x) = 1, \quad \sigma_{1n}(x) = \frac{e^{2\pi n x} + e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \sin 2\pi n x, \quad \sigma_{2n}(x) = -2 \cos 2\pi n x. \quad (14)$$

It should be noted that (11) and (14) are non-orthogonal system of functions. For example, for system (11), we have

$$(\vartheta_0(x), \vartheta_{1,n}(x))_0 = -4 \int_0^1 (1-x) \sin 2\pi n x dx = -\frac{2}{\pi n} \neq 0.$$

Let us study the issues of the basis of systems (11) and (14) in $L_2(0, 1)$.

Lemma 2. Systems of functions (11) and (14) are biorthogonal systems in $L_2(0, 1)$:

$$(\vartheta_0, \sigma_0)_0 = 1, \quad (\vartheta_{ik}, \sigma_{jn})_0 = \begin{cases} 1, & k = n, \quad i = j \\ 0, & k \neq n, \quad i \neq j \end{cases}, \quad i, j = 1, 2, \quad n, k = 1, 2, \dots$$

Proof. We present the proof of Lemma 2 for the functions $\vartheta_{1n}(x)$ and $\sigma_{1n}(x)$. We calculate the integral

$$(\vartheta_{1k}, \sigma_{1n})_0 = -\frac{2}{e^{2\pi n} - 1} \int_0^1 (e^{2\pi n x} + e^{2\pi n(1-x)}) \sin 2\pi k x dx + 2 \int_0^1 \sin 2\pi n x \sin 2\pi k x dx = I_{kn} + J_{kn},$$

where

$$I_{kn} = -\frac{2}{e^{2\pi n} - 1} \int_0^1 (e^{2\pi n x} + e^{2\pi n(1-x)}) \sin 2\pi k x dx = 0, \quad k, n \in N,$$

$$J_{kn} = 2 \int_0^1 \sin 2\pi k x \cdot \sin 2\pi n x dx = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}, \quad k, n \in N.$$

Consequently, $(\vartheta_{1n}, \sigma_{1n})_0 = 1$ for $k = n$ and $(\vartheta_{1n}, \sigma_{1n})_0 = 0$ for $k \neq n$. The Lemma 2 is proved. $\square$

Lemma 3. The systems of functions (11) and (14) are minimal in $L_2(0, 1)$.

The *proof* of Lemma 3 follows from the existence of a biorthonormal system which was established in Lemma 2.

Theorem 2. The system of functions (11) and (14) is complete in the space $L_2(0, 1)$.

Proof. First, we prove the completeness of (11). Assume the opposite, let the system of functions (11) be incomplete in $L_2(0, 1)$. Then there exists a function $\psi(x)$ from $L_2(0, 1)$, that is orthogonal to all functions of system (11). We will expand the function $\psi(x)$ into the Fourier series

$$\psi(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x + b_n \sin 2\pi n x),$$

which converges in $L_2(0, 1)$. Since $\psi(x)$ is orthogonal to the system $\{-2 \sin 2\pi n x\}_{n=1}^{\infty}$, the last expansion will take the form

$$\psi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n x. \quad (15)$$

By assumption, $\psi(x)$ is orthogonal to all functions of the form $\vartheta_0(x), \vartheta_{2k}(x)$. Next, multiplying the series (15) successively by these functions and integrating over the interval $[0, 1]$, we have

$$\begin{aligned} 0 &= 2 \int_0^1 \psi(x)(1-x) dx = 2a_0 \int_0^1 (1-x) dx + 2 \sum_{n=1}^{\infty} a_n \int_0^1 (1-x) \cos 2\pi n x dx = a_0, \\ 0 &= \sum_{n=1}^{\infty} a_n \int_0^1 \left(\frac{e^{2\pi k x} - e^{2\pi k(1-x)}}{e^{2\pi k} - 1} - \cos 2\pi k x \right) \cos 2\pi n x dx = -\frac{1}{2} a_k, \quad k = 1, 2, 3, \dots \end{aligned}$$

Hence, it follows that $a_k = 0$, $k = 0, 1, 2, \dots$. Therefore, from (15), we obtain that $\psi(x) = 0$ on $[0, 1]$ which contradicts the condition $\psi(x) \neq 0$. Thus, system (11) is complete in the space $L_2(0, 1)$.

Now, we will prove the completeness of system (14). Let there $\psi(x)$ be a function from $L_2(0, 1)$, different from zero, orthogonal to all functions of system (14). Since the function $\psi(x)$ is orthogonal to system $\{-2 \cos 2\pi n x\}_{n=0}^{\infty}$, it can be represented in $L_2(0, 1)$ as a sine series

$$\psi(x) = \sum_{n=1}^{\infty} b_n \sin 2\pi n x. \quad (16)$$

Next, multiplying the last series by $\sigma_{1k}(x)$ and integrating over the interval $[0, 1]$, taking into account the orthogonality of the functions $\psi(x)$ and $\sigma_{1k}(x)$, we obtain

$$0 = \sum_{n=0}^{\infty} b_n \int_0^1 \left(\frac{e^{2\pi kx} + e^{2\pi k(1-x)}}{e^{2\pi k} - 1} - \sin 2\pi kx \right) \sin 2\pi nx dx = -\frac{1}{2} b_k, \quad k = 1, 2, \dots,$$

i.e. $b_k = 0$, $n = 1, 2, \dots$. Then from (16), it follows that $\psi(x) = 0$ on $[0, 1]$. Consequently, the system (14) is complete in $L_2(0, 1)$. Theorem 2 is proved. $\square$

Theorem 3. The system of functions (13) and (14) forms the Riesz basis in $L_2(0, 1)$.

