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Original article

Inverse analysis of a loaded heat conduction equation

Umida Baltaeva[1,a], Praveen Agarwal[2,b], Bobur Khasanov[3,c], Hamrobek Hayitbayev[4,d], Florence Hubert[5,e]

[1]Department of Applied Mathematics and Mathematical Physics, Urgench State University, Urgench-220100, Uzbekistan

[2]Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India

[3]Department of Exact Sciences, Khorezm Mamun Academy, Khiva, Uzbekistan

[4]Department of Accounting and General Professional Sciences, Mamun university, Khiva, Uzbekistan

[5]Aix-Marseille Universite, I2M, Marseille, France

[a]umida_baltayeva@mail.ru, [b]goyal.praveen2011@gmail.com, [c]xasanovboburjon.1993@gmail.com, [d]hamrohayitboyev073@gmail.com, [e]florence.hubert@univ-amu.fr

Corresponding author: Baltaeva U., umida_baltayeva@mail.ru

ABSTRACT This work considers an inverse problem for a heat conduction equation that includes fractional loaded terms and coefficients varying with spatial coordinates. By reformulating the original equation into a system of equivalent loaded integro-differential equations, we establish sufficient conditions ensuring the existence and uniqueness of the solution. The study focuses on determining the multidimensional kernel associated with the fractional heat conduction operator. The approach is based on the contraction mapping principle and the use of Riemann-Liouville fractional integrals, providing a mathematical framework applicable to diffusion processes with spatial heterogeneity and memory effects.

KEYWORDS heat conduction, inverse problem, fractional calculus, kernel identification, fixed-point method.

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1. Introduction

Heat conduction refers to the transfer of internal energy, in the form of heat, between adjacent molecules within solids, liquids, or gases, as well as across the interfaces of materials in contact, without requiring bulk motion of the medium itself [1, 2]. The study of heat transfer processes has a long and rich history, spanning more than two centuries, and continues to play a vital role in both theoretical and applied research. In particular, understanding the mechanisms of heat dissipation and temperature regulation in modern high-speed and micro-structured systems remains a crucial challenge for engineering and materials science.

The classical heat conduction equation serves as a universal model in many branches of physics and applied mathematics. It arises naturally in the study of diffusion phenomena, such as the spreading of mass, charge, or vorticity in viscous fluids. From a mathematical perspective, the one-dimensional heat equation is one of the most extensively studied partial differential equations, while the extension to multidimensional and fractional-order formulations continues to attract significant attention due to its broad range of applications in complex physical systems.

In recent decades, the study of fractional differential equations - where the fractional derivative acts on the unknown function - has become increasingly important [3–8]. These equations provide a powerful framework for describing nonlocal and memory-dependent processes that cannot be adequately represented by classical integer-order models. Such features are particularly relevant in nanoscale systems, where energy transfer is strongly influenced by microstructural heterogeneity, phonon scattering, and boundary effects. At these scales, the conventional Fourier law of heat conduction is often insufficient, which motivates the use of fractional and integro-differential models.

In nanosystems, such as thin films, nanowires, and layered composites, thermal conductivity often varies spatially due to surface effects and structural anisotropy. To capture these phenomena, mathematical models with variable coefficients are required. Moreover, the presence of loaded or feedback terms in the governing equations can describe systems with internal heat sources, delayed responses, or coupling between local and nonlocal energy transfer mechanisms.

The present study focuses on an inverse problem for a loaded heat conduction equation with spatially variable coefficients and fractional integral operators. By reformulating the model into an equivalent system of loaded integro-differential equations, we establish conditions for the existence and uniqueness of the solution. The results obtained contribute to the development of mathematical methods for analyzing nanoscale heat transport processes characterized by nonlocality, memory, and structural heterogeneity.

The study of heat conduction has a history spanning over 200 years and remains a topic of significant scientific interest. For example, understanding the dissipation and transfer of heat from sources in high-speed machinery presents a critical technological challenge. This issue is of fundamental importance across various fields of physics.

The heat conduction equation is universal and arises in various contexts, such as modeling mass diffusion or describing the diffusion of vorticity in viscous fluids. Heat conduction problems also hold significant importance from a mathematical perspective. The theory of one-dimensional heat equations is currently the most well-developed and widely applied. Meanwhile, the study of heat conduction problems (of both integer and fractional order) in three-dimensional and multidimensional domains is a growing area within the modern theory of partial differential equations, driven by their relevance to real-world processes.

Fractional diffusion models generalize classical diffusion models by incorporating derivatives of non-integer order. Interest in their study arises from their application in modeling a wide range of phenomena in the physical sciences, medicine, and biology (see, for example, [9–13].

The two-dimensional and multidimensional space-fractional diffusion equations with variable coefficients present a complex challenge in both theoretical analysis and computational approaches [14]. When addressing such equations, it is not always feasible to establish the correctness of the problem using classical methods or to develop efficient numerical techniques. Consequently, the study of fractional diffusion equations with variable coefficients often combines analytical methods with various integration techniques to achieve meaningful results.

In [15], methods for the numerical approximation of the fractional diffusion equation with variable coefficients are presented. The integration of boundary value problems using the method of prior estimates for the fractional diffusion equation is discussed in [17], following an approach similar to that used in the classical case. Paper [16] introduces a finite difference approximation method for a spatial fractional convection-diffusion model governed by an equation with variable coefficients. In [18], the homotopy analysis method and the Adomian decomposition method are applied to high-order time-fractional partial differential equations. Additionally, works [19–23] investigate initial and boundary value problems for the fractional diffusion equation with variable coefficients.

