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Original article

On the existence of the maximum number of isolated eigenvalues for a lattice Schrödinger operator

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ABSTRACT This paper presents a detailed spectral analysis of the discrete Schrödinger operator $H_{\gamma\lambda\mu}(K)$, which describes a system of two identical bosons on a two-dimensional lattice, $\mathbb{Z}^2$. The operator's family is parameterized by the quasi-momentum $K \in \mathbb{T}^2$ and real interaction strengths: γ for on-site, λ for nearest-neighbor, and μ for next-nearest-neighbor interactions. A key finding of our study is that, under specific conditions on the interaction parameters, the operator $H_{\gamma\lambda\mu}(K)$ consistently possesses a total of seven eigenvalues that lie either below the bottom or above the top of its essential spectrum, over all $K \in \mathbb{T}^2$.

KEYWORDS two-particle system, discrete Schrödinger operator, essential spectrum, bound states, Fredholm determinant

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1. Introduction

Lattice models constitute a fundamental framework within mathematical physics [1]. Among these, the lattice N-body Hamiltonian provides a simplified representation of the corresponding Bose- or Fermi-Hubbard models, specifically focusing on the dynamics of a limited number, N, of identical particles. These Hamiltonians remain an area of significant research interest, particularly for low particle counts where $1 \leq N \leq 3$, and the associated lattice N-particle problems have been subject to intense scrutiny over the past decades [2–9].

A compelling motivation for studying these lattice Hamiltonians is their intrinsic connection to continuous systems; they naturally serve as a discrete approximation to the continuous N-body Schrödinger operators [10]. Formulating the N-body problem on a lattice offers the distinct advantage of placing the analysis within the established theory of bounded operators. It should be noted that the one-particle ($N = 1$) problem on a 1D lattice is largely addressed by the general perturbation theory applicable to infinite Jacobi matrices (see, for instance, [11, 12]). The bound state energies of one- and two-particle systems, situated in two adjacent 3D layers linked by a window, were numerically reported in [13].

Lattice N-body Schrödinger operators are essential models for systems describing N particles traveling through periodic structures, exemplified by ultracold atoms injected into optical crystals [14, 15]. The study of ultracold few-atom systems in optical lattices has been particularly active in recent decades due to the experimental control over critical parameters, including temperature, particle masses, and interaction potentials (see, e.g., [15–19] and references therein).

It is well known that the celebrated Efimov effect [20] was initially attributed to three-particle systems in the three-dimensional continuous space $\mathbb{R}^3$. A rigorous mathematical confirmation of the Efimov effect was established in [21–24]. Subsequently, it has been demonstrated that the Efimov effect also occurs in three-particle systems defined on lattices [25,26]. Consequently, lattice three-body problems represent another significant domain for Efimov physics research [27].

Furthermore, lattice Hamiltonians find application in fusion physics. For example, [28] utilized a 1D lattice-based Hamiltonian to successfully illustrate that arranging molecules of a specific type into a lattice structure can substantially enhance their nuclear fusion probability.

In contrast to the continuous setting, the center-of-mass motion of an N-particle system ($N \geq 2$) on a lattice cannot be fully decoupled. However, the inherent lattice translational invariance of the Hamiltonian permits the use of the Floquet-Bloch decomposition. Specifically, for the (quasi)momentum-space representation of the N-particle lattice Hamiltonian

H, one can employ the following von Neumann direct integral decomposition (see, e.g., [2, Sec. 4]):

$$\mathrm{H} \simeq \int\limits_{K\in\mathbb{T}^d}^{\oplus} H(K)\, dK, \tag{1}$$

where $\mathbb{T}^d$ denotes the d-dimensional torus, K is the center-of-mass quasimomentum, and $H(K)$ is referred to as the fiber Hamiltonian. For each $K \in \mathbb{T}^d$, the entry $H(K)$ operates within the functional Hilbert space associated with $\mathbb{T}^{(N-1)d}$. The decomposition (1) effectively reduces the problem of studying the total Hamiltonian H to analyzing the simpler fiber operators $H(K)$. We observe that the dependence of $H(K)$ on the quasimomentum $K \in \mathbb{T}^d$, although non-trivial, is confined solely to the kinetic energy part and does not involve the (pairwise) inter-particle interaction terms (see, e.g., [2,29]).

In this paper, we focus on the fiber Hamiltonians $H_{\gamma\lambda\mu}(K)$ on a 2D lattice, acting in the Hilbert space $L^{2,\mathrm{e}}(\mathbb{T}^2)$. The Hamiltonian is defined as

$$H_{\gamma\lambda\mu}(K) := H_0(K) + V_{\gamma\lambda\mu},$$

where $H_0(K)$ is the kinetic-energy operator and $V_{\gamma\lambda\mu}$ represents the interaction potential. The real parameters γ, λ, and μ describe interactions between particles at the same site, nearest-neighbor sites, and next-nearest-neighbor sites, respectively.