Proof. To prove the Riesz basis property of systems (11) and (14), according to Theorem 1, it is sufficient to establish the completeness of these systems and the convergence for any $\psi(x) \in L_2(0, 1)$ of the following series:

$$\begin{aligned} & (\psi(x), 2(1-x))_0^2 + \sum_{n=1}^{\infty} (\psi(x), -2 \sin 2\pi nx)_0^2 + \\ & + \sum_{n=1}^{\infty} \left(\psi(x), \frac{1}{e^{2\pi n} - 1} (e^{2\pi nx} - e^{2\pi n(1-x)}) - \cos 2\pi nx \right)_0^2, \end{aligned} \quad (17)$$

$$\begin{aligned} & (\psi(x), 1)_0^2 + \sum_{n=1}^{\infty} (\psi(x), -2 \cos 2\pi nx)_0^2 + \\ & + \sum_{n=1}^{\infty} \left(\frac{1}{e^{2\pi n} - 1} (e^{2\pi nx} + e^{2\pi n(1-x)}) - \sin 2\pi nx, \psi(x) \right)_0^2. \end{aligned} \quad (18)$$

The completeness of systems (11) and (14) follows from Lemma 2 and therefore we show the convergence of series (17) and (18). Let us consider the series (11) and use the notation

$$\begin{aligned} I_1 &= 4(\psi(x), (1-x))_0^2, \quad I_2 = 4 \sum_{n=1}^{\infty} (\psi(x), \sin 2\pi nx)_0^2, \\ I_3 &= \sum_{n=1}^{\infty} \left(\psi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi nx \right)_0^2. \end{aligned}$$

Applying the Cauchy-Schwartz inequality, for I_1 we obtain

$$I_1 = 4 \left(\int_0^1 (1-x) \psi(x) dx \right)^2 \leq 4 \int_0^1 (1-x)^2 dx \int_0^1 \psi^2(x) dx = \frac{4}{3} \|\psi(x)\|_{L_2(0,1)}^2 < \infty.$$

Further, we have

$$I_2 = 4 \sum_{n=1}^{\infty} (\psi(x), \sin 2\pi nx)_0^2 = 2 \sum_{n=1}^{\infty} (\psi(x), \sqrt{2} \sin 2\pi nx)^2 = 2 \sum_{n=1}^{\infty} c_n^2,$$

where $c_n = (\psi(x), \sqrt{2} \sin 2\pi nx)$ are the Fourier coefficients of a function $\psi(x)$ in an orthonormal system $\{\sqrt{2} \sin 2\pi nx\}$. Hence, applying Bessel's inequality, we obtain that

$$I_2 = 2 \sum_{n=1}^{\infty} c_n^2 \leq 2 \|\psi(x)\|_{L_2(0,1)}^2 < \infty.$$

Next, we consider I_3 . Since

$$A = \left(\left(\psi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} \right) - (\psi(x), \cos 2\pi nx) \right)_0^2,$$

applying the inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we obtain

$$A \leq 2 \left(\left(\psi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1} \right) - \left(\psi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1} \right) \right)_0^2 + 2(\psi(x), \cos 2\pi nx)_0^2.$$

Applying the previous inequality again, we get that

$$A \leq 4 \left(\psi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1} \right)_0^2 + 4 \left(\psi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1} \right)_0^2 + 2(\psi(x), \cos 2\pi nx)_0^2.$$

Hence, we obtain

$$I_3 \leq 4 \sum_{n=1}^{\infty} \left(\psi(x), \frac{e^{2\pi n x}}{e^{2\pi n} - 1} \right)_0^2 + 4 \sum_{n=1}^{\infty} \left(\psi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1} \right)_0^2 + 2 \sum_{n=1}^{\infty} (\psi(x), \cos 2\pi n x)_0^2 = J_1 + J_2 + J_3.$$

For J_3 , we have

$$J_3 = 2 \sum_{n=1}^{\infty} (\psi(x), \cos 2\pi n x)^2 = \sum_{n=1}^{\infty} a_n^2,$$

where $a_n = (\psi(x), \sqrt{2} \cos 2\pi n x)$ are the Fourier coefficients for function $\psi(x)$ in orthonormal system $\{\sqrt{2} \cos 2\pi n x\}$. Then, applying Bessel's inequality, we obtain that

$$J_3 = \sum_{n=1}^{\infty} a_n^2 \leq \|\psi(x)\|_{L_2(0,1)}^2 < \infty.$$

Since

$$\left(\psi(x), \frac{e^{2\pi n x}}{e^{2\pi n} - 1} \right)_0^2 \leq 4 \left(\int_0^1 \psi(x) e^{2\pi n(x-1)} dx \right)^2,$$

for J_1 , we derive

$$J_1 \leq 16 \sum_{n=1}^{\infty} \left(\int_0^1 \psi(x) e^{2\pi n(x-1)} dx \right)^2 = 16 \sum_{n=1}^{\infty} b_n^2, \quad b_n = \int_0^1 \psi(x) e^{2\pi n(1-x)} dx.$$

Hence, taking into account Lemma 1, we obtain that J_1 is finite. Similarly, we obtain that J_2 is also finite. Thus, the series I_1 and I_2 converge. Therefore, the series (17) also converges. The convergence of the series (18) is proved by similar way. Theorem 3 is proved. $\square$

