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Original article

Asymptotic expansion of Fredholm determinant associated to a family of Friedrichs models arising in quantum mechanics

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ABSTRACT In this paper, we consider a family of Friedrichs models that arise in quantum mechanics and corresponding to the Hamiltonian of a two-particle system on a one-dimensional lattice. The number, location, and existence conditions of eigenvalues of this family were analyzed. An asymptotic expansion for the associated Fredholm determinant in a neighborhood of the origin has been derived.

KEYWORDS Friedrichs model, quantum mechanics, lattice, two-particle system, Fredholm determinant, eigenvalue, expansion.

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1. Introduction

The Friedrichs model is among the most fundamental and classical frameworks in spectral theory and quantum mechanics [1–3]. It was introduced by Kurt Friedrichs as a simplified model to investigate how discrete spectra may become embedded in, or interact with, the continuous spectrum under perturbations. The Friedrichs model, also referred to as the Friedrichs-Lee model, is a quantum mechanical framework describing the interaction between a discrete energy level and a continuous spectrum. It is worth noting that the properties of the Friedrichs model find applications in various fields, including the quantum field theory of unstable particles, non-relativistic quantum electrodynamics, and quantum optics.

It should be noted that the Friedrichs model is among the most widely used and powerful theoretical tools for the mathematical analysis of quantum decay, resonances, and energy leakage into the continuum in nanosystems [4]. This model enables both analytical and numerical investigations of the stability of quantum dots, the lifetimes of resonance states, energy exchange processes, and decoherence phenomena.

In [5], a family of Friedrichs models $h_\mu(p)$, with $\mu > 0$ and $p \in (-\pi;\pi]^3$ of the generalized Friedrichs model featuring a rank-one perturbation, associated with a two-particle system moving on a one-dimensional lattice, is studied. The existence of a unique eigenvalue below the bottom of the essential spectrum of $h_\mu(p)$ for all non-trivial values of p is established under the assumption that $h_\mu(0)$ possesses either a threshold energy resonance (virtual level) or a threshold eigenvalue. The threshold energy expansion of the Fredholm determinant associated with a family of the Friedrichs models has also been derived.

In [6], a family $H_\mu(p)$, $\mu > 0$, $p \in (-\pi;\pi]$ of the generalized Friedrichs model with perturbation of rank one, associated with a system of two particles, moving on the one-dimensional lattice is considered. The existence of a unique eigenvalue $E(\mu,p)$, of the operator $H_\mu(p)$ lying below the essential spectrum, is established. Moreover, for all p in a neighborhood of the origin, the Puiseux series expansion of the eigenvalue $E(\mu,p)$ at the point $\mu = \mu(p) \to 0$ is derived. Furthermore, the asymptotic behavior of $E(\mu,p)$ as $\mu \to +\infty$ is established.

In [7], a family of the generalized Friedrichs models $H_\mu(p)$, $\mu > 0$, $p \in (-\pi;\pi]^3$ with the perturbation of rank one is investigated. An absolutely convergent expansion for the eigenvalue at the coupling constant threshold is obtained. It is shown that the form of this expansion crucially depends on whether the lower edge of the essential spectrum corresponds to a threshold resonance or a threshold eigenvalue.

In the present paper, we consider a family of Friedrichs models $h_\mu^{(\gamma)}(k)$, depending on the parameters $\mu, \gamma > 0$ and $k \in (-\pi;\pi]$ with the rank-one perturbation associated to a system of two arbitrary or identical quantum mechanical particles moving on the one-dimensional lattice. Here we note that the kinetic part of $h_\mu^{(\gamma)}(k)$ contains a parameter $\gamma > 0$ and differs from the above-mentioned works in this respect. One of the important aspect of studying such type Friedrichs models is that they describe the Hamiltonian for systems of both bosons and fermions (see, for example, [8–11]). It is important that when considering three particle model Hamiltonian on a lattice, the role of two-particle discrete Schrödinger operators is played by a family of Friedrich's models [12–14]. For each fixed number $\gamma > 0$, it is proved that there exists a critical point μ_γ^0 such that for any $\mu \in (0; \mu_\gamma^0]$ the Friedrichs model $h_\mu^{(\gamma)}(0)$ has no negative eigenvalues, and for any $\mu \in (\mu_\gamma^0; +\infty)$ the Friedrichs model $h_\mu^{(\gamma)}(0)$ has a unique simple negative eigenvalue. For any fixed $\mu > 0$ and $\gamma > 0$ we study the expansion for the Fredholm determinant $\Delta_\mu^{(\gamma)}(k\,; z)$ associated to $h_\mu^{(\gamma)}(k)$ with respect to the spectral parameters k and z.

