NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Akhmadova M.O., et al. *Nanosystems: Phys. Chem. Math.*, 2025, **16** (5), 577–585.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2025-16-5-577-585

Original article

Spectral analysis of two-particle Hamiltonians with short-range interactions

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ABSTRACT We analyze the spectral characteristics of lattice Schrödinger operators, denoted as $H_{\gamma\lambda\mu}(K)$, $K \in (-\pi,\pi]^3$, which represent a system of two identical bosons existing on $\mathbb{Z}^3$ lattice. The model includes on-site and nearest-neighbor interactions, parameterized by $\gamma,\lambda,\mu \in \mathbb{R}$. Our study of $H_{\gamma\lambda\mu}(0)$ reveals an invariant subspace on which its restricted form, $H^{\mathrm{ea}}_{\lambda\mu}(0)$, is solely dependent on λ and μ. To elucidate the mechanisms of eigenvalue birth and annihilation for $H^{\mathrm{ea}}_{\lambda\mu}(0)$, we define a critical operator. A detailed criterion is subsequently developed within the plane spanned by λ and μ. This involves: (i) the derivation of smooth critical curves that mark the onset of criticality for the operator, and (ii) the proof of exact conditions for the existence of precisely α eigenvalues below and β eigenvalues above the essential spectrum, where $\alpha,\beta \in \{0,1,2\}$ and $\alpha+\beta \leq 2$.

KEYWORDS Two-particle system, lattice Schrödinger operator, essential spectrum, bound states, Fredholm determinant.

ACKNOWLEDGEMENTS The authors are grateful to the referee for valuable insights. Support for this work was provided by the Fundamental Science Foundation of Uzbekistan, Grant No. FL-9524115052.

FOR CITATION Akhmadova M.O., Azizova M.A. Spectral analysis of two-particle Hamiltonians with short-range interactions. *Nanosystems: Phys. Chem. Math.*, 2025, **16** (5), 577–585.

1. Introduction

Lattice models are fundamental to mathematical physics, particularly few-body Hamiltonians [1], which simplify Bose-Hubbard models to finite particle interactions. Decades of research underscore their importance [2–16].

These discrete Hamiltonians also provide effective approximations for their continuous counterparts [17]. A prime illustration is the Efimov effect [18], rigorously proven for continuous three-particle systems [19–22], and demonstrably presented in lattice three-particle systems as well [2, 4, 9, 23]. The bound state energies of one- and two-particle systems, situated in two adjacent 3D layers linked by a window, were numerically reported in [24].

Beyond theoretical significance, discrete Schrödinger operators model few-particle systems in periodic structures, exemplified by ultracold atoms in optical crystals [25, 26]. Recent years have seen a surge in studying ultracold few-atom systems in optical lattices, driven by precise control over parameters like temperature and interaction potentials [25, 27–29]. This control facilitates experimental observation of phenomena like stable repulsively bound pairs [26, 30], which challenges the typical formation of stable objects through attractive forces.

Lattice Hamiltonians also find application in fusion physics. For example, [14] showed that a one-particle one-dimensional lattice Hamiltonian could enhance nuclear fusion probability in specific lattice structures.

A key challenge in lattice few-particle problems, unlike their continuous analogues, is the general inability to separate center-of-mass motion. However, for a lattice Hamiltonian H on $\mathbb{T}^{n\cdot d}$, the von Neumann direct integral decomposition provides a solution:

$$\mathrm{H} \simeq \int\limits_{K\in\mathbb{T}^d}^{\oplus} H(K)\,dK,$$

where $\mathbb{T}^d$ is the d-dimensional torus. This decomposition effectively transforms the problem into analyzing the fiber Hamiltonians $H(K)$, which act on $\mathbb{T}^{(n-1)d}$ and critically depend on the quasi-momentum $K \in \mathbb{T}^d$ [2, 3, 31].

In the present work, we focus on the spectral properties of $H_{\gamma\lambda\mu}(K)$ for $K \in \mathbb{T}^3$, acting on $L^{2,e}(\mathbb{T}^3)$ as

$$H_{\gamma\lambda\mu}(K) := H_0(K) + V_{\gamma\lambda\mu}.$$

The unperturbed operator $H_0(K)$ defined by multiplication with the dispersion function $\mathcal{E}_K$ defined in (3). The interaction potential $V_{\gamma\lambda\mu}$ is independent of the quasi-momentum $K \in \mathbb{T}^3$. Notably, the coupling constants γ, λ, and μ denote the on-site, first-neighbor, and second-nearest-neighbor interactions within the lattice system, respectively.

The operators $H_0(K)$ and $V_{\gamma\lambda\mu}$ (with $\gamma,\lambda,\mu \in \mathbb{R}$) are both bounded and self-adjoint. Because $V_{\gamma\lambda\mu}$ has finite rank, Weyl's theorem establishes that $\sigma_{ess}(H_{\gamma\lambda\mu}(K))$ coincides with $\sigma(H_0(K))$. This spectrum spans the interval $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$, defined by (5).

The Hilbert space $L^{2,\mathrm{e}}(\mathbb{T}^3)$ is decomposed into the direct sum of invariant subspaces for the operator $H_{\gamma\lambda\mu}(0)$:

$$L^{2,\mathrm{e}}(\mathbb{T}^3) = L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3) \oplus \left[L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right]^{\perp}. \tag{1}$$

Here, $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$ comprises functions in $L^{2,\mathrm{e}}(\mathbb{T}^3)$ that are antisymmetric in their first two variables, and $\left[L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right]^{\perp}$ is its orthogonal complement.

The decomposition (1) leads to the following spectral representation for $H_{\gamma\lambda\mu}(0)$:

$$\sigma(H_{\gamma\lambda\mu}(0)) = \sigma(H_{\gamma\lambda\mu}(0)|_{L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)}) \cup \sigma(H_{\gamma\lambda\mu}(0)|_{[L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)]^{\perp}}), \tag{2}$$

where $A\,\big|_{\mathcal{K}}$ denotes the restriction of A onto its reducing subspace $\mathcal{K}$. Consequently, the spectrum of the Hamiltonian $H_{\gamma\lambda\mu}(0)$ can be studied separately for its restrictions to $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$, and $\left[L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right]^{\perp}$.

Importantly, the restriction $V^{\mathrm{ea}}_{\lambda\mu}$ of $V_{\gamma\lambda\mu}$ to $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$ is independent of γ and has a rank of at most two. Consequently, the restriction $H^{\mathrm{ea}}_{\lambda\mu}(0)$ of $H_{\gamma\lambda\mu}(0)$ to $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$ is also independent of γ, and it possesses no more than two isolated eigenvalues.

To investigate the exact number of discrete eigenvalues of $H^{\mathrm{ea}}_{\lambda\mu}(0)$, we reduce the task of identifying these eigenvalues to determining the zeros of its Fredholm determinant $\Delta_{\lambda\mu}(z)$ by constructing a rank-two Lippmann-Schwinger operator $B^{\mathrm{ea}}_{\lambda\mu}(0)$.