The discrete eigenvalue problem for $H_{\gamma\lambda\mu}(K)$ is complex, but the operator has at most seven eigenvalues outside the essential spectrum, which is given by

$$\sigma_{\mathrm{ess}}(H_{\gamma\lambda\mu}(K)) = \left[2\sum_{i=1}^{2}\Big(1-\cos\tfrac{K_i}{2}\Big), 2\sum_{i=1}^{2}\Big(1+\cos\tfrac{K_i}{2}\Big)\right].$$

The space $L^{2,\mathrm{e}}(\mathbb{T}^2)$ can be decomposed into a direct orthogonal sum of invariant subspaces:

$$L^{2,\mathrm{e}}(\mathbb{T}^2) = L^{2,\mathrm{oos}}(\mathbb{T}^2) \oplus L^{2,\mathrm{ees}}(\mathbb{T}^2) \oplus L^{2,\mathrm{ea}}(\mathbb{T}^2).$$

This decomposition simplifies the spectral analysis of the full operator to studying its restrictions on these subspaces, as shown by the equality

$$\sigma(H_{\gamma\lambda\mu}(0)) = \sigma(H^{\mathrm{oos}}_{\mu}(0)) \cup \sigma(H^{\mathrm{ees}}_{\gamma\lambda\mu}(0)) \cup \sigma(H^{\mathrm{ea}}_{\lambda\mu}(0)). \tag{2}$$

Our primary objective is to find simple conditions on the parameters for which $H_{\gamma\lambda\mu}(0)$ possesses precisely seven isolated eigenvalues. We then apply this result to determine the exact count of discrete eigenvalues for $H_{\gamma\lambda\mu}(K)$ over all $K \in \mathbb{T}^2$. This work extends previous results on the ground state of $H_{\gamma\lambda\mu}(K)$ by providing a more comprehensive analysis of all eigenvalues.

In [30–35], similar spectral results were obtained for two-boson systems on $d = 1, 2$ lattices with on-site and nearest-neighbor interactions governed by real parameters γ and λ.

For a system of two identical bosons on a d-dimensional lattice $\mathbb{Z}^d$ ($d = 1, 2$) with on-site (γ), nearest-neighbor (λ), and next-nearest-neighbor (μ) interactions, the discrete spectrum of the associated two-particle Schrödinger operator $H_{\gamma\lambda\mu}(k)$, $k \in \mathbb{T}^d$ has been studied and determined the number and position of isolated eigenvalues for all values of the interaction parameters in [36–39].

The paper is structured as follows. In Section 2, we introduce the two-particle lattice Schrödinger operator. Section 3 presents our main results, and the proofs are provided in Section 4.

2. Discrete Schrödinger operators on lattices

2.1. Schrödinger operator for particle pairs with fixed quasimomentum and its essential spectrum

Let $\mathbb{T}^2$ be the 2D torus, and let $L^{2,e}(\mathbb{T}^2)$ denote the subspace of $L^2(\mathbb{T}^2)$ consisting of even functions.

For $\gamma, \lambda, \mu \in \mathbb{R}$ and $K \in \mathbb{T}^3$, the bounded and self-adjoint Schrödinger operator $H_{\gamma\lambda\mu}(K)$ describing interacting particle pairs ([2,35])is defined as:

$$H_{\gamma\lambda\mu}(K) := H_0(K) + V_{\gamma\lambda\mu}.$$

The unperturbed operator, $H_0(K)$, acts as

$$(H_0(K)f)(p) = \mathcal{E}_K(p)f(p),$$

where the dispersion function $\mathcal{E}_K(\cdot)$ is given by:

$$\mathcal{E}_K(p) = 2\sum_{i=1}^{2}\left(1-\cos\tfrac{K_i}{2}\,\cos p_i\right), \quad p = (p_1, p_2) \in \mathbb{T}^2.$$

The perturbation operator $V_{\gamma\lambda\mu}$ is given by

$$V_{\gamma\lambda\mu}f(p) = \frac{\gamma}{4\pi^2}\int_{\mathbb{T}^2} f(q)\,dq + \frac{\lambda}{4\pi^2}\sum_{i=1}^{2}\cos p_i\int_{\mathbb{T}^2}\cos q_i f(q)\,dq \tag{3}$$
$$+ \frac{\mu}{4\pi^2}\sum_{i=1}^{2}\cos 2p_i\int_{\mathbb{T}^2}\cos 2q_i f(q)\,dq$$
$$+ \frac{\mu}{2\pi^2}\cos p_1\cos p_2\int_{\mathbb{T}^2}\cos q_1\cos q_2 f(q)\,dq$$
$$+ \frac{\mu}{2\pi^2}\sin p_1\sin p_2\int_{\mathbb{T}^2}\sin q_1\sin q_2 f(q)\,dq.$$