The increased interest in the study of loaded differential equations [24] is attributed to their numerous applications and the fact that they form a distinct class of partial differential equations [25,26]. Notably, the pioneering works of Nakhushev provided one of the first generalized definitions of loaded equations, along with their classification and applications to various problems in mathematical physics and biology [27–34]. Boundary value problems for heat conductivity-loaded equations in both bounded and unbounded domains have been investigated in [28–31], particularly when the order of the derivative in the loaded term is greater than or equal to the order of the differential part of the equation.

In this work, we investigate both analytical and physical aspects of heat transfer at the microscale and nanoscale, focusing on an analogue of the fractional diffusion equation with variable coefficients and a fractional loading term. Such formulations provide a more accurate description of thermal processes in inhomogeneous media, where local and nonlocal interactions coexist and where the transport mechanisms are influenced by spatial variability in material properties.

It should be noted that significant progress in the theory of inverse problems for parabolic equations with variable coefficients was achieved by Beznoshenko, Yan, and Kamynin, who were among the first to investigate such problems systematically. Later, Prilepko and Kostin developed fundamental theorems on the existence and uniqueness of solutions for inverse problems associated with parabolic initial-boundary value problems involving variable coefficients. These classical studies laid the groundwork for the modern theory of inverse and ill–posed problems in heat conduction and diffusion.

Building upon these foundational contributions, the present work addresses an inverse problem for a loaded integro-differential heat conduction equation with spatially variable coefficients. In contrast to earlier models, we consider equations that include fractional integral operators and loading terms, which are essential for describing nonlocal and memory-dependent heat transfer phenomena observed in nanoscale materials. Following recent developments in this field [35–37], the main focus of our study is on determining an unknown coefficient function in the loaded heat conduction equation and establishing conditions for its unique reconstruction.

We introduce the following notations:

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let $\mathbb{R}^2_T$ represent a subset of three-dimensional Euclidean space, specified by a point (x, y, t), where $(x, y) \in \mathbb{R}^2$ and $t \in (0, T]$, with $T > 0$:

$$R^2_T = \left\{ (x, y, t) \,\middle|\, (x, y) \in R^2, 0 \le t \le T \right\}.$$

Let $f(x, y)$ be a function defined on $\mathbb{R}^2$.

Definition 1.1. Let $l \in (0, 1)$. If, for any two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, there exists a constant $A > 0$ such that

$$|f(x_1, y_1) - f(x_2, y_2)| \le A \left(|x_1 - x_2| + |y_1 - y_2| \right)^{l},$$

then the function $f(x, y)$ is said to satisfy the Holder condition with exponent l in $\mathbb{R}^2$. The class of all such functions is denoted by $H^{\,l}(\mathbb{R}^2)$.

Definition 1.2. Let $l \in (0,1)$ and $T > 0$. If, for any two points (x_1, y_1, t_1) and (x_2, y_2, t_2) in $\mathbb{R}^2_T := \mathbb{R}^2 \times [0,T]$, there exist positive constants $A_1, A_2, A_3 > 0$ such that

$$|f(x_1,y_1,t_1) - f(x_2,y_2,t_2)| \le A_1|x_1-x_2|^{\,l} + A_2|y_1-y_2|^{\,l} + A_3|t_1-t_2|^{\,l/2},$$

then the function $f(x,y,t)$ is said to satisfy the Holder condition with exponents l (in x, y) and $l/2$ (in t) in $\mathbb{R}^2_T$. The class of such functions is denoted by $H^{\,l,\,l/2}(\mathbb{R}^2_T)$.

Inverse problem. Find a pair of functions $u(x,y,t)$ and $k(x,0,t)$ in $(x,y,t) \in \mathbb{R}^2_T$, satisfying the following properties:

$$u_t - a\left(t\right)\left(u_{xx} + u_{yy}\right) = \lambda D_{0t}^{-\alpha} u\left(0,y,t\right) + \int_0^t k\left(x,0,\tau\right) u\left(x,y,t-\tau\right) d\tau, \tag{1.1}$$

$$\left. u\left(x,y,t\right)\right|_{t=0} = \varphi\left(x,y\right), \ \ (x,y) \in R^2, \tag{1.2}$$

$$\left. u\left(x,y,t\right)\right|_{y=0} = \chi\left(x,t\right), \ \ (x,t) \in R^1_T, \tag{1.3}$$

where $a(x),\ \varphi(x,y),\ \chi\left(x,t\right)$ are given functions and

$$a\left(t\right) \in I := \left\{a\left(t\right)\left|a\left(t\right) > 0,\, a\left(t\right) \in C^1\left[0,T\right] \cap C(0,T)\right.\right\}, \tag{1.4}$$

$$\varphi\left(x,y\right) \in H^{l+2}\left(R^2\right), \, \varphi\left(x,y\right) \le \varphi_0 = const > 0, \ \ \varphi\left(x,0\right) = \chi\left(x,0\right), \tag{1.5}$$

$$\chi\left(x,t\right) \in H^{l+4,(l+4)/2}\left(\bar{R}^1_T\right), \ l \in \left(0,1\right), \lambda \in R, \tag{1.6}$$

$D_{0t}^{-\alpha}$ is the Riemann-Liouville fractional integral operator [3] of order α and $\alpha > 0$. The inverse scattering problem is finding $u(x,y,t)$ and $k\left(x,0,t\right)$ from the equalities (1.1) - (1.3).