4. Construction of solution of the boundary value problem

Taking into account the formulas (11) and (14) we look for a solution to the problem (1)-(4) in the form of the following Fourier series:

$$U(t, x) = u_0(t) \vartheta_0(x) + \sum_{n=1}^{\infty} (u_{1,n}(t) \vartheta_{1,n}(x) + u_{2,n}(t) \vartheta_{2,n}(x)), \quad (19)$$

where

$$u_0(t) = \int_0^1 U(t, y) \sigma_0(y) dy, \quad u_{1,n}(t) = \int_0^1 U(t, y) \sigma_{1,n}(y) dy, \quad (20)$$

$$u_{2,n}(t) = \int_0^1 U(t, y) \sigma_{2,n}(y) dy. \quad (21)$$

Let the function $U(t, x)$ be a solution to the mixed problem (1)-(4). Then, substituting representation (19) into equation (1) and taking (11) and (14) into account, we obtain

$$\begin{aligned} & u'_0(t) \vartheta_0(x) + \sum_{n=1}^{\infty} (u'_{1,n}(t) \vartheta_{1,n}(x) + u'_{2,n}(t) \vartheta_{2,n}(x)) + \\ & + \sum_{n=1}^{\infty} \lambda_n [u'_{1,n}(t) \vartheta_{1,n}(x) + u'_{2,n}(t) \vartheta_{2,n}(x)] + \sum_{n=1}^{\infty} \lambda_n [u_{1,n}(t) \vartheta_{1,n}(x) + u_{2,n}(t) \vartheta_{2,n}(x)] + \\ & = f_0(t) \vartheta_0(x) + \sum_{n=1}^{\infty} (f_{1,n}(t) \vartheta_{2,n}(x) + f_{2,n}(t) \vartheta_{2,n}(x)), \end{aligned}$$

where

$$f_0(t) = \int_0^1 f(t, y) \sigma_0(y) dy, \quad (22)$$

$$f_{\kappa,n}(t) = \int_0^1 f(t, y) \sigma_{\kappa,n}(y) dy. \quad (23)$$

Hence, by virtue of (20)-(23), we obtain

$$u'_0(t) = f_0(t), \quad (24)$$

$$u'_{\kappa,n}(t) + \mu_n u_{\kappa,n}(t) = \frac{f_{\kappa,n}(t)}{1 + \lambda_n}, \quad \kappa = 1, 2, \quad (25)$$

where $\mu_n = \frac{\lambda_n}{1 + \lambda_n}$, $\lambda_n = (2n\pi)^4$. Taking into account the formulas (11) and (14), we consider the function $\varphi(x)$ as in the case of (19):

$$\varphi(t, x, p) = \varphi_0(t, p) \vartheta_0(x) + \sum_{n=1}^{\infty} \left(\varphi_{1,n}(t, p) \vartheta_{1,n}(x) + \varphi_{2,n}(t, p) \vartheta_{2,n}(x) \right). \quad (26)$$

The differential equation (24) is simple. The differential equation (25) is the countable system of differential equations. Taking (26) into account, from the two-point condition (2), we obtain

$$\begin{aligned} \alpha(t)u_0(0) + \omega\beta(t)u_0(T) &= \int_0^1 [\alpha(t)U_0(0, y) + \omega\beta(t)U_0(T, y)]\sigma_0(y)dy = \\ &= \int_0^1 \varphi_0(t, y, p(t))\sigma_0(y)dy = \varphi_0(t, p(t)), \end{aligned} \quad (27)$$

$$\begin{aligned} \alpha(t)u_{\kappa,n}(0) + \omega\beta(t)u_{\kappa,n}(T) &= \int_0^1 [\alpha(t)U_{\kappa}(0, y) + \omega\beta(t)U_{\kappa}(T, y)]\sigma_{\kappa,n}(y)dy = \\ &= \int_0^1 \varphi_{\kappa}(t, y, p(t))\sigma_{\kappa,n}(y)dy = \varphi_{\kappa,n}(t, p(t)), \quad \kappa = 1, 2. \end{aligned} \quad (28)$$

First, we solve equation (24):

$$u_0(t) = u_0(0) + \int_0^t f_0(s)ds. \quad (29)$$

Substituting the presentation (29) into two-point boundary condition (27), we obtain

$$u_0(0) = \frac{\varphi_0(t, p(t))}{\alpha(t) + \omega\beta(t)} - \omega \frac{\beta(t)}{\alpha(t) + \omega\beta(t)} \int_0^T f_0(s)ds. \quad (30)$$

Substituting the presentation (30) into (29), we have

$$u_0(t) = a_0(t)\varphi_0(t, p(t)) + \int_0^T K_0(t, s)f_0(s)ds, \quad (31)$$

where $a_0(t) = (\alpha(t) + \omega\beta(t))^{-1}$, $K_0(t, s) = \begin{cases} 1 - \omega a_0(t)\beta(t), & 0 \leq s \leq t, \\ -\omega a_0(t)\beta(t), & s \leq t \leq T. \end{cases}$

We consider such values of the parameter ω , for which $\alpha(t) + \omega\beta(t) \neq 0$. These values of the parameter ω are called the first regular values of the parameter ω . For the first regular values of the parameter ω , the presentation (31) is the solution of the problem (24), (27).

Now we solve the countable system (25) of ordinary differential equations with two-point boundary value condition (28). In this purpose, we rewrite the equation (20) as

$$u'_{\kappa,n}(t) = -\mu_n u_{\kappa,n}(t) + \frac{f_{\kappa,n}(t)}{1 + \lambda_n}, \quad \kappa = 1, 2. \quad (32)$$

Solving equation (32), we obtain

$$u_{\kappa,n}(t) = u_{\kappa,n}(0) \exp \{ -\mu_n t \} + \frac{1}{1 + \lambda_n} \int_0^t \exp \{ -\mu_n (t - s) \} f_{\kappa,n}(s) ds, \quad \kappa = 1, 2. \quad (33)$$

Substituting the presentation (33) into two-point boundary condition (30), we get

$$\begin{aligned} u_{\kappa,n}(0) &= \frac{\varphi_{\kappa,n}(t, p(t))}{\alpha(t) + \omega\beta(t) \exp \{ -\mu_n T \}} - \\ &- \frac{\omega}{1 + \lambda_n} \frac{\beta(t)}{\alpha(t) + \omega\beta(t) \exp \{ -\mu_n T \}} \int_0^T \exp \{ -\mu_n (T - s) \} f_{\kappa,n}(s) ds, \quad \kappa = 1, 2. \end{aligned}$$