Since the interaction potential $V_{\gamma\lambda\mu}$ has a rank of at most seven, it constitutes a compact perturbation to the kinetic-energy operator $H_0(K)$. According to Weyl's theorem, such a perturbation does not alter the essential spectrum of the operator. Consequently, the essential spectrum of the full operator $H_{\gamma\lambda\mu}(K)$ is identical to the spectrum of the unperturbed operator $H_0(K)$:

$$\sigma_{\text{ess}}(H_{\gamma\lambda\mu}(K)) = \sigma(H_0(K)).$$

This essential spectrum corresponds to the range of the kinetic energy function $\mathcal{E}_K(p)$ over the domain $p \in \mathbb{T}^2$, forming the interval $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$. The minimum and maximum energy values are given by:

$$\mathcal{E}_{\min}(K) = 2\sum_{i=1}^{2}\left(1-\cos\frac{K_i}{2}\right) \geq \mathcal{E}_{\min}(0) = 0,$$
$$\mathcal{E}_{\max}(K) = 2\sum_{i=1}^{2}\left(1+\cos\frac{K_i}{2}\right) \leq \mathcal{E}_{\max}(0) = 8.$$

3. Main results

The following results summarize and extend the findings presented in [2, Theorems 1 and 2] and [35, Theorem 3.1].

Theorem 1. *If, for some $\gamma, \lambda, \mu \in \mathbb{R}$, the operator $H_{\gamma\lambda\mu}(K)$ has at least n eigenvalues in $(-\infty, \mathcal{E}_{\min}(K))$ (or, respectively, $(\mathcal{E}_{\max}(K), +\infty)$), then for any quasi-momentum $K \in \mathbb{T}^2$, the operator $H_{\gamma\lambda\mu}(K)$ has at least n eigenvalues in $(-\infty, 0)$ (or, respectively, $(8, +\infty)$).*

This implies that the number of discrete eigenvalues observed at the zero quasi-momentum ($K = 0$) establishes the sharpest possible lower bound (across all $K \in \mathbb{T}^2$) for the total count of discrete eigenvalues of $H_{\gamma\lambda\mu}(K)$.

Our next findings detail the precise count of these discrete eigenvalues.

Theorem 2. *Let $K \in \mathbb{T}^2$ and $\gamma, \lambda, \mu \in \mathbb{R}$. The following assertions hold:*

(i) If the coupling constants are sufficiently negative ($\gamma < -12$, $\lambda < -12$, and $\mu < -12$), then $H_{\gamma\lambda\mu}(K)$ features exactly seven discrete eigenvalues positioned in $(-\infty, \mathcal{E}_{\min}(K))$.

(ii) If the coupling constants are sufficiently positive ($\gamma > 12$, $\lambda > 12$, and $\mu > 12$), then $H_{\gamma\lambda\mu}(K)$ features exactly seven discrete eigenvalues positioned in $(\mathcal{E}_{\max}(K), +\infty)$.

4. Proof of the main results

4.1. Invariant subspaces of the Schrödinger operators $H_{\gamma\lambda\mu}(0)$

Lemma 1. *The Hilbert space $L^{2,\text{e}}(\mathbb{T}^2)$ admits the orthogonal decomposition*

$$L^{2,e}(\mathbb{T}^2) = L^{2,\text{ees}}(\mathbb{T}^2) \oplus L^{2,\text{oos}}(\mathbb{T}^2) \oplus L^{2,ea}(\mathbb{T}^2), \tag{4}$$

where

$$L^{2,\text{ees}}(\mathbb{T}^2) = \{\phi \in L^{2,\text{e}}(\mathbb{T}^2) : \phi(t_1,t_2) = \phi(t_2,t_1) = \phi(-t_1,t_2),\ \forall t_1,t_2 \in \mathbb{T}\}$$
$$L^{2,\text{oos}}(\mathbb{T}^2) = \{\phi \in L^{2,\text{e}}(\mathbb{T}^2) : \phi(t_1,t_2) = \phi(t_2,t_1) = -\phi(-t_1,t_2),\ \forall t_1,t_2 \in \mathbb{T}\} \tag{5}$$
$$L^{2,\text{ea}}(\mathbb{T}^2) = \{\phi \in L^{2,\text{e}}(\mathbb{T}^2) : \phi(t_1,t_2) = -\phi(t_2,t_1),\ \forall t_1,t_2 \in \mathbb{T}\}.$$

Moreover, the action of the operator $H_{\gamma\lambda\mu}(0)$ preserves the invariance of every subspace defined in (4) .