2. Investigation of the problem

First of all, we will construct auxiliary problems equivalent to the inverse problem (1.1), (1.2), (1.3).

Let us introduce the following replacement in the problem (1.1) - (1.3):

$$\vartheta(x,y,t) = u_{yy}(x,y,t), \ (x,y,t) \in R^2_T. \tag{2.1}$$

Using the change of variable (2.1), the inverse problem (1.1), (1.2), (1.3), is equivalently reduced to the following problem:

Auxiliary problem: Find functions $\vartheta(x,y,t)$ and $k\left(x,0,t\right)$ in $(x,y,t) \in R^2_T$, possessing the following properties:

$$\vartheta_t - a\left(t\right)\Delta\vartheta = \lambda D_{0t}^{-\alpha}\vartheta\left(0,y,t\right) + \int_0^t k\left(x,0,\tau\right)\vartheta\left(x,y,t-\tau\right)d\tau, \tag{2.2}$$

$$\left.\vartheta\ \left(x,y,t\right)\right|_{t=0} = \varphi_{yy}\left(x,y\right), \ (x,y) \in R^2, \tag{2.3}$$

$$\begin{aligned}\left.\vartheta\ \left(x,y,t\right)\right|_{y=0} &= \frac{1}{a\left(t\right)}\chi_t\left(x,t\right) - \chi_{xx}\left(x,t\right) - \\ &- \frac{\lambda}{a\left(t\right)}D_{0t}^{-\alpha}\chi\left(0,0,t\right) - \frac{1}{a\left(t\right)}\int_0^t k\left(x,0,\tau\right)\chi\left(x,0,t-\tau\right)d\tau,\end{aligned} \tag{2.4}$$

moreover, from the initial condition (2.3) and (2.4) the following condition of agreement is satisfied:

$$\varphi_{yy}\left(x,0\right) = \frac{1}{a\left(0\right)}\chi_t\left(x,0\right) - \chi_{xx}\left(x,0\right). \tag{2.5}$$

Indeed, if the compatibility conditions (1.3) and (2.5) are satisfied, and the functions φ and χ are sufficiently smooth, it can be shown that the problems (2.2)–(2.4) are equivalent to the inverse problem (1.1)–(1.3):

First, integrating twice from the (2.1) substitution above from 0 to y, we get

$$u(x,y,t) = \chi\left(x,t\right) + y\varphi\left(x,0\right) + \int_0^y (y-\xi)\vartheta(x,\xi,t)d\xi, \tag{2.6}$$

and, consequently, in (2.1) for the function $u(x,y,t)$, taking into account the agreement condition (2.5) for $t = 0$, we have

$$\begin{aligned}\left.u\left(x,y,t\right)\right|_{t=0} &= \chi\left(x,0\right) + yu_y\left(x,0,0\right) + \int_0^y \left(y-\xi\right)\varphi_{\xi\xi}\left(x,\xi\right)d\xi = \\ &= \chi\left(x,0\right) + yu_y\left(x,0,0\right) + \int_0^y \left(y-\xi\right)d\varphi_\xi = \chi\left(x,0\right) + yu_y\left(x,0,0\right) + \\ &+ \left(y-\xi\right)\varphi_\xi\ \left(x,\xi\right)\big|_0^y + \int_0^y \varphi_\xi\left(x,\xi\right)d\xi = \chi\left(x,0\right) + yu_y\left(x,0,0\right) - \\ &- y\varphi_y\left(x,0\right) + \varphi\left(x,y\right) - \varphi\left(x,0\right) = y\left(u_y\left(x,0,0\right) - \varphi_y\left(x,0\right)\right) + \varphi\left(x,y\right) = \varphi\left(x,y\right).\end{aligned}$$

As can be seen from (2.6), on $y = 0$ the additional condition in (1.3) follows.

The procedure for obtaining equation (1.1) from equation (2.2) is as follows. As the first step, we integrate both sides of equation (2.2) twice from 0 to y:

$$\int_0^y (y-\xi)\,\vartheta_t(x,\xi,t)\,d\xi - a(t)\int_0^y (y-\xi)\left(\vartheta_{xx}(x,\xi,t)+\vartheta_{\xi\xi}(x,\xi,t)\right)d\xi =$$
$$= \int_0^y (y-\xi)\,d\xi \int_0^t k(x,0,\tau)\,\vartheta(x,\xi,t-\tau)\,d\tau +$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^y (y-\xi)\,d\xi \int_0^t (t-\tau)^{\alpha-1} u(0,\xi,\tau)\,d\tau,$$

and taking into account equality (2.6), i.e

$$\int_0^y (y-\xi)\vartheta(x,\xi,t)d\xi = u(x,y,t)-\chi(x,t)-y\varphi(x,0)$$

as a result, we have the following relations:

$$\frac{\partial}{\partial t}\left(u(x,y,t)-\chi(x,t)-y\varphi_y(x,0)\right) -$$
$$-a(t)\frac{\partial^2}{\partial x^2}\left(u(x,y,t)-\chi(x,t)-y\varphi_y(x,0)\right) -$$
$$-a(t)\int_0^y (y-\xi)\vartheta_{yy}(x,\xi,t)d\xi =$$
$$= \int_0^t k(x,0,\tau)\left(u(x,y,t-\tau)-\chi(x,t-\tau)-y\varphi_y(x,0)\right)d\tau +$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(u(0,y,\tau)-\chi(x,t-\tau)-y\varphi_y(x,0)\right)d\tau.$$

$$u_t(x,y,t)-\chi_t(x,t)-a(t)\frac{\partial^2}{\partial x^2}u(x,y,t)+a(t)\chi(x,t)+a(t)y\vartheta_y(x,0,t) -$$
$$-a(t)\vartheta(x,y,t)+a(t)\vartheta(x,0,t) =$$
$$= \int_0^t k(x,0,\tau)\left(u(x,y,t-\tau)-\chi(x,t-\tau)-y\varphi_y(x,0)\right)d\tau +$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(u(0,y,\tau)-\chi(x,t-\tau)-y\varphi_y(x,0)\right)d\tau.$$