Near the upper and lower threshold of $\sigma_{ess}(H_{\gamma\lambda\mu}(0))$, we have found the asymptotic expansions of $\Delta_{\lambda\mu}(z)$. The leading terms in these expansions are defined as the algebraic forms $P^{+}(\lambda,\mu)$ and $P^{-}(\lambda,\mu)$, respectively. As demonstrated by Lemma 5, the polynomial $P^{\pm}(\lambda,\mu)$ has a null set consisting of two separated smooth unbounded connected curves $\tau_0^{\pm},\tau_1^{\pm}$. As a result, the (λ,μ)-plane is split by these curves into three connected regions, denoted $\mathcal{C}_0^{\pm}$, $\mathcal{C}_1^{\pm}$, and $\mathcal{C}_2^{\pm}$.

Our first main result, Theorem 2 shows that the number of eigenvalues of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ lying in $(24,+\infty)$ (resp. $(-\infty,0)$) remains constant within each connected component $\mathcal{C}_\alpha^{+}$ (resp. $\mathcal{C}_\alpha^{-}$), $\alpha=0,1,2$. Theorem 3 subsequently provides the exact value of this constant. Moreover, Corollary 1 provides a criterion, expressed in terms of the perturbation parameters $\lambda,\mu\in\mathbb{R}$, for the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ to possess exactly α eigenvalues below or above $[0,24]$, where $\alpha\in\{0,1,2\}$. Notably, we have also derived a an imprecise lower boundary for the eigenvalue count of $H_{\gamma\lambda\mu}(K), K\in\mathbb{T}^3$ (detailed in Theorem 5), depending only on λ and μ.

In order to better explain the necessary and sufficient conditions on parameters λ and μ for eigenvalue birth and annihilation, near the upper and lower thresholds of $\sigma_{ess}(H_{\gamma\lambda\mu}(0))$, we introduce the concept of a *critical operator* (for more details see Definition 1). To conclude, we provide a clear characterization of this criticality in terms of interaction parameters (Theorem 1).

First, in [32], the authors studied discrete Schrödinger operators involving a particle under the influence of an external field on a three-dimensional lattice. The interaction energies of the system were non-negative and denoted by γ, λ, and the eigenvalue characteristics, including their count and whereabouts were determined as functions of these parameters.

Subsequent works [11,12,33] extended these results to two-boson systems with on-site and nearest-neighbor interactions for $d=1,2$, where the interactions are described by real parameters γ and λ.

For a system of two identical bosons on a d-dimensional lattice $\mathbb{Z}^d$ $(d=1,2)$ with on-site (γ), nearest-neighbor (λ), and next-nearest-neighbor (μ) interactions, the discrete spectrum of the associated two-particle Schrödinger operator $H_{\gamma\lambda\mu}(k)$, $k\in\mathbb{T}^d$ has been studied and determined the number and position of isolated eigenvalues for all values of the interaction parameters in [34–36].

Here's how this paper is structured. Section 2 introduces the fiber Schrödinger operators for particle pairs at a fixed quasi-momentum. Section 3 outlines the auxiliary information crucial for stating our primary findings. Our main results, along with the definition of critical operators, are presented in Section 4. Finally, Section 5 contains the proofs.

2. Two-boson system Hamiltonian on the $\mathbb{Z}^3$ lattice

2.1. Quasimomentum-fixed Schrödinger operator for particle pairs

For $\gamma,\lambda,\mu\in\mathbb{R}$ and $K\in\mathbb{T}^3$, the Schrödinger operator $H_{\gamma\lambda\mu}(K)$ describes interacting particle pairs. This operator is bounded and self-adjoint in $L^{2,e}(\mathbb{T}^3)$ ([3], [11], [35]), defined as:

$$H_{\gamma\lambda\mu}(K) := H_0(K) + V_{\gamma\lambda\mu}.$$

The unperturbed operator, $H_0(K)$, acts as a multiplication operator, defined by the quasi-momentum-dependent pair dispersion relation:

$$(H_0(K)f)(p) = \mathcal{E}_K(p)f(p),$$

where the dispersion function $\mathcal{E}_K(\cdot)$ is given by:

$$\mathcal{E}_K(p) = 4\sum_{i=1}^{3}\left(1-\cos\tfrac{K_i}{2}\,\cos p_i\right),\quad p=(p_1,p_2,p_3)\in\mathbb{T}^3. \tag{3}$$

The perturbation operator $V_{\gamma\lambda\mu}$ is given by:

$$
\begin{aligned}
\left(V_{\gamma\lambda\mu}f\right)(p) = &\frac{\gamma}{8\pi^3}\int\limits_{\mathbb{T}^3} f(q)\mathrm{d}q + \frac{\lambda}{4\pi^3}\sum_{i=1}^{3}\cos p_i\int\limits_{\mathbb{T}^3}\cos q_i f(q)\mathrm{d}q \\
&+ \frac{\mu}{4\pi^3}\sum_{i=1}^{3}\cos 2p_i\int\limits_{\mathbb{T}^3}\cos 2q_i f(q)\mathrm{d}q.
\end{aligned}
\tag{4}
$$

The rank of $V_{\gamma\lambda\mu}$ varies with $\gamma,\lambda,\mu\in\mathbb{R}$ but is always at most seven. By Weyl's theorem [37], $\sigma_{\mathrm{ess}}(H_{\gamma\lambda\mu}(K))$ is unaffected by these parameters. It coincides with $\sigma(H_0(K))$, $K\in\mathbb{T}^3$, forming the closed interval $[\mathcal{E}_{\min}(K),\mathcal{E}_{\max}(K)]$, where:

$$
\begin{aligned}
\mathcal{E}_{\min}(K) &:= \min_{p\in\mathbb{T}^3}\,\mathcal{E}_K(p) = 4\sum_{i=1}^{3}\left(1-\cos\tfrac{K_i}{2}\right)\geq\mathcal{E}_{\min}(0)=0,\\
\mathcal{E}_{\max}(K) &:= \max_{p\in\mathbb{T}^3}\,\mathcal{E}_K(p) = 4\sum_{i=1}^{3}\left(1+\cos\tfrac{K_i}{2}\right)\leq\mathcal{E}_{\max}(0)=24.
\end{aligned}
\tag{5}
$$

3. Auxiliary statements

3.1. Invariant subspaces of the fiber Schrödinger operators $H_{\gamma\lambda\mu}(0)$

We define $L^{2,\mathrm{e},a_{12}}(\mathbb{T}^3)$ as the set of functions within $L^{2,e}(\mathbb{T}^3)$ that are antisymmetric with respect to their first two coordinates, specifically,

$$
L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)=\{f\in L^{2,\mathrm{e}}(\mathbb{T}^3):\ f(p_1,p_2,p_3)=-f(p_2,p_1,p_3)\}.
$$

Let $\left(L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right)^{\perp}$ be its orthogonal complement within $L^{2,e}(\mathbb{T}^3)$. Notably, $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$ and its orthogonal complement $\left(L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right)^{\perp}$ are both closed subspaces of $L^{2,e}(\mathbb{T}^3)$. Then the Direct Sum Theorem yields the decomposition (1).