Proof. The definition (5) correspond to the standard decomposition of $L^{2,\text{e}}(\mathbb{T}^2)$ into irreducible subspaces under the action of the permutation group $\mathfrak{S}_2$. Orthogonality of these subspaces follows from symmetry considerations, and their direct sum exhausts $L^{2,\text{e}}(\mathbb{T}^2)$. Since $H_{\gamma\lambda\mu}(0)$ commutes with permutations of variables, each subspace is invariant under its action. □

From Lemma 1 (ii) it immediately follows Eq. (2). Therefore, it suffices to independently analyze the eigenvalue spectra of the restrictions of $H_{\gamma\lambda\mu}(0)$ to the reducing subspaces $L^{2,\mathrm{ees}}(\mathbb{T}^2)$, $L^{2,\mathrm{ea}}(\mathbb{T}^2)$, and $L^{2,\mathrm{oos}}(\mathbb{T}^2)$ to obtain the complete discrete spectrum of the total operator $H_{\gamma\lambda\mu}(0)$ on $L^{2,\mathrm{e}}(\mathbb{T}^2)$.

4.2. The Lippmann–Schwinger operator

Let $\{\alpha_1^{ees},\alpha_2^{ees},\alpha_3^{ees},\alpha_4^{ees}\}\subset L^{2,ees}(\mathbb{T}^2)$, resp. $\{\alpha_1^{ea},\alpha_2^{ea}\}\subset L^{2,ea}(\mathbb{T}^2)$ and $\{\alpha_1^{oos}\}\subset L^{2,oos}(\mathbb{T}^2)$ be a orthonormal system of vectors, with

$$\begin{aligned}&\alpha_1^{ees}(p)=\tfrac{1}{2\pi},\ \alpha_2^{ees}(p)=\tfrac{\cos p_1+\cos p_2}{2\pi},\ \alpha_3^{ees}(p)=\tfrac{\cos 2p_1+\cos 2p_2}{2\pi},\ \alpha_4^{ees}(p)=\tfrac{\cos p_1\cos p_2}{\pi},\\&\alpha_1^{oos}(p)=\tfrac{\sin p_1\sin p_2}{\pi},\ \alpha_1^{ea}(p)=\tfrac{\cos p_1-\cos p_2}{2\pi},\ \alpha_2^{ea}(p)=\tfrac{\cos 2p_1-\cos 2p_2}{2\pi}.\end{aligned}\tag{6}$$

We note that, the perturbation operator $V_{\gamma\lambda\mu}$ can be expressed in terms of the orthonormal systems (6):

$$\begin{aligned}(V_{\gamma\lambda\mu}f)(p)=\gamma(f,\alpha_1^{ees})\alpha_1^{ees}+\frac{\lambda}{2}(f,\alpha_2^{ees})\alpha_2^{ees}+\frac{\mu}{2}(f,\alpha_3^{ees})\alpha_3^{ees}+\frac{\mu}{2}(f,\alpha_4^{ees})\alpha_4^{ees}\\+\frac{\lambda}{2}(f,\alpha_1^{ea})\alpha_1^{ea}+\frac{\mu}{2}(f,\alpha_2^{ea})\alpha_2^{ea}+\frac{\mu}{2}(f,\alpha_1^{oos})\alpha_1^{oos}.\end{aligned}\tag{7}$$

By applying the representation (7) of $V_{\gamma\lambda\mu}$ one concludes that

$$\begin{aligned}H^{\mathrm{ees}}_{\gamma\lambda\mu}(0)&:=H_{\gamma\lambda\mu}(0)\big|_{L^{2,\mathrm{ees}}(\mathbb{T}^2)}=H_0(0)+V^{\mathrm{ees}}_{\gamma\lambda\mu},\\H^{\mathrm{oos}}_{\mu}(0)&:=H_{\gamma\lambda\mu}(0)\big|_{L^{2,\mathrm{oos}}(\mathbb{T}^2)}=H_0(0)+V^{\mathrm{oos}}_{\mu},\\H^{\mathrm{ea}}_{\lambda\mu}(0)&:=H_{\gamma\lambda\mu}(0)\big|_{L^{2,\mathrm{ea}}(\mathbb{T}^2)}=H_0(0)+V^{\mathrm{ea}}_{\lambda\mu},\end{aligned}$$