Hence, taking into account condition (2.4) for $\vartheta(x,y,t)$, it is easy to see that equation (1.1) has been obtained. Thus, the inverse problem of finding the functions $u(x,y,t)$ and $k(x,0,t)$, defined by (1.1) - (1.3), is equivalent to the inverse problem of finding the functions $\vartheta(x,y,t)$ and $k(x,0,t)$ from (2.2) - (2.4).

Auxiliary problem:

Now, in the second step, if we differentiate the resulting equations with respect to t and make the replacement $\vartheta_t(x,y,t) = \rho(x,y,t)$ in (2.2) - (2.4), we obtain the following auxiliary problem for determining the functions $\vartheta(x,y,t), k(x,0,t)$, and $\rho(x,y,t)$:

$$\rho_t - a(t)(\rho_{xx} + \rho_{yy}) = (\ln a(t))' \rho - (\ln a(t))' \int_0^t k(x,0,\tau)\vartheta(x,y,t-\tau)\,d\tau +$$

$$+ \int_0^t k(x,0,\tau)\rho(x,y,t-\tau)\,d\tau - \lambda(\ln a(t))' D_{0t}^{-\alpha}\vartheta(0,y,t) +$$

$$+\lambda D_{0t}^{-\alpha}\rho(0,y,t) + k(x,0,t)\varphi_{yy}(x,y) + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1}\varphi_{yy}(0,y,0), \tag{2.7}$$

$$\rho|_{t=0} = a(0)\Delta\varphi_{yy}(x,y), \tag{2.8}$$

$$\rho|_{y=0} = F_t(x,t) + \frac{a'(t)}{a^2(t)} \int_0^t k(x,0,\tau)\chi(x,t-\tau)\,d\tau -$$

$$-\frac{1}{a(t)}\int_0^t k(x,0,\tau)\chi_t(x,t-\tau)\,d\tau - \frac{1}{a(t)} k(x,0,t)\varphi(x',0), \tag{2.9}$$

where

$$F(x,t) = \frac{1}{a(t)}\chi_t(x,t) - \chi_{xx}(x,t) - \frac{\lambda}{a(t)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\chi(0,0,\tau)\,d\tau.$$

As a result, we obtain an auxiliary problem for determining the functions $\vartheta(x,y,t)$, $k(x,0,t)$, and $\rho(x,y,t)$.

In the next step, by integrating both sides of the last change of variable from 0 to t, we obtain the following equality:

$$\vartheta(x,y,t) = \varphi_{yy}(x,y) + \int_0^t \rho(x,y,\tau)\,d\tau. \tag{2.10}$$

If the function $\rho(x,y,t)$ is known, then the function $\vartheta(x,y,t)$ can be determined from (2.10). Thus, the problem (2.7) - (2.9) reduces to the problems (2.2) - (2.4), and the problems (2.2) - (2.4) lead to the inverse problem defined by (1.1) - (1.3). Therefore, finding the functions $\vartheta(x,y,t)$, $k(x,0,t)$, and $\rho(x,y,t)$ from problems (2.2) - (2.4) and (2.7) - (2.9) is equivalent to finding the functions $u(x,y,t)$ and $k(x,0,t)$ from the inverse problem (1.1) - (1.3).

Thus, we have proved the following lemma:

Lemma 2.1. *Suppose that $a(t) \in I$, $\varphi(x,y) \in H^{l+6}(\mathbb{R}^2)$, $\chi(x,t) \in H^{l+4,(l+4)/2}(\overline{\mathbb{R}}_T^1)$, and the matching conditions*

$$\chi(x,0) = \varphi(x,0), \qquad \varphi_{yy}(x,0) = \frac{1}{a(0)}\chi_t(x,0) - \chi_{xx}(x,0),$$

are satisfied. Then the problem (1.1) - (1.3) *is equivalent to the problem of determining the functions $\vartheta(x,y,t)$, $k(x,0,t)$, and $\rho(x,y,t)$ from equations* (2.2) - (2.4) *and* (2.7) - (2.9).

3. Reduction of problem (1.1) - (1.3) to a system of integral equations

Lemma 3.1. *The auxiliary problems* (2.2), (2.3) *and* (2.7) - (2.9) *is equivalent to finding the functions $\vartheta(x,y,t)$, $k(x,y,t)$, $\rho(x,y,t)$ from the following system of integral equations:*

$$\vartheta(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_{\eta\eta}(\xi,\eta)\,G d\xi d\eta + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \times$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)} k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\theta^{-1}(\tau)-\alpha\right) G d\alpha d\xi d\eta + \tag{3.1}$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)} \left(\theta^{-1}(\tau)-\beta\right)^{\alpha-1}\vartheta(0,\eta,\beta)\,G d\beta d\xi d\eta,$$

$$\rho(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(0)\Delta\varphi_{\eta\eta}(\xi,\eta)\,G d\xi d\eta + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \times$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\left(\ln a\left(\theta^{-1}(\tau)\right)\right)' \rho\left(\xi,\eta,\theta^{-1}(\tau)\right) -$$

$$- \left(\ln a\left(\theta^{-1}(\tau)\right)\right)' \int_0^{\theta^{-1}(\tau)} k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\theta^{-1}(\tau)-\alpha\right) d\alpha \Big) G d\xi d\eta +$$

$$+ \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)} k(\xi,0,\alpha)\rho\left(\xi,\eta,\theta^{-1}(\tau)-\alpha\right) G d\alpha d\xi d\eta +$$

$$+\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k\left(\xi,0,\theta^{-1}(\tau)\right)\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta+ \tag{3.2}$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)}\left(\theta^{-1}(\tau)-\beta\right)^{\alpha-1}\rho(0,\eta,\beta)\,Gd\beta d\xi d\eta-$$