Hence, taking into account (33), we derive that

$$u_{\kappa,n}(t) = \varphi_{\kappa,n}(t, p(t)) \frac{\exp \{ \mu_n(T-t) \}}{a_{2,n}(t)} + \frac{1}{1 + \lambda_n} \int_0^T K_n(t, s) f_{\kappa,n}(s) ds, \quad \kappa = 1, 2, \quad (34)$$

$$\text{where } K_n(t, s) = \begin{cases} \exp \{ -\mu_n(t-s) \} - \frac{\omega\beta(t)}{a_{2,n}(t)} \exp \{ -\mu_n(t-s) \}, & 0 \leq s \leq t, \\ -\frac{\omega\beta(t)}{a_{2,n}(t)} \exp \{ -\mu_n(t-s) \}, & t \leq s \leq T, \end{cases}$$

$$a_{2,n}(t) = \omega\beta(t) + \alpha(t) \exp \{ \mu_n T \}, \quad \mu_n = \frac{\lambda_n}{1 + \lambda_n}, \quad \lambda_n = (2n\pi)^4, \quad n = 1, 2, \dots$$

We consider such values of the parameter ω , for which $a_{2,n}(t) = \omega\beta(t) + \alpha(t) \exp \{ \mu_n T \} \neq 0$. These values of the parameter ω are called the second regular values of the parameter ω . For the second regular values of the parameter ω , presentation (34) is the representation of solution of the problem (32), (30).

The sets of the first and the second regular values of the parameter ω is denoted by Λ_1 and Λ_2 , respectively. We consider the intersection $\Lambda_1 \cap \Lambda_2$ of these values of the parameter ω . So, for regular values of the parameter ω from this intersection $\Lambda_1 \cap \Lambda_2$, we have to consider the Fourier series

$$\begin{aligned} U(t, x) = & \vartheta_0(\eta(t)) a_0(t) \varphi_0(t, p(t)) + \vartheta_0(x) \int_0^T K_0(t, s) f_0(s) ds + \\ & + \sum_{n=1}^{\infty} \sum_{\kappa=1}^2 \left[\vartheta_{\kappa,n}(\eta(t)) \varphi_{\kappa,n}(t, p(t)) a_{1,n}(t) + \vartheta_{\kappa,n}(x) \frac{1}{1 + \lambda_n} \int_0^T K_n(t, s) f_{\kappa,n}(s) ds \right]. \end{aligned} \quad (35)$$

Smoothness condition S. Let the functions $\varphi(t, x, p(t))$ and $f(t, x)$ be satisfying the conditions

$$\varphi_{\kappa}(t, x, p) \in C_{t,x}^{0,5}(\bar{\Omega}), \quad \varphi_{\kappa}(t, 1, p) = \frac{\partial^2}{\partial x^2} \varphi_{\kappa}(t, 0, p) = \frac{\partial^4}{\partial x^4} \varphi_{\kappa}(t, 1, p) = 0,$$

$$\frac{\partial}{\partial x} \varphi_{\kappa}(t, 0, p) = \frac{\partial}{\partial x} \varphi_{\kappa}(t, 1, p), \quad \frac{\partial^3}{\partial x^3} \varphi_{\kappa}(t, 0, p) = \frac{\partial^3}{\partial x^3} \varphi_{\kappa}(t, 1, p),$$

$$f(t, x) \in C_{t,x}^{0,1}(\bar{\Omega} \times R), \quad f(t, 0) = 0.$$

Then, we integrate by parts (27), (28) five times and (22), (23) one time with respect to variable x , respectively, and obtain

$$\varphi_{1,n}(t, p) = - \left(\frac{1}{2\pi} \right)^5 \frac{\varphi_{1,n}^{(V)}(t, p)}{n^5},$$

$$\varphi_{1,n}^{(V)}(t, p) = \int_0^1 \frac{\partial^5 \varphi(t, z, p)}{\partial(z)^5} \left(\frac{e^{2\pi n z} - e^{2\pi n(1-z)}}{e^{2\pi n} - 1} + \cos 2\pi n z \right) dz, \quad z = \eta(t),$$

$$\varphi_{2,n}(t, p) = \left(\frac{1}{2\pi} \right)^5 \frac{\varphi_{2,n}^{(V)}(t, p)}{n^5}, \quad \varphi_{2,n}^{(V)}(t, p) = 2 \int_0^1 \frac{\partial^5 \varphi(t, z, p)}{\partial(z)^5} \sin 2\pi n z dz, \quad z = \eta(t),$$

$$f_{1,n}(t) = -\frac{1}{2\pi} \frac{f_{1,n}^{(I)}(t)}{n}, \quad f_{1,n}^{(I)}(t) = \int_0^1 \frac{\partial f(t, y)}{\partial y} \left(\frac{e^{2\pi n y} - e^{2\pi n(1-y)}}{e^{2\pi n} - 1} + \cos 2\pi n y \right) dy,$$

$$f_{2,n}(t) = \frac{1}{2\pi} \frac{f_{2,n}^{(I)}(t)}{n}, \quad f_{2,n}^{(I)}(t) = 2 \int_0^1 \frac{\partial f(t, y)}{\partial y} \sin 2\pi n y dy.$$

In addition, we have

$$\left\| \tilde{\varphi}_{\kappa}^{(V)}(t, p) \right\|_{\ell_2} \leq C_1 \left\| \frac{\partial^5 \varphi_{\kappa}(t, z, p)}{\partial z^5} \right\|_{L_2[0,1]}, \quad \left\| \tilde{f}_{\kappa}^{(I)}(t) \right\|_{\ell_2} \leq C_2 \left\| \frac{\partial f_{\kappa}(t, x)}{\partial x} \right\|_{L_2(0,1)}.$$

5. Unique solvability of the problem (1)–(4)

Theorem 4. If there exists a solution of Problem (1)–(4), then it is unique for regular values of the parameter ω from the intersection $\Lambda_1 \cap \Lambda_2$.