where

$$\begin{aligned}V^{ees}_{\gamma\lambda\mu}f:=V_{\gamma\lambda\mu}(0)\big|_{L^{2,\mathrm{ees}}(\mathbb{T}^2)}f=&\ \gamma(f,\alpha_1^{ees})\alpha_1^{ees}+\frac{\lambda}{2}(f,\alpha_2^{ees})\alpha_2^{ees}\\&+\frac{\mu}{2}(f,\alpha_3^{ees})\alpha_3^{ees}+\frac{\mu}{2}(f,\alpha_4^{ees})\alpha_4^{ees},\\V^{oos}_{\mu}f:=H_{\gamma\lambda\mu}(0)\big|_{L^{2,\mathrm{oos}}(\mathbb{T}^2)}f=&\ \frac{\mu}{2}(f,\alpha_1^{oos})\alpha_1^{oos},\\V^{ea}_{\lambda\mu}f:=V_{\gamma\lambda\mu}(0)\big|_{L^{2,\mathrm{ea}}(\mathbb{T}^2)}f=&\ \frac{\lambda}{2}(f,\alpha_1^{ea})\alpha_1^{ea}+\frac{\mu}{2}(f,\alpha_2^{ea})\alpha_2^{ea},\end{aligned}\tag{8}$$

where $(\cdot,\cdot)$ is the inner product in $L^{2,e}(\mathbb{T}^2)$.

The Lippmann-Schwinger operators corresponding to $H^{\mathrm{ees}}_{\gamma\lambda\mu}$, H^{oos}_{μ}, and $H^{\mathrm{ea}}_{\lambda\mu}$ are defined for any $z\in\mathbb{C}\setminus[0,\,8]$ (and shown here in their transpose form, following, e.g., [41]) as:

$$\begin{aligned}B^{\mathrm{ees}}_{\gamma\lambda\mu}(0,z)&=-V^{\mathrm{ees}}_{\gamma\lambda\mu}R_0(0,z),\\B^{\mathrm{oos}}_{\mu}(0,z)&=-V^{\mathrm{oos}}_{\mu}R_0(0,z),\\B^{\mathrm{ea}}_{\lambda\mu}(0,z)&=-V^{\mathrm{ea}}_{\lambda\mu}R_0(0,z),\end{aligned}$$

Here, $R_0(0,z):=[H_0(0)-zI]^{-1}$ represents the resolvent of the free operator $H_0(0)$, defined for $z\in\mathbb{C}\setminus[0,\,8]$.

Lemma 2. *Let $\gamma,\lambda,\mu\in\mathbb{R}$. The number $z\in\mathbb{C}\setminus[0,\,8]$ is an eigenvalue of the operator $H^{ees}_{\gamma\lambda\mu}(0)$ (resp. $H^{oos}_{\mu}(0)$ and $H^{ea}_{\lambda\mu}(0)$), if and only if the number 1 is an eigenvalue for $B^{ees}_{\gamma\lambda\mu}(0,z)$ (resp. $B^{oos}_{\mu}(0,z)$ and $B^{ea}_{\lambda\mu}(0,z)$).*

The lemma's proof is standard, following well-known techniques (e.g., [40]), and is therefore omitted.

Due to the representation provided in (8), the eigenvalue equation

$$B^{\mathrm{ees}}_{\gamma\lambda\mu}(0,z)\varphi=\varphi,\quad \varphi\in L^{2,\mathrm{ees}}(\mathbb{T}^2)$$

can be transformed into the following algebraic linear system involving the component coefficients $x_i:=(\varphi,\alpha_i^{ees}),\quad i=1,2,3,4:$

$$\begin{cases}[1+2\gamma a_{11}(z)]x_1+\lambda a_{12}(z)x_2+\mu a_{13}(z)x_3+\mu a_{14}(z)x_4=0,\\2\gamma a_{12}(z)x_1+[1+\lambda a_{22}(z)]x_2+\mu a_{23}(z)x_3+\mu a_{24}(z)x_4=0,\\2\gamma a_{13}(z)x_1+\lambda a_{23}(z)x_2+[1+\mu a_{33}(z)]x_3+\mu a_{34}(z)x_4=0,\\2\gamma a_{14}(z)x_1+\lambda a_{24}(z)x_2+\mu a_{34}(z)x_3+[1+\mu a_{44}(z)]x_4=0,\end{cases}$$