Since $\mathcal{E}_0(p)=2\epsilon(p)$ exhibits symmetry in its first two coordinates, and $H_0(0)$ corresponds to multiplication by $\mathcal{E}_0(p)$, the subspace $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$, as well as $\left(L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right)^{\perp}$ is invariant for self-adjoint operator $H_0(0)$. Taking into account the identity

$$
\begin{aligned}
2\cos A\cos B+2\cos C\cos D &= (\cos A+\cos C)(\cos B+\cos D)\\
&\quad+(\cos A-\cos C)(\cos B-\cos D),\ \text{ for any }\ A,B,C,D\in\mathbb{R},
\end{aligned}
$$

from (4) one derives that

$$
\begin{aligned}
\left(V_{\gamma\lambda\mu}f\right)(p) = &\frac{\gamma}{8\pi^3}\int_{\mathbb{T}^3} f(q)\,\mathrm{d}q + \frac{\lambda}{8\pi^3}(\cos p_1+\cos p_2)\int_{\mathbb{T}^3}(\cos q_1+\cos q_2)f(q)\,\mathrm{d}q\\
&+\frac{\lambda}{8\pi^3}(\cos p_1-\cos p_2)\int_{\mathbb{T}^3}(\cos q_1-\cos q_2)f(q)\,\mathrm{d}q\\
&+\frac{\mu}{8\pi^3}(\cos 2p_1+\cos 2p_2)\int_{\mathbb{T}^3}(\cos 2q_1+\cos 2q_2)f(q)\,\mathrm{d}q\\
&+\frac{\mu}{8\pi^3}(\cos 2p_1-\cos 2p_2)\int_{\mathbb{T}^3}(\cos 2q_1-\cos 2q_2)f(q)\,\mathrm{d}q\\
&+\frac{\lambda}{4\pi^3}\cos p_3\int_{\mathbb{T}^3}\cos q_3 f(q)\,\mathrm{d}q+\frac{\mu}{4\pi^3}\cos 2p_3\int_{\mathbb{T}^3}\cos 2q_3 f(q)\,\mathrm{d}q,
\end{aligned}
\tag{6}
$$

which implies that $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$, and $\left(L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)\right)^{\perp}$ are invariant for $V_{\gamma\lambda\mu}$ and, hence, for $H_{\gamma\lambda\mu}(0)$. Thus, (1) implies the spectral decomposition (2).

Let us denote by $V^{\mathrm{ea}}_{\lambda\mu}$ and $H^{\mathrm{ea}}_{\lambda\mu}(0)$ the corresponding restrictions of the operators $V_{\gamma\lambda\mu}$ and $H_{\gamma\lambda\mu}(0)$ onto the subspace $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$. Then (6) implies that

$$
\begin{aligned}
\left(V^{\mathrm{ea}}_{\lambda\mu}f\right)(p) = &\frac{\lambda}{8\pi^3}(\cos p_1-\cos p_2)\int_{\mathbb{T}^3}(\cos q_1-\cos q_2)f(q)\,\mathrm{d}q\\
&+\frac{\mu}{8\pi^3}(\cos 2p_1-\cos 2p_2)\int_{\mathbb{T}^3}(\cos 2q_1-\cos 2q_2)f(q)\,\mathrm{d}q
\end{aligned}
$$

and

$$
H^{\mathrm{ea}}_{\lambda\mu}(0)=H_0(0)+V^{\mathrm{ea}}_{\lambda\mu}.
$$

3.2. The Lippmann–Schwinger operator

Let $\{\alpha_1, \alpha_2\}$ be a system of orthonormal vectors in $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$, with

$$\alpha_1(p) = \tfrac{\cos p_1 - \cos p_2}{\sqrt{8\pi^3}}, \quad \alpha_2(p) = \tfrac{\cos 2p_1 - \cos 2p_2}{\sqrt{8\pi^3}}, \tag{7}$$

By using the orthonormal system (7) one obtains

$$V^{\mathrm{ea}}_{\lambda\mu} f = \lambda(f, \alpha_1)\alpha_1 + \mu(f, \alpha_2)\alpha_2, \tag{8}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$. For any $z \in \mathbb{C}\setminus[0,\ 24]$, we define the Lippmann-Schwinger operator (or its transpose, see, e.g., [38]) as:

$$B^{\mathrm{ea}}_{\lambda\mu}(0,z) = -V^{\mathrm{ea}}_{\lambda\mu} R_0(0,z),$$

where $R_0(0,z) := [H_0(0) - zI]^{-1}$ is the resolvent of the operator $H_0(0)$.

Lemma 1. *Given $\lambda, \mu \in \mathbb{R}$, $z \in \mathbb{C} \setminus [0, 24]$ is an eigenvalue of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ with multiplicity m if and only if $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$ has an eigenvalue of 1 with multiplicity m.*

Proof. Let $\lambda, \mu \in \mathbb{R}$. For $z \in \mathbb{R} \setminus [0, 24]$, the operator $R_0(0,z) = [H_0(0) - zI]^{-1}$ is well-defined within the space $L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$. Therefore, the equation

$$H^{\mathrm{ea}}_{\lambda\mu}(0)\varphi = z\varphi, \ \ \varphi \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3)$$

is equivalent to $(H_0(0) - zI)\varphi = -V^{\mathrm{ea}}_{\lambda\mu}\varphi$, and further to

$$\varphi = -V^{\mathrm{ea}}_{\lambda\mu} R_0(0,z)\varphi, \ \ \varphi \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3).$$

This equivalence proves the above relationship between the eigenvalues. □

Remark 1. *Lemma 1 reduces the spectral analysis of the non-compact operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ to that of the compact operator $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$. Since $V^{\mathrm{ea}}_{\lambda\mu}$ has a rank of at most two, $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$ also has a rank of at most two. Consequently, the self-adjoint operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ possesses at most two (real) eigenvalues in $\mathbb{R} \setminus [0, 24]$.*

Equation (8) establishes the equivalence between the Lippmann-Schwinger equation,

$$B^{\mathrm{ea}}_{\lambda\mu}(0,z)\varphi = \varphi, \ \ \varphi \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^3), \tag{9}$$

and the following algebraic linear system:

$$\begin{cases} [1 + \lambda a_{11}(z)]x_1 + \mu a_{12}(z)x_2 = 0, \\ \lambda a_{21}(z)x_1 + [1 + \mu a_{22}(z)]x_2 = 0, \end{cases}$$

where

$$a_{ij}(z) := (\alpha_i, R_0(0,z)\alpha_j) = \int\limits_{\mathbb{T}^3} \frac{\alpha_i(p)\alpha_j(p)}{\mathcal{E}_0(p) - z}\mathrm{d}p, \ i,j \in \{1,2\}.$$

Let $z \in \mathbb{R} \setminus [0,\ 24]$ and

$$\Delta_{\lambda\mu}(z) := \det[I - B^{\mathrm{ea}}_{\lambda\mu}(0,z)] = \begin{vmatrix} 1 + \lambda a_{11}(z) & \mu a_{12}(z) \\ \lambda a_{12}(z) & 1 + \mu a_{22}(z) \end{vmatrix}.$$

The following lemma outlines the established relation between the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ and the function $\Delta_{\lambda\mu}(\cdot)$.