$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\left(\ln a\left(\theta^{-1}(\tau)\right)\right)'\int_0^{\theta^{-1}(\tau)}\left(\theta^{-1}(\tau)-\beta\right)^{\alpha-1}\times$$

$$\times\vartheta(0,\eta,\beta)\,d\beta-\left(\theta^{-1}(\tau)\right)^{\alpha-1}\varphi_{\eta\eta}(\xi,\eta)\Big]\,Gd\xi d\eta,$$

$$k(x,0,t)=\frac{a(t)}{\varphi(x,0)}\Bigg(F_t(x,t)-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta-$$

$$-\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\ln a\left(\theta^{-1}(\tau)\right)\right)'\rho\left(\xi,\eta,\theta^{-1}(\tau)\right)Gd\xi d\eta+$$

$$+\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\times$$

$$\times\int_{-\infty}^{\infty}\left(\left(\ln a\left(\theta^{-1}(\tau)\right)\right)'\int_0^{\theta^{-1}(\tau)}k(\xi,0,\alpha)\vartheta\left(\xi,\eta,\theta^{-1}(\tau)-\alpha\right)d\alpha\right)Gd\xi d\eta-$$

$$-\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)}k(\xi,0,\alpha)\rho\left(\xi,\eta,\theta^{-1}(\tau)-\alpha\right)Gd\alpha d\xi d\eta-$$

$$-\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k\left(\xi,0,\theta^{-1}(\tau)\right)\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta- \tag{3.3}$$

$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\times$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)}\left(\theta^{-1}(\tau)-\beta\right)^{\alpha-1}\rho(0,\eta,\beta)\,Gd\beta d\xi d\eta+$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\times$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\left(\ln a\left(\theta^{-1}(\tau)\right)\right)'\int_0^{\theta^{-1}(\tau)}\left(\theta^{-1}(\tau)-\beta\right)^{\alpha-1}\vartheta(0,\eta,\beta)\,d\beta\right)Gd\xi d\eta-$$

$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}(\tau)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\theta^{-1}(\tau)\right)^{\alpha-1}\varphi_{\eta\eta}(\xi,\eta)\,Gd\xi d\eta\Bigg)+$$

$$+\frac{1}{\varphi(x,0)}\int_0^t\Big((\ln a(t))'\chi(x,t-\tau)-\chi_t(x,t-\tau)\Big)k(x,0,\tau)\,d\tau,$$

where, $G=G\left(x-\xi,y-\eta,\theta(t)\right)$ and, respectively, from $\theta(t)-\tau$.

Proof. In problems (2.2) - (2.3), taking into account formula (2.8) as in the correct problem from [23]

$$F(x,y,t)=\lambda D_{0t}^{-\alpha}\vartheta(0,y,t)+\int_0^t k(x,0,\tau)\,\vartheta(x,y,t-\tau)\,d\tau$$

in the above form, we ontain the integral equation (3.1) correspondingly equivalently. In the same way, taking into account (2.7) - (2.8), we get formula (3.2). Then, taking into account the resulting integral equations and using (2.9), we get the integral equation (3.3)

From problems (2.3),(2.4) and (2.7), (2.8), integral equations (3.1), (3.2) are obtained analogously to the equation (1.3). Equation (3.3) follows from equations (2.9) and (3.2).

Now we will prove the existence and uniqueness of the solution to problem (2.2) - (2.4). The proof is based on the contraction mapping principle.

Theorem 3.2. *If conditions (1.4), (1.5), (1.6), and, (2.5) are satisfied, then there exists a sufficiently small number $T_0>0$ such that, for $T\in(0,T_0]$, there exists a unique solution to the integral equations* (3.1) - (3.3) *belonging to the classes*

$$\{\vartheta(x,y,t),\ \rho(x,y,t)\}\in H^{l+2,(l+2)/2}\left(\bar{R}_T^2\right),\ k(x,0,t)\in H^{l,l/2}\left(\bar{R}_T^1\right).$$