Analogously, the Lippmann-Schwinger equation $B^{ea}_{\lambda\mu}(0,z)\varphi=\varphi,\ \varphi\in L^{2,ea}(\mathbb{T}^2)$ respectively $B^{oos}_{\mu}(0,z)\varphi=\varphi,\ \varphi\in L^{2,oos}(\mathbb{T}^2)$ is equivalent to

$$\begin{cases}[1+\lambda b_{11}(z)]y_1+\mu b_{12}(z)y_2=0,\\\lambda b_{12}(z)y_1+[1+\mu b_{22}(z)]y_2=0\end{cases},\quad y_i:=(\varphi,\alpha_i^{ea}),\ i=1,2$$

respectively

$$(1+\mu c(z))(\varphi, \alpha^{oos}) = 0,$$

where

$$\begin{aligned} a_{ij}(z) :=& \frac{1}{2}\int_{\mathbb{T}^2} \frac{\alpha_i^{\mathrm{ees}}(p)\alpha_j^{\mathrm{ees}}(p)\underset{\cdot}{\mathrm{p}}}{\mathcal{E}_0(p)-z},\ i,j=1,2,3,4,\\ b_{ij}(z) :=& \frac{1}{2}\int_{\mathbb{T}^2} \frac{\alpha_i^{\mathrm{ea}}(p)\alpha_j^{\mathrm{ea}}(p)\underset{\cdot}{\mathrm{p}}}{\mathcal{E}_0(p)-z},\ i,j=1,2,\\ c(z) :=& \frac{1}{2}\int_{\mathbb{T}^2} \frac{\left(\alpha^{\mathrm{oos}}(p)\right)^2 \underset{\cdot}{\mathrm{p}}}{\mathcal{E}_0(p)-z}. \end{aligned} \tag{9}$$

Let us introduce the determinant functions $\Delta^{\mathrm{ees}}_{\gamma\lambda\mu}(z)$, $\Delta^{\mathrm{ea}}_{\lambda\mu}(z)$, $\Delta^{\mathrm{oos}}_{\mu}(z)$ for $z \in \mathbb{R}\setminus[0,\,8]$:

$$\begin{aligned} \Delta^{\mathrm{ees}}_{\gamma\lambda\mu}(z) &:= \det[I - B^{\mathrm{ees}}_{\gamma\lambda\mu}(0,z)] = \begin{vmatrix} 1+2\gamma a_{11}(z) & \lambda a_{12}(z) & \mu a_{13}(z) & \mu a_{14}(z)\\ 2\gamma a_{12}(z) & 1+\lambda a_{22}(z) & \mu a_{23}(z) & \mu a_{24}(z)\\ 2\gamma a_{13}(z) & \lambda a_{23}(z) & 1+\mu a_{33}(z) & \mu a_{34}(z)\\ 2\gamma a_{14}(z) & \lambda a_{24}(z) & \mu a_{34}(z) & 1+\mu a_{44}(z) \end{vmatrix},\\ \Delta^{\mathrm{ea}}_{\lambda\mu}(z) &:= \det[I - B^{\mathrm{ea}}_{\lambda\mu}(0,z)] = \begin{vmatrix} 1+\lambda b_{11}(z) & \mu b_{12}(z)\\ \lambda b_{12}(z) & 1+\mu b_{22}(z) \end{vmatrix},\\ \Delta^{\mathrm{oos}}_{\mu}(z) &:= \det[I - B^{\mathrm{oos}}_{\mu}(0,z)] = 1+\mu c(z). \end{aligned} \tag{10}$$

We state the well-known lemma connecting the eigenvalues of the restricted operators $H^{\mathrm{ees}}_{\gamma\lambda\mu}(0)$, $H^{\mathrm{oos}}_{\mu}(0)$, and $H^{\mathrm{ea}}_{\lambda\mu}(0)$ to the zeros of the corresponding determinants.

Lemma 3. *A number $z\in\mathbb{R}\setminus[0,8]$ is an eigenvalue of $H^{\mathrm{ees}}_{\gamma\lambda\mu}(0)$ (resp. $H^{\mathrm{oos}}_{\mu}(0)$, $H^{\mathrm{ea}}_{\lambda\mu}(0)$) with multiplicity $m\ge 1$ if and only if z is a zero of $\Delta^{\mathrm{ees}}_{\gamma\lambda\mu}(z)$ (resp. $\Delta^{\mathrm{oos}}_{\mu}(z)$, $\Delta^{\mathrm{ea}}_{\lambda\mu}(z)$) with multiplicity m. Moreover, the maximum number of zeros in $\mathbb{R}\setminus[0,8]$ for $\Delta^{\mathrm{ees}}_{\gamma\lambda\mu}(z)$, $\Delta^{\mathrm{oos}}_{\mu}(z)$, and $\Delta^{\mathrm{ea}}_{\lambda\mu}(z)$ is four, one, and two, respectively.*

The proof for this lemma follows from routine methods (see [35]), so we proceed without including it.