Lemma 2. *Given $\lambda, \mu \in \mathbb{R}$, a real number $z \in \mathbb{R} \setminus [0, 24]$ is an eigenvalue of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ with the multiplicity m precisely when it is a zero of $\Delta_{\lambda\mu}(\cdot)$ of the multiplicity m. Additionally, $\Delta_{\lambda\mu}(\cdot)$ has at most two zeros within the interval $\mathbb{R} \setminus [0, 24]$.*

Proof. Assume that $z \in \mathbb{R} \setminus [0, 24]$ is an eigenvalue of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ with multiplicity $m \geq 1$, i.e. 1 is an eigenvalue of compact operator $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$ with the same multiplicity.

First, an eigenvalue $z \in \mathbb{R}\setminus[0,\ 24]$ of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ corresponds to 1 being an eigenvalue of the Birman-Schwinger operator $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$, which implies $\Delta_{\lambda\mu}(z) = 0$ (see [37, Chapter XIII.14]).

Since $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$ is compact operator of rank at most two, for any its isolated eigenvalue, its algebraic multiplicity equals its geometric multiplicity (see [39, Chapter IV, Sections 3.1-3.5]). Furthermore, the multiplicity of a zero z of the Fredholm determinant $\det(I - B^{\mathrm{ea}}_{\lambda\mu}(0,z))$ corresponds precisely to the algebraic multiplicity of the number 1 as an eigenvalue of the compact operator $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$, provided $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$ depends analytically on z (for more details, see [39, Chapter IV, Section 5.3]). Combining these results, one directly establishes that the multiplicity of z as an eigenvalue of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ is identical to its multiplicity as a zero of $\Delta_{\lambda\mu}(z)$.

Finally, the fact that $\Delta_{\lambda\mu}(\cdot)$ has at most two zeros in $\mathbb{R} \setminus [0,\ 24]$ follows directly from $B^{\mathrm{ea}}_{\lambda\mu}(0,z)$ being of rank at most two. □

The forthcoming lemma details the global properties and asymptotic expansions of $a_{ij}(z)$.

Lemma 3. *The functions $a_{ij}(z)$, $i, j \in \{1, 2\}$ are real-valued and exhibit the following behavior:*

(i) *The functions $a_{ii}(z)$, $i = 1, 2$ are strictly increasing and positive on $(-\infty, 0]$, and strictly increasing and negative on $[24, +\infty)$.*

(ii) *The equality*

$$\lim_{z\to\pm\infty} a_{ij}(z) = 0$$

holds for all $i, j = 1, 2$.

(iii) *The limits $a_{ij}(0) = \lim\limits_{z\nearrow 0} a_{ij}(z)$ and $a_{ij}(24) = \lim\limits_{z\searrow 24} a_{ij}(z)$ both exist and satisfy the relations*

$$\begin{aligned} &a_{ij}(24) = (-1)^{i+j+1} a_{ij}(0), \\ &a_{11}(0) > 0, \quad a_{22}(0) > 0, \; a_{11}(0)a_{22}(0) > a_{12}^2(0). \end{aligned}$$

(iii) *The functions $a_{ij}(z)$ admit the following asymptotics:*

$$\begin{aligned} a_{ij}(z) =&a_{ij}(0) + O(-z), \quad z \nearrow 0, \\ a_{ij}(z) =&(-1)^{i+j+1} \left[a_{ij}(0) + O(z-24)\right], \quad z \searrow 24. \end{aligned}$$

Proof. The equality

$$a_{11}(0)a_{22}(0) - a_{12}^2(0) = \frac{1}{2} \int\limits_{\mathbb{T}^3\times\mathbb{T}^3} \tfrac{[\alpha_1(p)\alpha_2(q)-\alpha_1(q)\alpha_2(p)]^2}{\mathcal{E}_0(p)\mathcal{E}_0(q)} \mathrm{d}p\mathrm{d}q$$

(where α_1, α_2 are as in (7)), along with the monotonicity of the Lebesgue integral, directly implies $a_{11}(0)a_{22}(0) - a_{12}^2(0) > 0$. For the remaining statements of this lemma, a demonstration similar to Proposition 1 in [35] can be employed. □

Lemma 4. *For any $\lambda, \mu \in \mathbb{R}$ the function $\Delta_{\lambda\mu}(z)$ is holomorphic in $z \in \mathbb{R} \setminus [0, 24]$. Furthermore, this function is real analytic for $z \in \mathbb{R} \setminus [0, 24]$ and possesses the following asymptotics:*

(i) $\lim\limits_{z\to\pm\infty} \Delta_{\lambda\mu}(z) = 1,$

(ii) $\lim\limits_{z\nearrow 0} \Delta_{\lambda\mu}(z) = P^-(\lambda, \mu),$

(iii) $\lim\limits_{z\searrow 24} \Delta_{\lambda\mu}(z) = P^+(\lambda, \mu),$

where

$$P^\pm(\lambda, \mu) = a\left[(\lambda \mp \lambda_0)(\mu \mp \mu_0) - b\right], \tag{10}$$

and

$$\begin{aligned} a = a_{11}(0)a_{22}(0) - a_{12}^2(0), \quad & b = \left[\frac{a_{12}(0)}{a_{11}(0)a_{22}(0) - a_{12}^2(0)}\right]^2, \\ \lambda_0 = \frac{a_{22}(0)}{a_{11}(0)a_{22}(0) - a_{12}^2(0)}, \quad & \mu_0 = \frac{a_{11}(0)}{a_{11}(0)a_{22}(0) - a_{12}^2(0)} \end{aligned}$$

are positive real numbers.