2. Statement of the problem

Let $\mathbb{T}$ be the one-dimensional torus and $L_2(\mathbb{T})$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}$.

For any fixed $\mu > 0$ and $\gamma > 0$, we consider a family of Friedrichs models $h_\mu^{(\gamma)}(k)$, $k \in \mathbb{T}$, acting in the Hilbert space $L_2(\mathbb{T})$, given by

$$h_\mu^{(\gamma)}(k) := h_0^{(\gamma)}(k) - \mu v,$$

where $h_0^{(\gamma)}(k)$ is the multiplication operator by the function $E_\gamma(k; \cdot)$:

$$(h_0^{(\gamma)}(k)f)(x) = E_\gamma(k, x)f(x), \quad f \in L_2(\mathbb{T}),$$
$$E_\gamma(k, x) := \varepsilon(k) + \varepsilon(x) + \gamma\varepsilon(k + x), \quad \varepsilon(x) := 1 - \cos x,$$

and v is a non-local interaction operator:

$$(vf)(x) = \sin(x) \int_{\mathbb{T}} \sin(t)f(t)dt.$$

It is clear that the family of Friedrichs models $h_\mu^{(\gamma)}(k)$, $k \in \mathbb{T}$ is a linear, bounded and self-adjoint operator in $L_2(\mathbb{T})$.

We note that in [15, 16], by analyzing the spectra of two Friedrichs models with rank-two perturbations, the existence of eigenvalues located inside the spectrum, within the spectral gap, and below the bottom of the essential spectrum of the tensor sum of these Friedrichs models has been established.

Throughout the paper, the notations $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$, $\sigma_{\rm p}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$ are employed to represent the spectrum, the essential spectrum, the point spectrum, and the discrete spectrum, respectively, of a bounded self-adjoint operator. For the reader's convenience, we provide the definitions of the essential and discrete spectra for a linear, bounded, and self-adjoint operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$. The set of all isolated eigenvalues of finite multiplicity of the operator $\mathcal{A}$ is called the discrete spectrum of the operator $\mathcal{A}$. The complement set $\sigma(\mathcal{A}) \setminus \sigma_{\rm disc}(\mathcal{A})$ is called the essential spectrum of the operator $\mathcal{A}$.

The perturbation operator v corresponding to the unperturbed operator $h_0^{(\gamma)}(k)$ is a self-adjoint operator of rank one. Hence, according to the Weyl theorem regarding the invariance of the essential spectrum under finite-rank perturbations, the essential spectra of the operators $h_\mu^{(\gamma)}(k)$ and $h_0^{(\gamma)}(k)$ coincide. It is evident that the unperturbed operator $h_0^{(\gamma)}(k)$ has a purely essential spectrum and the essential spectrum $\sigma_{\rm ess}(h_0^{(\gamma)}(k))$ equals $[m_\gamma(k); M_\gamma(k)]$, where the numbers $m_\gamma(k)$ and $M_\gamma(k)$ are defined by

$$m_\gamma(k) := \min_{x\in\mathbb{T}} E_\gamma(k, x), \quad M_\gamma(k) := \max_{x\in\mathbb{T}} E_\gamma(k, x).$$

Consequently, the essential spectrum $\sigma_{\rm ess}(h_\mu^{(\gamma)}(k))$ equals $[m_\gamma(k); M_\gamma(k)]$, and does not depend on the parameter $\mu > 0$. Notably, when $k = 0$ the following equality

$$\sigma_{\rm ess}(h_\mu^{(\gamma)}(0)) = [0; 2(1 + \gamma)]$$

holds.