Theorem 3. *For $z\in\mathbb{R}\setminus[0,8]$, the functions $a_{ij}(z)$ ($i,j=1,2,3,4$) are real-valued. They are strictly increasing on both $(-\infty,0)$ (where they are positive) and $(8,+\infty)$ (where they are negative). Additionally, they satisfy the following asymptotic relations:*

$$\begin{aligned} a_{ij}(z) &= \begin{cases} -a^{(0)}_{ij}\dfrac{\ln(\frac{-z}{32})}{8\pi} + a^{(1)}_{ij} + o(1), & \text{as} \quad z \nearrow 0,\\ a^{(0)}_{ij}\dfrac{\ln(\frac{z-8}{32})}{8\pi} - a^{(1)}_{ij} + o(1), & \text{as} \quad z \searrow 8, \end{cases}\\ b_{ij}(z) &= \begin{cases} -b^{(1)}_{ij} + o(1), & \text{as} \quad z \nearrow 0,\\ b^{(1)}_{ij} + o(1), & \text{as} \quad z \searrow 8, \end{cases}\\ c(z) &= \begin{cases} \dfrac{3\pi-8}{3\pi} + o(1), & \text{as} \quad z \nearrow 0,\\ -\dfrac{3\pi-8}{3\pi} + o(1), & \text{as} \quad z \searrow 8. \end{cases} \end{aligned}$$

The functions $\ln(-z)$ and $\ln(z-8)$ are understood to be the specific branches chosen to be real when $z<0$ and $z>8$, respectively. The coefficient matrices $a^{(0)}=(a^{(0)}_{ij})$, $a^{(1)}=(a^{(1)}_{ij})$ ($i,j=1,2,3,4,$) and $b^{(1)}=(b^{(1)}_{ij})$ ($i,j=1,2$), are given by

$$a^{(0)} = \begin{pmatrix} 1 & 2 & 2 & 2\\ 2 & 4 & 4 & 4\\ 2 & 4 & 4 & 4\\ 2 & 4 & 4 & 4 \end{pmatrix}, \quad a^{(1)} = \begin{pmatrix} 0 & \dfrac{-1}{4} & \dfrac{-\pi+2}{\pi} & -\dfrac{1}{\pi}\\ \dfrac{-1}{\pi} & -\dfrac{1}{2} & \dfrac{-2\pi+4}{\pi} & -\dfrac{2}{\pi}\\ \dfrac{-\pi+2}{\pi} & \dfrac{-\pi+2}{\pi} & \dfrac{-10\pi+\frac{88}{3}}{\pi} & \dfrac{2\pi-\frac{26}{3}}{\pi}\\ -\dfrac{1}{\pi} & -\dfrac{2}{\pi} & \dfrac{2\pi-\frac{26}{3}}{\pi} & \dfrac{-\pi+\frac{4}{3}}{2\pi} \end{pmatrix},$$

$$b^{(1)} = \begin{pmatrix} \dfrac{4-\pi}{\pi} & \dfrac{32-9\pi}{\pi}\\ \dfrac{32-9\pi}{\pi} & \dfrac{2(32-9\pi)}{\pi} \end{pmatrix}.$$

Proof. Theorem 3 can be proven by adapting the proof of Proposition 4.4 in [35]. □

Lemma 4. *The asymptotic behavior of the real-valued functions $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$, $\Delta^{\rm ea}_{\lambda\mu}(z)$ and $\Delta^{\rm oos}_{\mu}(z)$ is given by:*

(i) $\lim\limits_{z\to\pm\infty} \Delta^{\rm ees}_{\gamma\lambda\mu}(z) = \lim\limits_{z\to\pm\infty} \Delta^{\rm ea}_{\lambda\mu}(z) = \lim\limits_{z\to\pm\infty} \Delta^{\rm oos}_{\mu}(z) = 1;$

(ii)

$$\Delta^{\rm ees}_{\gamma\lambda\mu}(z) = \begin{cases} -\frac{1}{4\pi}Q^-(\gamma,\lambda,\mu)ln(-z) + D^-_{\gamma\lambda\mu} + o(1), & as \quad z \nearrow 0, \\ -\frac{1}{4\pi}Q^+(\gamma,\lambda,\mu)ln(-z) + D^+_{\gamma\lambda\mu} + o(1), & as \quad z \searrow 8, \end{cases}$$

$$\Delta^{\rm ea}_{\lambda\mu}(z) = \begin{cases} 1 + \frac{(4-\pi)}{\pi}\lambda + \frac{2(32-9\pi)}{\pi}\mu + \frac{(32-9\pi)}{4\pi}\lambda\mu + o(1), & as \quad z \nearrow 0, \\ 1 - \frac{(4-\pi)}{\pi}\lambda - \frac{2(32-9\pi)}{\pi}\mu + \frac{(32-9\pi)}{4\pi}\lambda\mu + o(1), & as \quad z \searrow 8, \end{cases}$$