Proof. In view of Lemma 3, the proof is obtained by a simple inspection. □

Lemma 5. *In $\mathbb{R}^2$, the set of points satisfying $P^\pm(\lambda, \mu) = 0$ precisely forms the graph of the function*

$$\mu^\pm(\lambda) = \frac{b}{\lambda \mp \lambda_0} \pm \mu_0.$$

This graph consists of two isolated smooth unbounded simple connected curves:

$$\begin{aligned} \tau_0^\pm = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu = \frac{b}{\lambda \mp \lambda_0} \pm \mu_0, \; \pm\lambda < \lambda_0\}, \\ \tau_1^\pm = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu = \frac{b}{\lambda \mp \lambda_0} \pm \mu_0, \; \pm\lambda < \lambda_0\}. \end{aligned}$$

and separates the (λ, μ)-parameter plane into three unbounded, contiguous, and connected components:

$$\begin{aligned}
\mathcal{C}_0^{\pm} =&\{(\lambda,\mu)\in\mathbb{R}^2 : \pm\mu < \frac{b}{\pm\lambda-\lambda_0}+\mu_0,\ \pm\lambda<\lambda_0\},\\
\mathcal{C}_1^{\pm} =&\{(\lambda,\mu)\in\mathbb{R}^2 : \pm\mu > \frac{b}{\pm\lambda-\lambda_0}+\mu_0,\ \pm\lambda<\lambda_0\}\cup\{(\pm\lambda_0,\mu)\in\mathbb{R}^2\},\\
&\{(\lambda,\mu)\in\mathbb{R}^2 : \pm\mu < \frac{b}{\pm\lambda-\lambda_0}+\mu_0,\ \pm\lambda>\lambda_0\},\\
\mathcal{C}_2^{\pm} =&\{(\lambda,\mu)\in\mathbb{R}^2 : \pm\mu > \frac{b}{\pm\lambda-\lambda_0}+\mu_0,\ \pm\lambda>\lambda_0\}
\end{aligned} \tag{11}$$

The proof of Lemma 5 resembles that of Lemma 1 in [40].

4. Main results

4.1. Critical operators

To elucidate the mechanisms of eigenvalue birth and annihilation, we introduce the concept of a *critical operator*.

Definition 1. *A parameter point (λ_0,μ_0) is lower-critical for $H^{\mathrm{ea}}_{\lambda\mu}(0)$ if the number of discrete eigenvalues below the essential spectrum is non-constant in every neighborhood of (λ_0,μ_0). Upper-critical points are defined analogously for eigenvalues above the essential spectrum.*

The following theorem characterizes criticality of the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ through its interaction parameters.

Theorem 1. *A parameter point $(\lambda_0,\mu_0)\in\mathbb{R}^2$ is:*

(i) *Lower-critical for $H^{\mathrm{ea}}_{\lambda\mu}(0)$ iff $P^-(\lambda_0,\mu_0)=0$*

(ii) *Upper-critical for $H^{\mathrm{ea}}_{\lambda\mu}(0)$ iff $P^+(\lambda_0,\mu_0)=0$*

The following theorem, establishes that the number of isolated eigenvalues of the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ lying above (resp. below) its essential spectrum is constant within each connected component $\mathcal{C}_\alpha^-$ (resp. $\mathcal{C}_\alpha^+$), $\alpha=0,1,2$.

Theorem 2. *Let $\mathcal{C}^-$ resp. $\mathcal{C}^+$ be one of the open connected components $\mathcal{C}_\alpha^-$ resp. $\mathcal{C}_\alpha^+$,$\alpha=0,1,2$, of the (λ,μ)-plane defined in* (11)*. Then for any $(\lambda,\mu)\in\mathcal{C}^-$ resp. $(\lambda,\mu)\in\mathcal{C}^+$ the number of eigenvalues of $H^{\mathrm{ea}}_{\lambda\mu}(0)$, lying below resp. above the essential spectrum remains constant (counting multiplicities).*

Proof. Theorem 2 can be proved analogously to [36, Theorem 3.2]. □

We will now determine the exact number of eigenvalues of the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ lying below and above the essential spectrum within the connected components $\mathcal{C}_\zeta^-$ and $\mathcal{C}_\zeta^+$, $\zeta=0,1,2$.

Theorem 3. *Let $\lambda,\mu\in\mathbb{R}$ and $\zeta=0,1,2$. If $(\lambda,\mu)\in\mathcal{C}_\zeta^-$ (resp. $(\lambda,\mu)\in\mathcal{C}_\zeta^+$) then the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has exactly ζ eigenvalues lying below (resp. above) its essential spectrum.*

The next theorem demonstrates the exact number of eigenvalues of the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ that lie below and above the essential spectrum within different unbounded smooth curves τ_ζ^- and τ_ζ^+, $\zeta=0,1$, respectively.

Theorem 4. *Let $\lambda,\mu\in\mathbb{R}$ and $\zeta=0,1$. If $(\lambda,\mu)\in\tau_\zeta^-$ (resp. $(\lambda,\mu)\in\tau_\zeta^+$) then the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has ζ eigenvalues, which lie below (resp. above) the essential spectrum.*

Let us define the partitions $\{\mathcal{P}_\zeta^-\}_{\zeta=0}^2$ and $\{\mathcal{P}_\zeta^+\}_{\zeta=0}^2$ of the plane $\mathbb{R}^2$:

$$\mathcal{P}_\zeta^\pm := \mathcal{C}_\zeta^\pm\cup\tau_\zeta^\pm,\ \zeta=0,1 \quad \text{and} \quad \mathcal{P}_2^\pm := \mathcal{C}_2^\pm.$$

Now, we present a criterion, based on the perturbation parameters $\lambda,\mu\in\mathbb{R}$, detailing when the operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has exactly α eigenvalues in $(-\infty,0)$ or $(24,+\infty)$, $\alpha\in\{0,1,2\}$.

Corollary 1. *Let $\lambda,\mu\in\mathbb{R}$ and $\zeta\in\{0,1,2\}$. The operator $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has exactly ζ eigenvalues in $(-\infty,0)$ if and only if $(\lambda,\mu)\in\mathcal{P}_\zeta^-$. Similarly, it has exactly ζ eigenvalues in $(24,+\infty)$ if and only if $(\lambda,\mu)\in\mathcal{P}_\zeta^+$.*

Let us define

$$\mathcal{G}_{\alpha\beta}=\mathcal{P}_\alpha^-\cap\mathcal{P}_\beta^+,\quad \alpha,\beta=0,1,2. \tag{12}$$

Corollary 2. *Let $\lambda,\mu\in\mathbb{R}$ and the number $\alpha,\beta=0,1,2$ satisfies $\alpha+\beta\le 2$. Then, $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has α eigenvalues below and β eigenvalues above its essential spectrum if and only if $(\lambda,\mu)\in\mathcal{G}_{\alpha\beta}$.*

Proof. Corollary 2 can be proved by combining Corollary 1 and (12). □

The following theorem offers a lower estimate for the number of eigenvalues of the operator $H_{\gamma\lambda\mu}(K), K \in \mathbb{T}^3$, dependent only on λ, μ.

Theorem 5. *Let $\gamma, \lambda, \mu \in \mathbb{R}$ and $\alpha, \beta \in \{0, 1, 2\}$. If $(\lambda, \mu) \in \mathcal{G}_{\alpha\beta}$, then for each $K \in \mathbb{T}^3$, the operator $H_{\gamma\lambda\mu}(K)$ has at least α eigenvalues below and at least β eigenvalues above its essential spectrum.*

Proof of Theorem 5. Theorem 5 can be proved as Theorem 3 in [40]. □

5. Proof of the main results

Proof of Theorem 1. Due to symmetry, we only prove the statement for the upper threshold.