For any $\mu, \gamma > 0$ and $k \in \mathbb{T}$ we define an analytic function $\Delta_\mu^{(\gamma)}(k\,; \cdot)$ (the Fredholm determinant corresponding to the Friedrichs model $h_\mu^{(\gamma)}(k)$) in $\mathbb{C} \setminus [m_\gamma(k); M_\gamma(k)]$ by

$$\Delta_\mu^{(\gamma)}(k\,; z) := 1 - \mu \int_{\mathbb{T}} \frac{\sin^2(t)\, dt}{E_\gamma(k, t) - z}.$$

The following lemma is a straightforward consequence of the Birman-Schwinger principle and the Fredholm theorem.

Lemma 2.1. *Let $\mu, \gamma > 0$ and $k \in \mathbb{T}$ be fixed. The Friedrichs model $h_\mu^{(\gamma)}(k)$ has an eigenvalue $z_{\mu,\gamma}(k) \in \mathbb{C} \setminus [m_\gamma(k); M_\gamma(k)]$ if and only if $\Delta_\mu^{(\gamma)}(k\,; z_{\mu,\gamma}(k)) = 0$.*

Lemma 2.1 thus yields the following equality

$$\sigma_{\rm disc}(h_\mu^{(\gamma)}(k)) = \{z \in \mathbb{C} \setminus [m_\gamma(k); M_\gamma(k)] :\ \Delta_\mu^{(\gamma)}(k\,; z) = 0\}$$

for the discrete spectrum of $h_\mu^{(\gamma)}(k)$.

At this point, we specify the number and location of the eigenvalues of the Friedrichs model $h_\mu^{(\gamma)}(k)$.

Lemma 2.2. *For any fixed $\mu, \gamma > 0$ and $k \in \mathbb{T}$, the Friedrichs model $h_\mu^{(\gamma)}(k)$ has at most one simple eigenvalue to the left of $m_\gamma(k)$ and no eigenvalues to the right of $M_\gamma(k)$.*

The proof of Lemma 2.2 relies on the monotonicity of the function $\Delta_\mu^{(\gamma)}(k\,;\cdot)$ over $(-\infty; m_\gamma(k))$, the inequality $\Delta_\mu^{(\gamma)}(k\,;z) > 1$ for all $z > M_\gamma(k)$, and Lemma 2.1.

Since for any $\gamma > 0$ the function $E_\gamma(\cdot,\cdot)$ has non-degenerate global minimum equal to zero at the point $(0,0) \in \mathbb{T}^2$, the following expansion

$$E_\gamma(x,y) = \frac{1}{2}\left(\frac{\partial^2 E_\gamma(0,0)}{\partial x^2}x^2 + 2\frac{\partial^2 E_\gamma(0,0)}{\partial x \partial y}xy + \frac{\partial^2 E_\gamma(0,0)}{\partial y^2}y^2\right) + o(x^2) + o(y^2)$$

holds as $x, y \to 0$. Hence, one can find positive constants $C_1(\gamma), C_2(\gamma) > 0$ and $\delta > 0$ for which the following estimates

$$C_1(\gamma)(x^2+y^2) \le E_\gamma(x,y) \le C_2(\gamma)(x^2+y^2), \quad (x,y) \in (-\delta;\delta)\times(-\delta;\delta), \tag{2.1}$$

$$E_\gamma(x,y) \ge C_1(\gamma), \quad (x,y) \notin (-\delta;\delta)\times(-\delta;\delta) \tag{2.2}$$

are valid. Using inequalities (2.1), (2.2), together with the asymptotic relation $\sin x \sim x$ as $x \to 0$ one can easily see that the following integral

$$\int_{\mathbb{T}} \frac{\sin^2(t)dt}{E_\gamma(k,t)}$$

is positive and finite for any $\gamma > 0$ and $k \in \mathbb{T}$. Thus, the Lebesgue dominated convergence theorem implies that $\Delta_\mu^{(\gamma)}(0\,;0) = \lim\limits_{k\to 0} \Delta_\mu^{(\gamma)}(k\,;0)$, and, consequently, the function $\Delta_\mu^{(\gamma)}(\cdot\,;0)$ is continuous on $\mathbb{T}$.