Proof. We consider the regular values of the parameter ω from the intersection $\Lambda_1 \cap \Lambda_2$. Suppose that there exist two different solutions $\bar{U}_1(t, x)$ and $\bar{U}_2(t, x)$ to the problem (1)–(4). Then the difference $U(t, x) = \bar{U}_1(t, x) - \bar{U}_2(t, x)$ is a solution of equation (1), satisfying the conditions (2)–(4) with functions $\varphi(t, x, p) \equiv 0$, $f(t, x) \equiv 0$. Then, it follows

from formulas (20)–(23) and (35) that $u_{\kappa,n}(t) = \int_0^1 U(t, y) \sigma_{\kappa,n}(y) dy \equiv 0$, $\kappa = 1, 2$. From this, due to the completeness of system (14) in the space $L_2(0, 1)$, it follows that $U(t, x) = 0$ is valid almost everywhere on $[0, 1]$ for any $t \in [0, T]$. Since $U \in C(\bar{\Omega})$, it follows that $U(x, t) \equiv 0$ in $\bar{\Omega}$. Theorem 4 is proved. $\square$

Theorem 5. Let the smoothness conditions S be satisfied. Then, for the regular values of the parameter ω from the intersection $\Lambda_1 \cap \Lambda_2$ and fixed values of control function $p(t)$, the function (35) belongs to the class of functions (6).

Proof. We consider the series (35) and

$$\begin{aligned} \frac{\partial^4}{\partial t \partial x^3} U_{\kappa}(t, x) = \sum_{n=1}^{\infty} \left[\vartheta_{\kappa,n}'''(z) (a_{1,n}(t))' \varphi_{\kappa,n}(t, p(t)) + \vartheta_{\kappa,n}'''(z) a_{1,n}(t) \frac{d}{dt} \varphi_{\kappa,n}(t, p(t)) + \right. \\ \left. + \frac{\vartheta_{\kappa,n}'''(x)}{1 + \lambda_n} \int_0^T K_n'(t, s) f_{\kappa,n}(s) ds \right], \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial^5}{\partial t \partial x^4} U_{\kappa}(t, x) = \sum_{n=1}^{\infty} \lambda_n \left[\vartheta_{\kappa,n}(z) (a_{1,n}(t))' \varphi_{\kappa,n}(t, p(t)) + \vartheta_{\kappa,n}(z) a_{1,n}(t) \frac{d}{dt} \varphi_{\kappa,n}(t, p(t)) + \right. \\ \left. + \frac{\vartheta_{\kappa,n}(x)}{1 + \lambda_n} \int_0^T K_n'(t, s) f_{\kappa,n}(s) ds \right], \end{aligned} \quad (37)$$

$$\frac{\partial^4}{\partial x^4} U_{\kappa}(t, x) = \sum_{n=1}^{\infty} \lambda_n \left[\vartheta_{\kappa,n}(z) a_{1,n}(t) \varphi_{\kappa,n}(t, p(t)) + \frac{\vartheta_{\kappa,n}(x)}{1 + \lambda_n} \int_0^T K_n'(t, s) f_{\kappa,n}(s) ds \right], \quad (38)$$

where $\kappa = 1, 2$, $z = \eta(t)$.

The proofs of convergence of the series (36)–(38) are similar. So, we will prove of convergence for the series (37). Applying the smoothness conditions to (37), we have:

$$\begin{aligned} \left| \frac{\partial^5 U_{\kappa}(t, x)}{\partial t \partial x^4} \right| \leq \chi_0 \max_{0 \leq t \leq T} \left[\chi_1 \frac{1}{(2\pi)^5} \left(\left\| \frac{\partial^5 \varphi(t, x, p(t))}{\partial x^5} \right\|_{L_2(0,1)} + \left\| \frac{d}{dt} \frac{\partial^5 \varphi(t, x, p(t))}{\partial x^5} \right\|_{L_2(0,1)} \right) + \right. \\ \left. + \frac{1}{2\pi} \left\| \frac{\partial f(t, x)}{\partial x} \right\|_{L_2(0,1)} \right] < \infty, \end{aligned} \quad (39)$$

$$\text{where } \chi_0 = \max \left\{ \|a_{1,n}(t)\|; \|(a_{1,n}(t))'\|; \max_{t \in [0, T]} \int_0^T |K_n'(t, s)| ds \right\}, \chi_1 = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}},$$

$$\chi_2 = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \sqrt{\frac{\pi^2}{6}}.$$

From estimate (39), it implies the convergence of the series (37). The convergence of the remaining series is proved similarly. Theorem 5 is proved. $\square$

6. Optimal control function

Let $p^*(t)$ is optimal control function: $\Delta J[p^*(t)] = J[p^*(t) + \Delta p^*(t)] - J[p^*(t)] \geq 0$, where $p^*(t) + \Delta p^*(t) \in C[0, T]$. We consider the following function

$$Q_0(t, \eta(t)) \vartheta_0(\eta(t)) [a_0(t) \varphi_0(t, p_0^*(t)) + b_0(t)] = \gamma_1 [p_0^*(t)]^2 + \gamma_2 [\eta(t)]^2, \quad (40)$$

$$\begin{aligned} Q_{\kappa}(t, \eta(t)) \left[\sum_{n=1}^{\infty} a_{1,n}(t) \vartheta_{\kappa,n}(\eta(t)) \varphi_{\kappa,n}(t, p_{\kappa}^*(t)) + \sum_{n=1}^{\infty} \frac{b_{\kappa,n}(t)}{1 + \lambda_n} \vartheta_{\kappa,n}(\eta(t)) \right] = \\ = \gamma_1 [p_{\kappa}^*(t)]^2 + \gamma_2 [\eta(t)]^2, \end{aligned} \quad (41)$$

where

$$a_0(t) = \frac{1}{\alpha(t) + \omega\beta(t)}, \quad a_{1,n}(t) = \frac{\exp\{\mu_n(T-t)\}}{a_{2,n}(t)},$$

$$b_0(t) = \int_0^T K_0(t,s) f_0(s) ds, \quad b_{\kappa,n}(t) = \int_0^T K_n(t,s) f_{\kappa,n}(s) ds, \quad \kappa = 1, 2.$$