Assume that $P^+(\lambda, \mu) = 0$. Then Lemma 5 shows that $(\lambda, \mu) \in \tau^+_\alpha$ for some $\alpha \in \{0, 1\}$, and any neighborhood of a point $(\lambda, \mu) \in \tau^+_\alpha$ contains points from both $\mathcal{C}^+_\alpha$ and $\mathcal{C}^+_{\alpha+1}$. Moreover, Lemma 4 states that:

$$\lim_{z\to+\infty} \Delta_{\lambda\mu}(z) = 1 \quad \text{and} \quad \lim_{z\searrow 24} \Delta_{\lambda\mu}(z) = P^+(\lambda, \mu).$$

Since $P^+(\lambda, \mu)$ exhibits different signs within $\mathcal{C}^+_\alpha$ and $\mathcal{C}^+_{\alpha+1}$ for each $\alpha = 0, 1$, this implies that $\Delta_{\lambda\mu}(\cdot)$ has a different number of zeros on $(24, +\infty)$ in these regions. Consequently, by Lemma 2, the number of eigenvalues of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ lying in $(24, +\infty)$ is not constant in any neighborhood of (λ, μ). This indicates that $H^{\mathrm{ea}}_{\lambda\mu}(0)$ is *critical* at the upper threshold.

We now assume that $H^{\mathrm{ea}}_{\lambda\mu}(0)$ is critical at the upper threshold for some $(\lambda, \mu) \in \mathbb{R}^2$, and proceed to prove that $P^+(\lambda, \mu) = 0$. Assume for contradiction, that $P^+(\lambda, \mu) \neq 0$. Then, Lemma 5 implies $(\lambda, \mu) \in \mathcal{C}^+_\alpha$ for some $\alpha \in \{0, 1, 2\}$. Since $\mathcal{C}^+_\alpha$ is an open set, there exists a neighborhood $U_\delta(\lambda, \mu) \subseteq \mathcal{C}^+_\alpha$. Within this neighborhood $U_\delta(\lambda, \mu)$, Theorem 2 ensures a constant number of eigenvalues of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ lying above the essential spectrum and therefore $H^{\mathrm{ea}}_{\lambda\mu}(0)$ is not critical at its upper threshold, leading to a contradiction.

□

Note that $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has at most two discrete eigenvalues. Let us denote these eigenvalues, arranged in increasing order, by $z_1(H^{\mathrm{ea}}_{\lambda\mu}(0))$ and $z_2(H^{\mathrm{ea}}_{\lambda\mu}(0))$.

Lemma 6. (i) *For each fixed $\mu \in \mathbb{R}$, the maps*

$$\lambda \mapsto z_1(H^{\mathrm{ea}}_{\lambda\mu}(0)) \quad \textit{and} \quad \lambda \mapsto z_2(H^{\mathrm{ea}}_{\lambda\mu}(0))$$

are non-decreasing on $\mathbb{R}$.

(ii) *Analogously, for each fixed $\lambda \in \mathbb{R}$, the maps*

$$\mu \mapsto z_1(H^{\mathrm{ea}}_{\lambda\mu}(0)) \quad \textit{and} \quad \mu \mapsto z_2(H^{\mathrm{ea}}_{\lambda\mu}(0))$$

are non-decreasing on $\mathbb{R}$.

Proof. (i) Let $\mu \in \mathbb{R}$ be fixed and $\lambda_1 < \lambda_2$ be an arbitrary real numbers. Then, the representation (8) and the inequality $\lambda_1 < \lambda_2$ imply that

$$(H^{\mathrm{ea}}_{\lambda_1\mu}(0)\psi, \psi) - (H^{\mathrm{ea}}_{\lambda_2\mu}(0)\psi, \psi) = (\lambda_1 - \lambda_2)(\psi, \alpha_1)^2 \leq 0, \text{ i.e.}$$
$$(H^{\mathrm{ea}}_{\lambda_1\mu}(0)\psi, \psi) \leq (H^{\mathrm{ea}}_{\lambda_2\mu}(0)\psi, \psi), \ \forall \psi \in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^2).$$

For each $n = 1, 2$, the last inequality leads that

$$\begin{aligned} z_n(H^{\mathrm{ea}}_{\lambda_1\mu}(0)) &:= \sup_{\phi_1,\ldots,\phi_{n-1}\in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^2)} \ \inf_{\psi\in[\phi_1,\ldots,\phi_{n-1}]^\perp,\, \|\psi\|=1} (H^{\mathrm{ea}}_{\lambda_1\mu}(0)\psi, \psi) \\ &\leq \sup_{\phi_1,\ldots,\phi_{n-1}\in L^{2,\mathrm{e},\mathrm{a}_{12}}(\mathbb{T}^2)} \ \inf_{\psi\in[\phi_1,\ldots,\phi_{n-1}]^\perp,\, \|\psi\|=1} (H^{\mathrm{ea}}_{\lambda_2\mu}(0)\psi, \psi) = z_n(H^{\mathrm{ea}}_{\lambda_2\mu}(0)). \end{aligned}$$

(ii) For every fixed $\lambda \in \mathbb{R}$, the case of $\mu \in \mathbb{R} \mapsto z_n(H^{\mathrm{ea}}_{\lambda\mu}(0))$, $n = 1, 2$ can be proved similarly. □

Proof of Theorem 3. We'll prove the "plus" case, as the "minus" case follows a similar logic and its proof is omitted for brevity.

Let us start the proof with the case $\alpha = 1$ and assume that $(\lambda, \mu) \in \mathcal{C}^+_1$. From (10) and (11), we deduce that

$$P^+(\lambda, \mu) = a\,[(\lambda - \lambda_0)(\mu - \mu_0) - k] < 0.$$

Then, Lemma 4 demonstrates that

$$\lim_{z\to+\infty} \Delta_{\lambda\mu}(z) = 1 \text{ and } \lim_{z\searrow 24} \Delta_{\lambda\mu}(z) = P^+(\lambda, \mu) < 0.$$

Since $\Delta_{\lambda\mu}(\cdot)$ changes sign on $(24, +\infty)$, it has at least one there. If there were more zeros, the endpoint sign changes would require at least three, contradicting Lemma 2. Thus, $\Delta_{\lambda\mu}(\cdot)$ has exactly one zero in $(24, +\infty)$, which Lemma 2 then implies to a unique eigenvalue of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ in the same interval.