Proof. To prove the theorem using the contraction mapping principle, we rewrite the system of equations (3.1) - (3.3) as a nonlinear operator:

$$\sigma = L\sigma,\ \sigma = (\sigma_1, \sigma_2, \sigma_3)^* = (\vartheta, \rho, k(x,0,t))^*, \tag{3.4}$$

where $*$ is the transposition symbol, $L\sigma = [(L\sigma)_1, (L\sigma)_2, (L\sigma)_3]^*$. Thus, according to the right sides of equations (3.1) - (3.3), $(L\sigma)_i\,(i = 1,2,3)$, we have

$$(L\sigma)_1 = \sigma_{01}(x,y,t) + \int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})} \times$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} \sigma_3(\xi,0,\alpha)\sigma_1(\xi,\eta,\widehat{\tau}-\alpha)\, G d\alpha d\xi d\eta +$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} (\widehat{\tau}-\beta)^{\alpha-1}\sigma_1(0,\eta,\beta)\, G d\beta d\xi d\eta,$$

where

$$(L\sigma)_2 = \sigma_{02}(x,y,t) + \int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[(\ln a(\widehat{\tau}))'\,\sigma_2(\xi,\eta,\widehat{\tau}) -$$
$$-(\ln a(\widehat{\tau}))'\int_0^{\widehat{\tau}} \sigma_3(\xi,0,\alpha)\sigma_1(\xi,\eta,\widehat{\tau}-\alpha)\, d\alpha\Big] G d\xi d\eta +$$
$$+\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} \sigma_3(\xi,0,\alpha)\sigma_2(\xi,\eta,\widehat{\tau}-\alpha)\, G d\alpha d\xi d\eta +$$
$$+\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_3(\xi,0,\widehat{\tau})\,\varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta +$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} (\widehat{\tau}-\beta)^{\alpha-1}\sigma_2(0,\eta,\beta)\, G d\beta d\xi d\eta -$$
$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[(\ln a(\widehat{\tau}))'\int_0^{\widehat{\tau}} (\widehat{\tau}-\beta)^{\alpha-1}\sigma_1(0,\eta,\beta)\, d\beta\Big] G d\xi d\eta.$$

$$(L\sigma)_3 = \sigma_{03}(x,y,t) -$$
$$-\frac{a(t)}{\varphi(x,0)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\ln a(\widehat{\tau}))'\,[\sigma_2(\xi,\eta,\widehat{\tau}) -$$
$$-\int_0^{\widehat{\tau}} \sigma_3(\xi,0,\alpha)\sigma_1(\xi,\eta,\widehat{\tau}-\alpha)\, d\alpha\Big] G d\xi d\eta -$$
$$-\frac{a(t)}{\varphi(x,0)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} \sigma_3(\xi,0,\alpha)\sigma_2(\xi,\eta,\widehat{\tau}-\alpha)\, G d\alpha d\xi d\eta -$$
$$-\frac{a(t)}{\varphi(x,0)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_3(\xi,0,\widehat{\tau})\,\varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta -$$
$$-\frac{\lambda a(t)}{\Gamma(\alpha)\varphi(x,0)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}} (\widehat{\tau}-\beta)^{\alpha-1}\sigma_2(0,\eta,\beta)\, G d\beta d\xi d\eta -$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\ln a(\widehat{\tau}))'\int_0^{\widehat{\tau}} (\widehat{\tau}-\beta)^{\alpha-1}\sigma_1(0,\eta,\beta)\, d\beta G d\xi d\eta\Big) +$$
$$+\frac{1}{\varphi(x,0)}\int_0^{t}\Big((\ln a(t))'\,\chi(x,t-\tau) - \chi_t(x,t-\tau)\Big)\sigma_3(x,0,\tau)\, d\tau,$$

where $\theta^{-1}(\tau) = \widehat{\tau}$, $\sigma_{01}(x,y,t)$, $\sigma_{02}(x,y,t)$ and $\sigma_{03}(x,y,t)$ depends on the given functions, i.e.

$$\sigma_{01}(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta,$$
$$\sigma_{02}(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta +$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)} \frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\widehat{\tau})^{\alpha-1}\varphi_{\eta\eta}(0,\eta)\, G d\xi d\eta,$$
$$\sigma_{03}(x,y,t) = \frac{a(t)}{\varphi(x,0)}\Big(F_t(x,t) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(0)\,\Delta\varphi_{\eta\eta}(\xi,\eta)\, G d\xi d\eta -$$

$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^{\theta(t)}\frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\widehat{\tau})^{\alpha-1}\varphi_{\eta\eta}(0,\eta)\,Gd\xi d\eta\Bigg).$$

Introducing the notation $|\sigma|_T^{l,l/2}=\max\left(|\sigma_1|_{T_0}^{l.l/2},|\sigma_2|_{T_0}^{l.l/2},|\sigma_3|_{T_0}^{l.l/2}\right)$ in $H^{l,l/2}\left(R_T^2\right)$, we introduce the following condition

$$S(T)=|\sigma_1-\sigma_0|_T^{l,l/2}\leq|\sigma_0|_{T_0}^{l,l/2}, \tag{3.5}$$

where $\sigma_0=(\sigma_{01},\sigma_{02},\sigma_{03})$ and $|\sigma_0|_{T_0}^{l,l/2}=\max\left(|\sigma_{01}|_{T_0}^{l,l/2},|\sigma_{02}|_{T_0}^{l,l/2},|\sigma_{03}|_{T_0}^{l,l/2}\right)$. Thus, for any function σ from $S(T)$, $T<T_0$, when (3.5) is executed, the following inequality is true:

$$|\sigma_i|_T^{l,l/2}\leq 2\,|\sigma_0|_{T_0}^{l,l/2},\, i=1,2,3.$$

As is known, for $\varphi(x,y)\in H^{l+2}\left(R^2\right)$, the Cauchy problem for the classical heat conduction equation has a solution. In problem (1.1) - (1.3), taking into account the auxiliary problem, the initial conditions must satisfy $\varphi(x,y)\in H^{l+6}(\mathbb{R}^2)$, since the auxiliary problem involves fourth-order derivatives.