Set

$$\mu_\gamma^0 := (1+\gamma)\left(\int_{\mathbb{T}} \frac{\sin^2(t)\,dt}{\varepsilon(t)}\right)^{-1}.$$

This implies that $\Delta_\mu^{(\gamma)}(0\,;0) = 0$ if and only if $\mu = \mu_\gamma^0$.

We now examine the eigenvalues of the Friedrichs model $h_\mu^{(\gamma)}(k)$ when $k = 0$.

Theorem 2.3. *Suppose $\gamma > 0$ is fixed. When $\mu \in (0; \mu_\gamma^0]$, the Friedrichs model $h_\mu^{(\gamma)}(0)$ does not have any eigenvalues. Whenever $\mu > \mu_\gamma^0$, the Friedrichs model $h_\mu^{(\gamma)}(0)$ admits single negative eigenvalue.*

Proof. Assume $\mu \in (0; \mu_\gamma^0]$. To start, let us prove the following inequality

$$\Delta_\mu^{(\gamma)}(0\,;z) > \Delta_\mu^{(\gamma)}(0\,;0) \ge \Delta_{\mu_\gamma^0}^{(\gamma)}(0\,;0)$$

for all $z < 0$.

Since the function $\Delta_\mu^{(\gamma)}(0\,;\cdot)$ is monotonically decreasing on the interval $(-\infty;0)$, it follows that

$$\Delta_\mu^{(\gamma)}(0\,;z) > \Delta_\mu^{(\gamma)}(0\,;0)$$

for every $z < 0$.

Now, we turn to the proof of the inequality

$$\Delta_\mu^{(\gamma)}(0\,;0) \ge \Delta_{\mu_\gamma^0}^{(\gamma)}(0\,;0)$$

for any $\mu \in (0; \mu_\gamma^0]$. As a matter of fact, we have

$$\Delta_\mu^{(\gamma)}(0\,;0) = 1 - \mu \int_{\mathbb{T}} \frac{\sin^2(t)\,dt}{(1+\gamma)\varepsilon(t)} \ge 1 - \mu_\gamma^0 \int_{\mathbb{T}} \frac{\sin^2(t)\,dt}{(1+\gamma)\varepsilon(t)} = \Delta_{\mu_\gamma^0}^{(\gamma)}(\,0;0).$$

Alternatively, using the definition of μ_γ^0, we find

$$\Delta_{\mu_\gamma^0}^{(\gamma)}(0;0) = 1 - (1+\gamma)\left(\int_{\mathbb{T}} \frac{\sin^2(t)\,dt}{\varepsilon(t)}\right)^{-1} \int_{\mathbb{T}} \frac{\sin^2(t)\,dt}{(1+\gamma)\varepsilon(t)} = 0.$$

Employing the definition of $\Delta_\mu^{(\gamma)}(0;\cdot)$, we deduce that

$$\lim_{z\to-\infty} \Delta_\mu^{(\gamma)}(0;z) = 1 \tag{2.3}$$

and

$$\lim_{z\to 0^-} \Delta_\mu^{(\gamma)}(0;z) = \Delta_\mu^{(\gamma)}(0;0).$$

Since $\Delta_\mu^{(\gamma)}(0;z) > 0$ for all $z < 0$, it follows from Lemma 2.1 that the Friedrichs model $h_\mu^{(\gamma)}(0)$ has no eigenvalues in $(-\infty;0)$ (see Figure 1).

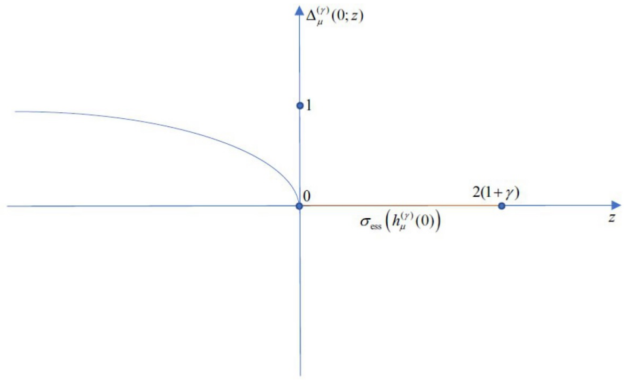

FIG. 1. An absence of the eigenvalue of $h_\mu^{(\gamma)}(0)$ for $0 < \mu \le \mu_\gamma^0$.