Case $\alpha = 0$. Assuming $(\lambda, \mu) \in \mathcal{C}^+_0$, we have from (11) that

$$\mu < \frac{k}{\lambda - \lambda_0} - \mu_0 \quad \text{and} \quad \lambda < \lambda_0, \tag{13}$$

which implies $P^+(\lambda,\mu)>0$. Lemma 4 then allows us to obtain

$$\lim_{z\to+\infty}\Delta_{\lambda\mu}(z)=1 \text{ and } \lim_{z\searrow 24}\Delta_{\lambda\mu}(z)=P^+(\lambda,\mu)>0. \tag{14}$$

Relation (14) and Lemma 2 imply that $\Delta_{\lambda\mu}(z)$ possesses either zero or two zeros above the essential spectrum. Assume, for contradiction, that there are two such zeros, $z_1(H^{\mathrm{ea}}_{\lambda\mu}(0))$ and $z_2(H^{\mathrm{ea}}_{\lambda\mu}(0))$, satisfying

$$24<z_1(H^{\mathrm{ea}}_{\lambda\mu}(0))\le z_2(H^{\mathrm{ea}}_{\lambda\mu}(0)). \tag{15}$$

In other hand, (11) states that

$$(\lambda_0,\mu)\in\mathcal{C}_1^+ \ \forall\mu\in\mathbb{R}.$$

Further, the inequality $\lambda<\lambda_0$ in (13), the relation (15), and Lemma 6 yield

$$z_2(H^{\mathrm{ea}}_{\lambda_0\mu}(0))\ge z_2(H^{\mathrm{ea}}_{\lambda\mu}(0))>24,$$

indicating that $H^{\mathrm{ea}}_{\lambda_0\mu}(0)$ has at least two eigenvalues located in $(24,+\infty)$, contradicting the previous result. Thus, for all $(\lambda,\mu)\in\mathcal{C}_0^+$, $H^{\mathrm{ea}}_{\lambda\mu}(0)$ lacks eigenvalues in $(24,+\infty)$.

Take any $(\lambda,\mu)\in\mathcal{C}_2^+$. Then, (10) and (11) provide the inequalities

$$\begin{aligned}&P(\lambda,\mu)=a\,[(\lambda-\lambda_0)(\mu-\mu_0)-k]>0,\\&\mu>\mu_0,\quad \lambda>\lambda_0.\end{aligned} \tag{16}$$

Using Lemma 4 and (16), we find that

$$\lim_{z\to+\infty}\Delta_{\lambda\mu}(z)=1 \quad\text{and}\quad \lim_{z\searrow 24}\Delta_{\lambda\mu}(z)=P^+(\lambda,\mu)>0. \tag{17}$$

Note that $(\lambda_0,\mu)\in\mathcal{C}_1^+$ and so $H^{\mathrm{ea}}_{\lambda_0\mu}(0)$ has exactly one eigenvalue $z_1(H^{\mathrm{ea}}_{\lambda_0\mu}(0))>24$. Then the relation $\lambda>\lambda_0$ in (16) and Lemma 6 give that

$$z_1(H^{\mathrm{ea}}_{\lambda\mu}(0))\ge z_1(H^{\mathrm{ea}}_{\lambda_0\mu}(0))>24,$$

which yields that $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has at least one eigenvalue in $(24,+\infty)$. Then, Lemma 2 implies that $\Delta_{\lambda\mu}(\cdot)$ has at least one zero in $(24,+\infty)$. On the other hand, (17) indicates that $\Delta_{\lambda\mu}$ has the same sign at the endpoints of $(24,+\infty)$; therefore, it must possess an even number of zeros (counting multiplicities). Thus, $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has exactly two (simple) eigenvalues located in $(24,+\infty)$. □

Proof of Theorem 4. The function $(\lambda,\mu)\to n_+(H^{\mathrm{ea}}_{\lambda\mu}(0))$ (resp. $(\lambda,\mu)\to n_-(H^{\mathrm{ea}}_{\lambda\mu}(0))$) is continuous on each $\mathcal{P}_\zeta^+$ (respectively, on each $\mathcal{P}_\zeta^-$), for $\zeta=0,1,2$, where, $n_+(H^{\mathrm{ea}}_{\lambda\mu}(0))$ (respectively, $n_-(H^{\mathrm{ea}}_{\lambda\mu}(0))$) denotes the number of isolated eigenvalues of $H^{\mathrm{ea}}_{\lambda\mu}(0)$ located in $(24,+\infty)$ (resp. $(-\infty,0)$). This proves the assertion. □

Proof of Corollary 1. Let $(\lambda,\mu)\in\mathcal{P}_\alpha^+$. Then Theorem 3 implies that $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has exactly α eigenvalues in $(24,+\infty)$.

For the converse, assume $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has precisely α eigenvalues in $(24,+\infty)$. We proceed by contradiction to establish that $(\lambda,\mu)\in\mathcal{P}_\alpha^+$. Suppose $(\lambda,\mu)\notin\mathcal{P}_\alpha^+$. Since $\{\mathcal{P}_0^+,\mathcal{P}_1^+,\mathcal{P}_2^+\}$ forms a partition of $\mathbb{R}^2$, there exists $\beta\ne\alpha$ such that $(\lambda,\mu)\in\mathcal{P}_\beta^+$. Theorem 3 then implies that $H^{\mathrm{ea}}_{\lambda\mu}(0)$ has exactly $\beta\ne\alpha$ eigenvalues above its essential spectrum, directly contradicting our initial assumption. □

6. Conclusion

This paper meticulously analyzed the spectral characteristics of the lattice Schrödinger operators, $H_{\gamma\lambda\mu}(K)$, which model two identical bosons on $\mathbb{Z}^3$ with on-site and nearest-neighbor interactions. Our study of $H_{\gamma\lambda\mu}(0)$ revealed an invariant subspace where its restricted form, $H^{\mathrm{ea}}_{\lambda\mu}(0)$, depends only on λ and μ. To elucidate eigenvalue birth and annihilation for $H^{\mathrm{ea}}_{\lambda\mu}(0)$, we defined a critical operator and developed a detailed criterion in the $\lambda-\mu$ plane. This involved: (i) deriving smooth critical curves that delineate the onset of criticality; and (ii) proving exact conditions for the existence of precisely α eigenvalues below and β eigenvalues above the essential spectrum, with $\alpha,\beta\in\{0,1,2\}$ and $\alpha+\beta\le 2$.

References

[1] Mattis D. The few-body problem on a lattice. *Rev. Mod. Phys.*, **58**, P. 361–379, 1986.

[2] Albeverio S., Lakaev S.N., Muminov Z.I. Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics. *Ann. Henri Poincaré*, 2004, **5**, P. 743–772.

[3] Albeverio S., Lakaev S.N., Makarov K.A., Muminov Z.I. The Threshold Effects for the Two-particle Hamiltonians on Lattices. *Comm. Math. Phys.*, 2006, **262**, P. 91–115.

[4] Albeverio S., Lakaev S.N., Khalkhujaev A.M. Number of Eigenvalues of the Three-Particle Schrödinger Operators on Lattices. *Markov Process. Relat. Fields*, 2012, **18**, P. 387–420.

[5] Bach V., W. de Siqueira Pedra, Lakaev S.N. Bounds on the discrete spectrum of lattice Schrödinger operators. *J. Math. Phys.*, 2017, **59**(2), P. 022109.

[6] P.A. Faria Da Veiga, Ioriatti L., O'Carroll M. Energy-momentum spectrum of some two-particle lattice Schrödinger Hamiltonians. *Phys. Rev. E*, 2002, **66**, P. 016130.

[7] Hiroshima F., Muminov Z., Kuljanov U. Threshold of discrete Schrödinger operators with delta-potentials on N-dimensional lattice. *Linear and Multilinear Algebra*, 2020, **68**(11), P. 2267–2279.