If the Cauchy condition $\varphi(x,y)\in H^{l+2}(\mathbb{R}^2)$ holds for the classical heat conduction equation, then the Cauchy problem for the classical heat diffusion equation has a solution [39]. Since we consider the class $\varphi(x,y)\in H^{l+6}(\mathbb{R}^2)$, the fourth-order derivatives are also involved in the corresponding auxiliary problem. Based on this, we introduce the following notation:

$$a_0:=\max_{t\in[0,T]}\left|(\ln a(t))'\right|,\varphi_1:=|\varphi|^{l+6},\chi_0:=|\chi|_T^{l+4,(l+4)/2}.$$

Contraction Mapping Principle. Any contraction mapping defined on a complete metric space has a unique fixed point; that is, the equation $x=Ax$ has a unique solution $x_0\in S$.

At the initial stage, using the estimates of thermal volume potentials [38][pp. 318-325], we can easily obtain the following inequalities:

$$|(L\sigma)_1-\sigma_{01}|_T^{l,l/2}=$$
$$=\left|\int_0^t\frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\sigma_3(\xi,0,\alpha)\sigma_1(\xi,\eta,\widehat{\tau}-\alpha)\,Gd\alpha d\xi d\eta\right|_T^{l,l/2}+$$
$$+\left|\frac{\lambda}{\Gamma(\alpha)}\int_0^t\frac{d\tau}{a(\widehat{\tau})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}(\widehat{\tau}-\beta)^{\alpha-1}\sigma_1(0,\eta,\beta)\,Gd\beta d\xi d\eta\right|_T^{l,l/2}\leq$$
$$\leq\bar{\beta}_0(T)\,|(\sigma_3(\xi,\eta,t_0)\,\sigma_1(\xi,\eta,z-t_0))|_T^{l,l/2}+\bar{\beta}_1(T)\,|(\sigma_1(0,\eta,t_0))|_T^{l,l/2}\leq$$
$$\leq 4\beta_0(T)\left(|\sigma_0|_{T_0}^{l,l/2}\right)^2+2\beta_1(T)\left(|\sigma_0|_{T_0}^{l,l/2}\right),$$
$$|(L\sigma)_2-\sigma_{02}|_T^{l,l/2}\leq$$
$$\leq 4\beta_1(T)(a_0+1)\left(|\sigma_0|_{T_0}^{l,l/2}\right)^2+2\beta_2(T)(2a_0+\varphi_1+1)\left(|\sigma_0|_{T_0}^{l,l/2}\right),$$
$$|(L\sigma)_3-\sigma_{03}|_T^{l,l/2}\leq$$
$$\leq 2\left(\beta_1(T)\,a_1\varphi_0^{-1}(2a_0+\varphi_1+1)+\chi_0\varphi_0^{-1}T_0(a_0+1)\right)|\sigma_0|_{T_0}^{l,l/2}+$$
$$+4\beta_2(T)\,a_1\varphi_0^{-1}(a_0+1)\left(|\sigma_0|_{T_0}^{l,l/2}\right)^2.$$

Therefore, if we choose T_0 so that the following inequalities should be satisfied:

$$\begin{gathered}4\beta_0(T_0)\left(|\sigma_0|_{T_0}^{l,l/2}\right)^2+2\beta_1(T_0)\left(|\sigma_0|_{T_0}^{l,l/2}\right)\leq 1,\\ 4\beta_1(T_0)(a_0+1)\left(|\sigma_0|_{T_0}^{l,l/2}\right)^2+2\beta_2(T_0)(2a_0+\varphi_1+1)\left(|\sigma_0|_{T_0}^{l,l/2}\right)\leq 1,\\ 2\left(\beta_1(T_0)\,a_1\varphi_0^{-1}(2a_0+\varphi_1+1)+\chi_0\varphi_0^{-1}T_0(a_0+1)\right)|\sigma_0|_{T_0}^{l,l/2}+\\ +4\beta_2(T_0)\,a_1\varphi_0^{-1}(a_0+1)\left(|\sigma_0|_{T_0}^{l,l/2}\right)^2\leq 1,\end{gathered} \tag{3.6}$$

then the operator L for $T<T_0$ has the first property of a contraction mapping operator, that is, $L\sigma\in S(T)$.

Now consider the second property of the contraction mapping for the operator L. Let $\sigma^{(1)}=\left(\sigma_1^{(1)},\sigma_2^{(1)},\sigma_3^{(1)}\right)\in S(T)$, $\sigma^{(2)}=\left(\sigma_1^{(2)},\sigma_2^{(2)},\sigma_3^{(2)}\right)\in S(T)$, then, following evaluation

$$\left|\sigma_2^{(1)}\sigma_1^{(1)}-\sigma_2^{(2)}\sigma_1^{(2)}\right|_T^{l,l/2}=\left|\left(\sigma_2^{(1)}-\sigma_2^{(2)}\right)\sigma_1^{(1)}+\sigma_2^{(2)}\left(\sigma_1^{(1)}-\sigma_1^{(2)}\right)\right|_T^{l,l/2}\leq$$

$$\leq 2\left|\sigma^{(1)}-\sigma^{(2)}\right|_T^{l,l/2}\max\left(\left|\sigma_1^{(1)}\right|_T^{l,l/2},\left|\sigma_2^{(2)}\right|_T^{l,l/2}\right)\leq 4\left|\sigma_0\right|_T^{l,l/2}\left|\sigma^{(1)}-\sigma^{(2)}\right|_T^{l,l/2},$$

we have

$$\left|\left((L\sigma)^{(1)}-(L\sigma)^{(2)}\right)_1\right|_T^{l,l/2}=\left|\int_0^{\theta(t)}\frac{d\tau}{a\left(\widehat{\tau}\right)}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\right.$$