Let us now assume that $\mu > \mu_\gamma^0$. It can be easily seen from simple calculations that

$$\mu > \mu_\gamma^0 \Longrightarrow \mu(\mu_\gamma^0)^{-1} > 1 \Longrightarrow 1 - \mu(\mu_\gamma^0)^{-1} < 0.$$

The last inequality is expressed equivalently by the following relation

$$\Delta_\mu^{(\gamma)}(0\,;0) = 1 - \mu \int_{\mathbb{T}} \frac{\sin^2(t)\,dt}{(1+\gamma)\varepsilon(t)} < 0.$$

Since the function $\Delta_\mu^{(\gamma)}(0\,;\cdot)$ is monotone decreasing on the interval $(-\infty;0)$ and $\Delta_\mu^{(\gamma)}(0\,;z) < 0$, it follows from equality (2.3) that $\Delta_\mu^{(\gamma)}(0\,;\cdot)$ has unique negative zero $z_{\mu,\gamma}$ (see Figure 2).

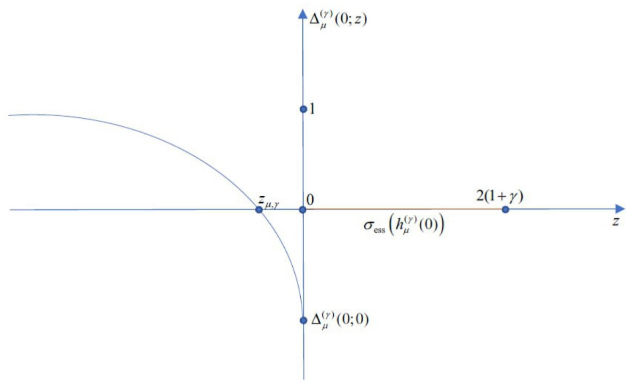

FIG. 2. An existence of the eigenvalue of $h_\mu^{(\gamma)}(0)$ for $\mu > \mu_\gamma^0$.

According to 2.1 the number $z_{\mu,\gamma}$ is an eigenvalue of the Friedrichs model $h_\mu^{(\gamma)}(0)$. $\square$

3. Expansion for the Fredholm determinant

In this section, we obtain an asymptotic expansion for the Fredholm determinant, which is important in analyzing the number of eigenvalues of the model operator corresponding to the energy operator of a system of three particles on a lattice.

Now we formulate the main result of the present paper.

Theorem 3.1. *Let $\mu, \gamma > 0$ be a fixed. The following expansion*

$$\Delta_\mu^{(\gamma)}(k\,;z) = \Delta_\mu^{(\gamma)}(0\,;0) + \frac{2\mu\pi(1+2\gamma-\gamma^2)}{(1+\gamma)^2\sqrt{1+2\gamma}} \sqrt{k^2 - \frac{2(1+\gamma)}{1+2\gamma}z} + O(k^2) + O(\sqrt{|z|})$$

holds as $k \to 0$ and $z \to -0$.

Proof. Let $\delta>0$ be sufficiently small and $\mathbb{T}_\delta := \mathbb{T}\setminus(-\delta;\delta)$. We rewrite the function $\Delta_\mu^{(\gamma)}(\cdot\,;\cdot)$ in the from $\Delta_\mu^{(\gamma)}(k\,;z) = \Delta_\mu^{(\gamma,1)}(k\,;z)+\Delta_\mu^{(\gamma,2)}(k\,;z)$, where

$$\Delta_\mu^{(\gamma,1)}(k\,;z) := 1-\mu\int_{\mathbb{T}_\delta}\frac{\sin^2(t)\,dt}{E_\gamma(k,t)-z},$$

$$\Delta_\mu^{(\gamma,2)}(k\,;z) := -\mu\int\limits_{-\delta}^{\delta}\frac{\sin^2(t)\,dt}{E_\gamma(k,t)-z}.$$