[8] Kholmatov Sh.Yu., Lakaev S.N., Almuratov F.M. On the spectrum of Schrödinger-type operators on two dimensional lattices. *J. Math. Anal. Appl.*, 2022, **504**(2), P. 126363.
[9] Lakaev S.N. The Efimov's effect of the three identical quantum particle on a lattice. *Funct. Anal. Appl.*, 1993, **27**, P. 15–28.
[10] Lakaev S.N., Abdukhakimov S.Kh. Threshold effects in a two-fermion system on an optical lattice. *Theoret. and Math. Phys.*, 2020, **203**(2), P. 251–268.
[11] Lakaev S.N., Kholmatov Sh.Yu., Khamidov Sh.I. Bose-Hubbard model with on-site and nearest-neighbor interactions; exactly solvable case. *J. Phys. A: Math. Theor.*, 2021, **54**, P. 245201.
[12] Lakaev S.N., Özdemir E. The existence and location of eigenvalues of the one particle Hamiltonians on lattices. *Hacettepe J. Math. Stat.*, 2016, **45**, P. 1693–1703.
[13] Lakaev S.N., Khalkhuzhaev A.M., Lakaev Sh.S. Asymptotic behavior of an eigenvalue of the two-particle discrete Schrödinger operator. *Theoret. and Math. Phys.*, 2012, **171**(3), P. 438–451.
[14] Motovilov A.K., Sandhas W., Belyaev V.B. Perturbation of a lattice spectral band by a nearby resonance. *J. Math. Phys.*, 2001, **42**, P. 2490–2506.
[15] Muminov M.I., Khurramov A.M., Bozorov I.N. On eigenvalues and virtual levels of a two-particle Hamiltonian on a d-dimensional lattice. *Nanosystems: Phys. Chem. Math.*, 2023, **14**(3), P. 295–303.
[16] Lakaev S.N., Kurbanov Sh.Kh., Alladustov Sh.U. Convergent Expansions of Eigenvalues of the Generalized Friedrichs Model with a Rank-One Perturbation. *Complex Analysis and Operator Theory*, 2021, **15**, P. 121.
[17] Faddeev L.D., Merkuriev S.P. *Quantum Scattering Theory for Several Particle Systems*. Kluwer Academic Publishers, Doderecht, 1993.
[18] Efimov V.N. Weakly Bound States of Three Resonantly Interacting Particles. *Yad. Fiz.*, 1970, **12**, P. 1080. [*Sov. J. Nucl. Phys.*, 1970, **12**, P. 589].
[19] Ovchinnikov Y.N., Sigal I.N. Number of bound states of three-body systems and Efimov's effect. *Ann. Phys.*, 1979, **123**(2), P. 274–295.
[20] Sobolev A.V. The Efimov effect. Discrete spectrum asymptotics. *Commun. Math. Phys.*, 1993, **156**(1), P. 101–126.
[21] Tamura H. The Efimov effect of three-body Schrödinger operators. *J. Funct. Anal.*, 1991, **95**(2), P. 433–459.
[22] Yafaev D.R. On the theory of the discrete spectrum of the three-particle Schrödinger operator. *Mat. Sb.*, 1974, **94**(136), P. 567–593.
[23] Dell'Antonio G., Muminov Z.I., Shermatova V.M. On the number of eigenvalues of a model operator related to a system of three particles on lattices. *Journal of Physics A: Mathematical and Theoretical*, 2011, **44**, P. 315302.
[24] Bagmutov A.S., Popov I.Y. (2020). Window-coupled nanolayers: window shape influence on one-particle and two-particle eigenstates. *Nanosystems: Physics, Chemistry, Mathematics*, **11**(6), P. 636–641.
[25] Bloch I. Ultracold quantum gases in optical lattices. *Nat. Phys.*, 2005, **1**, P. 23–30.
[26] Winkler K., Thalhammer G., Lang F., Grimm R., J. Hecker Denschlag, Daley A.J., Kantia A.n, Büchler H.P., Zoller P. Repulsively bound atom pairs in an optical lattice. *Nature*, 2006, **441**, P. 853–856.
[27] Jaksch D., Bruder C., Cirac J., Gardiner C.W., Zoller P. Cold bosonic atoms in optical lattices. *Phys. Rev. Lett.*, 1998, **81**, P. 3108–3111.
[28] Jaksch D., Zoller P. The cold atom Hubbard toolbox. *Ann. Phys.*, 2005, **315**, P. 52–79.
[29] Lewenstein M., Sanpera A., Ahufinger V. *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-body Systems*. Oxford University Press, Oxford, 2012.
[30] Ospelkaus C., Ospelkaus S., Humbert L., Ernst P., Sengstock K., Bongs K. Ultracold heteronuclear molecules in a 3d optical lattice. *Phys. Rev. Lett.*, 2006, **97**, P. 120402.
[31] Reed M., Simon B. *Methods of Modern Mathematical Physics. III: Scattering Theory*. Academic Press, N.Y., 1978.
[32] Lakaev S.N., Bozorov I.N. The number of bound states of a one-particle Hamiltonian on a three-dimensional lattice. *Theoret. and Math. Phys.*, 2009, **158**, P. 360–376.
[33] Lakaev S.N., Alladustova I.U. The exact number of eigenvalues of the discrete Schrödinger operators in one-dimensional case. *Lobachevskii J. Math.*, 2021, **42**, P. 1294–1303.
[34] Akhmadova M.O., Alladustova I.U., Lakaev S.N. On the Number and Locations of Eigenvalues of the Discrete Schrödinger Operator on a Lattice. *Lobachevskii Journal of Mathematics*, 2023, **44**, P. 1091–1099.
[35] Lakaev S.N., Akhmadova M.O. The number and location of eigenvalues for the two-particle Schrödinger operators on lattices. *Complex Analysis and Operator Theory*, 2023, **17**, P. 104.
[36] Lakaev S.N., Motovilov A.K., Abdukhakimov S.Kh. Two-fermion lattice Hamiltonian with first and second nearest-neighboring-site interactions. *J. Phys. A: Math. Theor.*, 2023, **56**, P. 315202.
[37] Reed M., Simon B. *Methods of Modern Mathematical Physics. IV: Analyses of Operators*. Academic Press, N.Y., 1978.
[38] Lippmann B.A., Schwinger J. Variational principles for scattering processes. I. *Phys. Rev.*, 1950, **79**, P. 469.
[39] Kato T. *Perturbation Theory for Linear Operators* (2nd ed.). Springer-Verlag, 1995.
[40] Lakaev S.N., Khamidov Sh.I., Akhmadova M.O. Number of bound states of the Hamiltonian of a lattice two-boson system with interactions up to the next neighbouring sites. *Lobachevskii Journal of Mathematics*, 2024, **45**(12), P. 6409–6420.

Submitted 1 July 2025; revised 21 August 2025; accepted 22 August 2025

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Conflict of interest: the authors declare no conflict of interest.