$$\left.\int_0^{\widehat{\tau}}\left[\sigma_3^{(1)}(\xi,0,\alpha)\sigma_1^{(1)}\left(\xi,\eta,\widehat{\tau}-\alpha\right)-\sigma_3^{(2)}(\xi,0,\alpha)\sigma_1^{(2)}\left(\xi,\eta,\widehat{\tau}-\alpha\right)\right]Gd\alpha\right|_T^{l,l/2}+$$

$$+\left|\int_0^{\theta(t)}\frac{d\tau}{a\left(\theta^{-1}\left(\tau\right)\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\widehat{\tau}}\frac{\lambda\left(\widehat{\tau}-\beta\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[\sigma_1^{(1)}\left(0,\eta,\beta\right)-\sigma_1^{(2)}\left(0,\eta,\beta\right)\right]\times\right.$$

$$\left.\times\,Gd\beta d\xi d\eta\right|_T^{l,l/2}\leq\left[8\beta_0\left(T\right)\left|\sigma_0\right|_{T_0}^{l,l/2}+4\beta_1\left(T\right)\right]\left|\sigma^{(1)}-\sigma^{(2)}\right|_T^{l,l/2},$$

Similarly, estimating the second and third components of $L\sigma$ we have:

$$\left|\left((L\sigma)^{(1)}-(L\sigma)^{(2)}\right)_2\right|_T^{l,l/2}\leq$$

$$\leq\left[2\beta_1\left(T\right)\left(2a_0+\varphi_1+1\right)+8\beta_2\left(T\right)\left(a_0+1\right)\left|\sigma_0\right|_{T_0}^{l,l/2}\right]\left|\sigma^{(1)}-\sigma^{(2)}\right|_{T_0}^{l,l/2},$$

$$\left|\left((L\sigma)^{(1)}-(L\sigma)^{(2)}\right)_3\right|_T^{l,l/2}\leq$$

$$\leq\left[2\left(\beta_1\left(T\right)a_1\varphi_0^{-1}\left(2a_0+\varphi_1+1\right)+\chi_0\varphi_0^{-1}T_0\left(a_0+1\right)\right)\right]\left|\sigma^{(1)}-\sigma^{(2)}\right|_{T_0}^{l,l/2}+$$

$$+\left[8\beta_2\left(T\right)a_1\varphi_0^{-1}\left(a_0+1\right)\left(\left|\sigma_0\right|_{T_0}^{l,l/2}\right)\right]\left|\sigma^{(1)}-\sigma^{(2)}\right|_{T_0}^{l,l/2}.$$

Therefore $\left|\left(L\sigma^{(1)}-L\sigma^{(2)}\right)\right|_T^{l,l/2}<\rho\left|\sigma^{(1)}-\sigma^{(2)}\right|_T^{l,l/2}$, where $\rho\leq 1$, if satisfied

$$\begin{gathered}\left[8\beta_0\left(T\right)\left|\sigma_0\right|_{T_0}^{l,l/2}+2\beta_1\left(T\right)\right]\leq\rho<1,\\ \left[2\beta_1\left(T\right)\left(2a_0+\varphi_1+1\right)+8\beta_2\left(T\right)\left(a_0+1\right)\left|\sigma_0\right|_{T_0}^{l,l/2}\right]\leq\rho<1,\\ \left[2\left(\beta_1\left(T\right)a_1\varphi_0^{-1}\left(2a_0+\varphi_1+1\right)+f_0\varphi_0^{-1}T_0\left(a_0+1\right)\right)\right]+\\ +\left[8\beta_2\left(T\right)a_1\varphi_0^{-1}\left(a_0+1\right)\left(\left|\sigma_0\right|_{T_0}^{l,l/2}\right)\right]\leq\rho<1,\end{gathered}\tag{3.7}$$

then the operator L is also contraction on $S\left(T\right)$.

From the satisfaction of inequality (3.7), it directly follows that (3.6) also holds. Furthermore, since T_0 satisfies $T<T_0$ and condition (3.7), the properties of a contraction mapping operator are fully satisfied. Consequently, by the Banach fixed-point theorem, equation (3.4) has a unique solution. Using the method of successive approximations for the system of equations (3.1)–(3.3), obtain a unique solution within the function space $H^{l+2,(l+2)/2}\left(\bar{R}_T^2\right)$.

Thus, the existence and uniqueness of the solution to the system of integral equations (3.1)–(3.3) imply the existence and uniqueness of the solution to the equivalent problems (1.1)–(1.3).

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Information about the authors:

Umida Baltaeva – Department of Applied Mathematics and mathematical physics, Urgench State University, Urgench-220100, Uzbekistan; ORCID 0000-0001-9687-5220; umida_baltayeva@mail.ru

Praveen Agarwal – Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India; ORCID 0000-0001-7556-8942; goyal.praveen2011@gmail.com

Bobur Khasanov – Department of Exact sciences, Khorezm Mamun Academy, Khiva, Uzbekistan; ORCID 0009-0003-5079-7150; xasanovboburjon.1993@gmail.com

Hamrobek Hayitbayev – Department of Accounting and General Professional Sciences, Mamun university, Khiva, Uzbekistan; ORCID 0009-0002-0491-5104; hamrohayitboyev073@gmail.com

Florence Hubert – Aix-Marseille Universite, CNRS, I2M, Marseille, France; ORCID 0000-0002-7553-9698; florence.hubert@univ-amu.fr

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