Since $\Delta_\mu^{(\gamma,1)}(\cdot\,;z)$ is an even analytic function on $\mathbb{T}$ for any $z\le 0$, we have

$$\Delta_\mu^{(\gamma,1)}(k\,;z) = \Delta_\mu^{(\gamma,1)}(0\,;0)+O(k^2)+O(|z|) \tag{3.1}$$

as $k\to 0$ and $z\to -0$. Using the relations

$$\sin k = k+O(k^3), \quad 1-\cos k = \frac{1}{2}k^2+O(k^4)$$

as $k\to 0$, we obtain

$$\Delta_\mu^{(\gamma,2)}(k\,;z) := -2\mu\int\limits_{-\delta}^{\delta}\frac{t^2dt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z}+O(k^2)+O(|z|)$$

as $k\to 0$ and $z\to -0$. For the convenience, we rewrite the latter integral as

$$\int\limits_{-\delta}^{\delta}\frac{t^2dt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z} =$$

$$= \frac{2\delta}{1+\gamma}-\frac{\gamma k}{1+\gamma}\int\limits_{-\delta}^{\delta}\frac{2tdt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z}-$$

$$-\frac{(1+\gamma)k^2-2z}{1+\gamma}\int\limits_{-\delta}^{\delta}\frac{dt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z}.$$

Now, we will analyze each integral in the previous equality. Evaluating the integral in the second summand, we obtain

$$\int\limits_{-\delta}^{\delta}\frac{2tdt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z} = \frac{1}{1+\gamma}\log\left|1+\frac{4\gamma\delta k}{(1+\gamma)k^2-2\gamma\delta k+(1+\gamma)\delta^2-2z}\right|$$

$$-\frac{2\gamma k}{1+\gamma}\int\limits_{-\delta}^{\delta}\frac{dt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z}.$$

Since

$$\log\left|1+\frac{4\gamma\delta k}{(1+\gamma)k^2-2\gamma\delta k+(1+\gamma)\delta^2-2z}\right| = O(k)$$

as $k\to 0$, from the comparison of the last expressions, we derive

$$\int\limits_{-\delta}^{\delta}\frac{t^2dt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z} = \frac{2\delta}{1+\gamma}-$$

$$-\left(\frac{1+2\gamma-\gamma^2}{(1+\gamma)^2}k^2-\frac{2}{1+\gamma}z\right)\int\limits_{-\delta}^{\delta}\frac{dt}{(1+\gamma)k^2+2\gamma kt+(1+\gamma)t^2-2z}+O(k^2)+O(|z|)$$

as $k\to 0$ and $z\to -0$. Applying the identity

$$\int\limits_{a}^{b}\frac{dt}{k^2+t^2} = \frac{1}{|k|}\left(\arctan\frac{b}{|k|}-\arctan\frac{a}{|k|}\right),$$

we obtain

$$\int_{-\delta}^{\delta} \frac{dt}{(1+\gamma)k^2 + 2\gamma kt + (1+\gamma)t^2 - 2z} = \frac{1}{1+\gamma} \int_{-\delta}^{\delta} \frac{dt}{\left(t + \frac{\gamma}{1+\gamma}k\right)^2 + \frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z} =$$

$$\frac{1}{(1+\gamma)\sqrt{\frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z}} \left(\arctan \frac{\delta + \frac{\gamma}{1+\gamma}k}{\sqrt{\frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z}} + \arctan \frac{\delta - \frac{\gamma}{1+\gamma}k}{\sqrt{\frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z}} \right).$$