The functions $Q_j(t, x)$ are defined by solving the following mixed problem

$$Q_{j,t}(t, x) + Q_{j,txxxx}(t, x) - Q_{jxxxx}(t, x) = 0, \quad j = 0, 1, 2, \quad (t, x) \in \Omega, \quad (42)$$

$$Q_j(T, x) = -2[U_j(T, x) - \xi_j(x)], \quad Q_j(t, 1) = 0, \quad Q_{jxx}(t, 0) = 0, \quad (43)$$

$$Q_{jx}(t, 0) = Q_{jx}(t, 1), \quad Q_{jxxx}(t, 0) = Q_{jxxx}(t, 1), \quad 0 \leq t \leq T, \quad (44)$$

which is conjugated to problem (1)-(4). We note that

$$\xi(x) = \xi_0 \vartheta_0(x) + \sum_{n=1}^{\infty} [\xi_{1,n} \vartheta_{1,n}(x) + \xi_{2,n} \vartheta_{2,n}(x)], \quad \xi_0 = \int_0^1 \xi_0(y) \sigma_0(y) dy,$$

$$\xi_{\kappa,n} = \int_0^1 \xi_{\kappa}(y) \sigma_{\kappa,n}(y) dy, \quad |\xi_0| + \sum_{n=1}^{\infty} |\xi_{1,n}| + \sum_{n=1}^{\infty} |\xi_{2,n}| < \infty.$$

We will find the partial derivatives with respect to the control function in (40) and (41) and come to the following necessary conditions for optimality

$$Q_0(t, \eta(t)) \vartheta_0(\eta(t)) a_0(t) \frac{\partial}{\partial p_0^*(t)} \varphi_0(t, p_0^*(t)) = 2\gamma_1 p_0^*(t), \quad (45)$$

$$Q_{\kappa}(t, \eta(t)) \sum_{n=1}^{\infty} a_{1,n}(t) \vartheta_{\kappa,n}(\eta(t)) \frac{\partial}{\partial p_{\kappa}^*(t)} \varphi_{\kappa,n}(t, p_{\kappa}^*(t)) = 2\gamma_1 p_{\kappa}^*(t), \quad (46)$$

where $\frac{\partial}{\partial p(t)} \varphi_0(t, p(t))$ means that $\frac{\partial}{\partial \xi} \varphi_0(t, \xi) \Big|_{\xi=p(t)}$.

Calculating partial derivatives in (45) and (46) again with respect to control function, we obtain another necessary conditions for optimality

$$Q_0(t, \eta(t)) \vartheta_0(\eta(t)) a_0(t) \frac{\partial^2}{\partial [p_0^*(t)]^2} \varphi_0(t, p_0^*(t)) < 2\gamma_1, \quad (47)$$

$$Q_{\kappa}(t, \eta(t)) \sum_{n=1}^{\infty} a_{1,n}(t) \vartheta_{\kappa,n}(\eta(t)) \frac{\partial^2}{\partial [p_{\kappa}^*(t)]^2} \varphi_{\kappa,n}(t, p_{\kappa}^*(t)) < 2\gamma_1. \quad (48)$$

We solve the adjoint problem (42)-(44) by the same way as we used for solving the problem (1)-(4). According to the second condition of (43), the nonzero solution of the equation (42) are found from the system of differential equations

$$q'_0(t) = 0, \quad (49)$$

$$q'_{\kappa,n}(t) = \mu_n q_{\kappa,n}(t), \quad (50)$$

where $q_{k,n}(t) = \int_0^1 Q_k(t, y) \sigma_{k,n}(y) dy$.

The solution of equation (49) is the arbitrary constant

$$q_0(t) = C_0, \quad C_0 = \text{const}. \quad (51)$$

The solution of equation (50) is

$$q_{\kappa,n}(t) = C_{\kappa} \exp\{\mu_n t\}, \quad C_{\kappa} = \text{const}, \quad \kappa = 1, 2. \quad (52)$$

To find C_0 and C_{κ} in (49) and (50), we rewrite (43) in the following form

$$q_{j,n}(T) = -2 \int_0^1 [U_j(T, y) - \xi_j(y)] \sigma_{j,n}(y) dy, \quad j = 0, 1, 2. \quad (53)$$

Taking (31), (34), from condition (53), we obtain

$$C_0 = -2 \int_0^1 \frac{\varphi_0(T, z, p_0(T))}{a_0(T)} \sigma_0(z) dz - 2 \int_0^1 [b_0(T, y) - \xi_0(y)] \sigma_0(y) dy, \quad z = \eta(t), \quad (54)$$

$$C_\kappa = -2 \exp \{ -\mu_n T \} \int_0^1 \frac{\varphi_\kappa(T, z, p_\kappa(T))}{a_{2,n}(T)} \sigma_{\kappa,n}(z) dz +$$

$$+ 2 \exp \{ -\mu_n T \} \int_0^1 [\xi_\kappa(y) + b_\kappa(T, y)] \sigma_{\kappa,n}(y) dy, \quad \kappa = 1, 2, \quad z = \eta(t), \quad (55)$$

where $a_{2,n}(t) = \omega \beta(t) + \alpha(t) \exp \{ \mu_n T \}$.