From the properties of the arctan function listed below

$$\arctan y + \arctan \frac{1}{y} = \frac{\pi}{2}, \quad y \geq 0$$

and $\arctan y = O(y)$, $y \to 0$, we conclude that

$$\int_{-\delta}^{\delta} \frac{dt}{(1+\gamma)k^2 + 2\gamma kt + (1+\gamma)t^2 - 2z} =$$

$$\left(\frac{1+2\gamma-\gamma^2}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z \right) \frac{\pi}{(1+\gamma)\sqrt{\frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z}} + O\left(\sqrt{\frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z} \right)$$

as $k \to 0$ and $z \to -0$. In view of the fact that

$$\left(\frac{1+2\gamma-\gamma^2}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z \right) \frac{\pi}{(1+\gamma)\sqrt{\frac{1+2\gamma}{(1+\gamma)^2}k^2 - \frac{2}{1+\gamma}z}} =$$

$$\pi \frac{1+2\gamma-\gamma^2}{(1+\gamma)^2\sqrt{1+2\gamma}} \sqrt{k^2 - \frac{2(1+\gamma)}{2\gamma+1}z} + O(\sqrt{-z}),$$

we obtain

$$\Delta_\mu^{(\gamma,2)}(k\,;z) := \Delta_\mu^{(\gamma,2)}(0\,;0) + \frac{2\mu\pi(1+2\gamma-\gamma^2)}{(1+\gamma)^2\sqrt{1+2\gamma}} \sqrt{k^2 - \frac{2(1+\gamma)}{2\gamma+1}z} + O(k^2) + O(\sqrt{-z}) \tag{3.2}$$

as $k \to 0$ and $z \to -0$. The equalities (3.1) and (3.2) finalize the proof of the theorem. □

Since $\Delta_\mu^{(\gamma)}(0\,;0) = 0$ if and only if $\mu = \mu_\gamma^0$, it follows from Theorem 3.1 that the following assertion holds.

Corollary 3.2. *Assume that $\gamma > 0$ is fixed. If $\mu = \mu_\gamma^0$, then the following expansion*

$$\Delta_{\mu_\gamma^0}^{(\gamma)}(k\,;z) = \frac{2\pi\mu_\gamma^0(1+2\gamma-\gamma^2)}{(1+\gamma)^2\sqrt{1+2\gamma}} \sqrt{k^2 - \frac{2(1+\gamma)}{1+2\gamma}z} + O(k^2) + O(\sqrt{|z|})$$

holds as $k \to 0$ and $z \to -0$.

As a consequence of Corollary 3.2, the following estimates for $\Delta_{\mu_\gamma^0}^{(\gamma)}(k\,;0)$ are obtained.

Corollary 3.3. *Let $\gamma > 0$ be a fixed parameter. If $\mu = \mu_\gamma^0$, then there exist the numbers $C_1(\gamma), C_2(\gamma) > 0$ and $\delta > 0$ such that the inequalities*
(i) $C_1(\gamma)|k| \leq \Delta_{\mu_\gamma^0}^{(\gamma)}(k\,;0) \leq C_2(\gamma)|k|$ *for any* $k \in (-\delta;\delta)$;
(ii) $\Delta_{\mu_\gamma^0}^{(\gamma)}(k\,;0) \geq C_1(\gamma)$ *for any* $k \in \mathbb{T} \setminus (-\delta;\delta)$
are satisfied.

Conclusion. In the present paper, we investigate a class (family) of Friedrichs models that arise in quantum mechanical problem. It represents the energy operator (Hamiltonian) for a two-particle system defined on a one-dimensional lattice. We analyze the number, distribution, and existence criteria for the eigenvalues associated with this family. As the main result, we derive an asymptotic expansion of the associated Fredholm determinant in a neighborhood of the origin. This asymptotic expansion, along with Corollaries 3.2 and 3.3, plays a crucial role in proving the infiniteness (respectively, finiteness) of the number of eigenvalues lying below the essential spectrum of the corresponding three-particle lattice model Hamiltonian. It should be noted that the results on the infinite number of eigenvalues of the three-particle discrete Schrödinger operators and the corresponding model Hamiltonians are very important in quantum mechanics, modern mathematical physics, and the spectral theory of operators. The eigenvalues correspond to bound states in the quantum system. If the number of eigenvalues is infinite, then this means that there are infinitely many energy levels in the system. In [17] it was shown that the number of eigenvalues of the three-particle discrete Schrödinger operator is infinite in the case where the masses of two particles in a three-particle system are infinite.

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