Substituting (54) and (55) into (51) and (52), we, respectively, obtain

$$q_0(t) = -2 \int_0^1 \left[\frac{\varphi_0(T, y, p_0(T))}{a_0(T)} + b_0(T, y) - \xi_0(y) \right] \sigma_0(y) dy, \quad (56)$$

$$q_{\kappa,n}(t) = -2 \exp \{ -\mu_n(T-t) \} \int_0^1 \left[\frac{\varphi_\kappa(T, y, p_\kappa(T))}{a_{2,n}(T)} + b_\kappa(T, y) - \xi_\kappa(y) \right] \sigma_{\kappa,n}(y) dy. \quad (57)$$

From (56) and (57), we obtain the series

$$Q_0(t, y) = -2 \vartheta_0(y) \int_0^1 \left[\frac{\varphi_0(T, y, p_0(T))}{a_0(T)} + b_0(T, y) - \xi_0(y) \right] \sigma_0(y) dy, \quad (58)$$

$$Q_\kappa(t, y) = -2 \sum_{n=1}^{\infty} \vartheta_{\kappa,n}(y) \exp \{ -\mu_n(T-t) \} \times$$

$$\times \int_0^1 \left[\frac{\varphi_\kappa(T, y, p_\kappa(T))}{a_{2,n}(T)} + b_\kappa(T, y) - \xi_\kappa(y) \right] \sigma_{\kappa,n}(y) dy. \quad (59)$$

Substituting presentations (58) and (59) into (45) and (46), respectively, we have

$$p_0(t) = J_0(t; p_0(t)) \equiv -\frac{\vartheta_0^2(\eta(t))}{\gamma_1} \left[\frac{\varphi_0(T, p_0(T))}{a_0(T)} + b_0(T) - \xi_0 \right] a_0(t) \frac{\partial}{\partial p_0} \varphi_0(t, p_0(t)), \quad (60)$$

$$p_\kappa(t) = J_\kappa(t; p_\kappa(t)) \equiv -\frac{1}{\gamma_1} \sum_{n=1}^{\infty} \vartheta_{\kappa,n}(\eta(t)) \exp \{ -\mu_n(T-t) \} \times$$

$$\times \left[\frac{\varphi_{\kappa,n}(T, p_\kappa(T))}{a_{2,n}(T)} + b_{\kappa,n}(T) - \xi_{\kappa,n} \right] \sum_{n=1}^{\infty} a_{1,n}(t) \vartheta_{\kappa,n}(\eta(t)) \frac{\partial}{\partial p_\kappa} \varphi_{\kappa,n}(t, p_\kappa(t)). \quad (61)$$

For determination of control functions $p_0(t)$ and $p_\kappa(t)$ ($\kappa = 1, 2$), we have two functional equations (60) and (61).

Theorem 6. Let the following conditions be fulfilled:

- 1). $\| \varphi_0(t, p_0(t)) \|_{C[0,T]} \leq M_{01}, \quad 0 < M_{01} = \text{const};$
- 2). $\left\| \frac{\partial \varphi_0(t, p_0(t))}{\partial p_0(t)} \right\|_{C[0,T]} \leq M_{02}, \quad 0 < M_{02} = \text{const};$
- 3). $|\varphi_0(t, p_0(t)) - \varphi_0(t, \bar{p}_0(t))| \leq L_{01} |p_0(t) - \bar{p}_0(t)|, \quad 0 < L_{01} = \text{const};$
- 4). $\left| \frac{\partial}{\partial p_0(t)} \varphi_0(t, p_0(t)) - \frac{\partial}{\partial \bar{p}_0(t)} \varphi_0(t, \bar{p}_0(t)) \right| \leq L_{02} |p_0(t) - \bar{p}_0(t)|, \quad 0 < L_{02} = \text{const};$
- 5). $\rho_0 = \max \left\{ \frac{L_{02}}{\gamma_1 a_0} M_{01}; \rho_{01} \frac{L_{01}}{\gamma_1 a_0} M_{02} \right\} < 1$, where $\rho_{01} = \frac{1}{\gamma_1 a_0} (L_{01} M_{02} + L_{02} M_{01})$.

Then, for the first regular values of the parameter ω , from the set Λ_1 , the functional equation (60) has a unique control function $p_0(t)$ (solution) on the segment $[0, T]$. This solution can be found by the iteration process:

$$p_0^0(t) = -\frac{\vartheta_0^2(\eta(t))}{\gamma_1} a_0(t) (b_0(T) - \xi_0), \quad (62)$$

$$p_0^{m+1}(t) = J_0(t; p_0^m(t)).$$

Proof. By virtue of conditions of the theorem 6, from (62), we have

$$\| p_0^0(t) \|_{C[0,T]} \leq \frac{4(1-\eta(t))^2}{\gamma_1} a_0(b_0(T) + |\xi_0|) \leq \frac{4}{\gamma_1} a_0(b_0(T) + |\xi_0|) < \infty, \quad (63)$$

where $a_0 = \max_{0 \leq t \leq T} |a_0(t)|$;

$$\| p_0^{m+1}(t) - p_0^m(t) \| \leq \frac{L_{01}}{\gamma_1 a_0} M_{02} \| p_0^m(T) - p_0^{m-1}(T) \| + \frac{L_{02}}{\gamma_1 a_0} M_{01} \| p_0^m(t) - p_0^{m-1}(t) \|. \quad (64)$$

For $t = T$ from (64), we have

$$\begin{aligned} \|p_0^{m+1}(T) - p_0^m(T)\|_{C[0,T]} &\leq \frac{L_{01}}{\gamma_1 a_0} M_{02} \|p_0^m(T) - p_0^{m-1}(T)\|_{C[0,T]} + \\ &+ \frac{L_{02}}{\gamma_1 a_0} M_{01} \|p_0^m(T) - p_0^{m-1}(T)\|_{C[0,T]} \leq \rho_{01} \|p_0^m(T) - p_0^{m-1}(T)\|_{C[0,T]}, \end{aligned} \quad (65)$$

where $\rho_{01} = \frac{1}{\gamma_1 a_0} (L_{01} M_{02} + L_{02} M_{01})$.

From the estimates (64) and (65), we obtain that there holds the estimate

$$\begin{aligned} &\|p_0^{m+1}(t) - p_0^m(t)\|_{C[0,T]} + \|p_0^{m+1}(T) - p_0^m(T)\|_{C[0,T]} \leq \\ &\leq \rho_0 \left[\|p_0^{m+1}(t) - p_0^m(t)\|_{C[0,T]} + \|p_0^{m+1}(T) - p_0^m(T)\|_{C[0,T]} \right], \end{aligned} \quad (66)$$

where $\rho_0 = \max \left\{ \frac{L_{02}}{\gamma_1 a_0} M_{01}; \rho_{01} \frac{L_{01}}{\gamma_1 a_0} M_{02} \right\}$. From the estimates (63) and (66), it implies that the operator on the right-hand side of equation (60) is compressing mapping and the functional equation has unique solution on the interval $[0, T]$. $\square$

Theorem 7. Let the smoothness conditions S and the following conditions be fulfilled:

- 1). $\|\vec{\varphi}_\kappa(t, p_\kappa(t))\|_{\ell_2} \leq M_{\kappa 1}, \quad 0 < M_{\kappa 1} = \text{const}, \quad \kappa = 1, 2;$
- 2). $\left\| \frac{\partial}{\partial p_\kappa} \vec{\varphi}_\kappa(t, p_\kappa(t)) \right\|_{\ell_2} \leq M_{\kappa 2}, \quad 0 < M_{\kappa 2} = \text{const};$
- 3). $|\varphi_\kappa(t, p_\kappa(t)) - \varphi_\kappa(t, \bar{p}_\kappa(t))| \leq L_{\kappa 1} |p_\kappa(t) - \bar{p}_\kappa(t)|, \quad 0 < L_{\kappa 1} = \text{const};$
- 4). $\left| \frac{\partial}{\partial p_\kappa(t)} \varphi_\kappa(t, p_\kappa(t)) - \frac{\partial}{\partial \bar{p}_\kappa(t)} \varphi_\kappa(t, \bar{p}_\kappa(t)) \right| \leq L_{\kappa 2} |p_\kappa(t) - \bar{p}_\kappa(t)|, \quad 0 < L_{\kappa 2} = \text{const};$
- 5). $\rho_\kappa = \max \left\{ \frac{L_{\kappa 2}}{\gamma_1 a_\kappa} M_{\kappa 1}; \rho_{\kappa 1} \frac{L_{\kappa 1}}{\gamma_1 a_\kappa} M_{\kappa 2} \right\} < 1$, where $\rho_{\kappa 1} = \frac{1}{\gamma_1 a_\kappa} (L_{\kappa 1} M_{\kappa 2} + L_{\kappa 2} M_{\kappa 1})$.

Then, for the second regular values of the parameter ω from the set Λ_2 , functional equation (61) has a unique solution on the segment $[0, T]$. This solution can be found by the iteration process:

$$\begin{aligned} p_\kappa^0(t) &= -\frac{1}{\gamma_1} \sum_{n=1}^{\infty} \vartheta_{\kappa, n}(\eta(t)) \exp \{ -\mu_n(T-t) \} (b_{\kappa, n}(T) - \xi_{\kappa, n}), \\ p_\kappa^{m+1}(t) &= J_\kappa(t; p_\kappa^m(t)). \end{aligned}$$

Proof of the theorem 7 is similar to the proof of the Theorem 6.

We substitute the found values of the control functions $p_j(t)$, $j = 0, 1, 2$ into the functions (35). Then we uniquely determine the state function $U(t, x)$. The proof of this statement is similar to the proofs of the Theorems 4 and 5.

7. Conclusion

In the domain $\Omega = \{t \in (0, T), x \in (0, 1)\}$, we consider a pseudoparabolic type differential equation with nonlinear boundary conditions in regard to time variable. Samarskii–Ionkin type boundary value conditions in regard to spatial variable x are used in solving the nonlocal optimal movable point control problem.

The scalar and countable system of nonlinear functional integral equations are obtained. The properties of eigenfunctions and unique solvability of scalar and countable system of nonlinear functional integral equations with respect to state function is proved for fixed values of the control function. The optimality conditions are found. The equations with product of nonlinear functions are obtained for control function. The Picar iteration process is built. The uniform convergence of the Fourier series is proved. The results of this work make it possible to determine the solutions of huge set of problems.

Moreover, the results obtained in this work will allow us in the future to investigate direct and inverse optimal control problems for other kind of partial differential equations of mathematical physics with different quality functionals. Parabolic type differential equations, such as the heat equations or the diffusion equations, have different applications. In [14], the problem of fast forward evolution of the processes described in terms of the heat equation is considered. The matter is considered on an adiabatically expanding time-dependent box. Attention is paid to acceleration of heat transfer processes. As the physical realization, the heat transport in harmonic crystals is considered. In [11], Sibatov R. T., Svetukhin V. V. are studied the subdiffusion kinetics of nanoprecipitate growth and destruction in solid solutions. The authors of the work [58] are proposed a numerical method for estimating the effective thermal conductivity coefficient of hydrate-bearing rock samples using synchrotron-based microtomography data. In the work [59], is studied the development of magnetorotation instability in the accreting envelope of a protostar in the non-isothermal case. The paper [60] is devoted to the development of a multiscale approach to calculating gas flows near solid surfaces taking into account microscopic effects. The work [61] discusses approaches to conducting comprehensive optimization of devices that protect against

ultrawideband interference, known as modal filters. These devices can take the form of strip N-conductor structures with a various number of conducting layers, as well as cables. Optimization is one of the important stages of their design.

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Submitted 11 June 2025; revised 28 July 2025; accepted 20 August 2025

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Conflict of interest: the authors declare no conflict of interest.