$$\Delta^{\rm oos}_{\mu}(z) = \begin{cases} 1 + \dfrac{3\pi-8}{3\pi}\mu + o(1), & as \quad z \nearrow 0, \\ 1 - \dfrac{3\pi-8}{3\pi}\mu + o(1), & as \quad z \searrow 8, \end{cases}$$

where

$$Q^{\pm}(\gamma,\lambda,\mu) = (\gamma \mp 4)\Big(Q^{\pm}_0(\mu)\lambda \mp Q^{\pm}_1(\mu)\Big) - 8Q^{\pm}_0(\mu)$$

and

$$Q^{\pm}_0(\mu) := \tfrac{16-5\pi}{4\pi}\mu^2 \mp \tfrac{4(10-3\pi)}{3\pi}\mu + \tfrac{1}{2},$$
$$Q^{\pm}_1(\mu) := \tfrac{2(16-5\pi)}{\pi}\mu^2 \mp \tfrac{80-21\pi}{3\pi}\mu + 1.$$

Proof of Lemma 4. The proof is facilitated by the Lebesgue dominated convergence theorem (for the first part) and Proposition 3 (for the final part). □

The following lemmas provide the exact count of the zeros for the determinant functions $\Delta^{\rm oos}_{\mu}(z)$, $\Delta^{\rm ea}_{\lambda\mu}(z)$, and $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$ outside the essential spectrum $[0,8]$.

Lemma 5. *The following assertions hold for the determinant $\Delta^{\rm oos}_{\mu}(z)$:*

(i) If $\mu < -12$, the function $\Delta^{\rm oos}_{\mu}(z)$ has precisely one zero in $(-\infty,0)$.
(ii) If $\mu > 12$, the function $\Delta^{\rm oos}_{\mu}(z)$ has precisely one zero in $(8,+\infty)$.

Proof. The result follows directly from [35, Theorem 4.5]. □

Lemma 6. *The following assertions hold for the determinant $\Delta^{\rm ea}_{\lambda\mu}(z)$:*

(i) If $\lambda < -12$ and $\mu < -12$, the function $\Delta^{\rm ea}_{\lambda\mu}(z)$ has precisely two zeros in $(-\infty,0)$.
(ii) If $\lambda > 12$ and $\mu > 12$, the function $\Delta^{\rm ea}_{\lambda\mu}(z)$ has precisely two zeros in $(8,+\infty)$.

Proof. The proof for Lemma 6 follows established techniques, such as those demonstrated in [37, Theorem 1] and [39, Theorem 2]. □

Lemma 7. *The following assertions hold for the determinant $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$:*

(i) If $\gamma < -12$, $\lambda < -12$, and $\mu < -12$, the determinant function $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$ has exactly four zeros in $(-\infty,0)$.
(ii) If $\gamma > 12$, $\lambda > 12$, and $\mu > 12$, the determinant function $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$ has exactly four zeros in $(8,+\infty)$.

Proof of Lemma 7. i) Let $\gamma < -12, \lambda < -12$ and $\mu < -12$. Assuming μ is negative ($\mu < 0$), the function

$$\delta(z) := 1 + \mu a_{44}(z)$$

—where $a_{44}(z)$ is defined in (9)—is continuous and strictly decreasing for $z \in (-\infty,0)$. From the explicit definition of a_{44} in (9) it follows that

$$\lim_{z\to-\infty} \delta(z) = 1.$$

At the same time, the asymptotic expression for $a_{44}(z)$ in Proposition 3 implies that

$$\lim_{z\nearrow 0} \delta(z) = -\infty.$$

Therefore the function $\delta(z) = 1 + \mu a_{44}(z)$ has exactly one zero z_{11} within the half-axis $(-\infty,0)$ and, thus,

$$\begin{aligned} 1 + \mu a_{44}(z) > 0 \quad &\text{if} \quad z < z_{11}, \\ 1 + \mu a_{44}(z) < 0 \quad &\text{if} \quad z_{11} < z < 0. \end{aligned} \tag{11}$$

Notice that the equality $1+\mu a_{44}(z_{11})=0$ implies that

$$\begin{aligned}\Delta^{\rm ees}_{00\mu}(z_{11}) &= (1+\mu a_{33}(z_{11}))\,(1+\mu a_{44}(z_{11})) - \mu^2(a_{34}(z_{11}))^2\\ &= -\mu^2(a_{34}(z_{11}))^2<0.\end{aligned} \tag{12}$$

The inequality $\mu<-12$ implies that

$$Q^-(0,0,\mu)=\tfrac{6(16-5\pi)}{\pi}\mu\Big[\mu+\tfrac{2\pi}{3(16-5\pi)}\Big]>0.$$

The inequality $Q^-(0,0,\mu)>0$ and Lemma 4 yield that

$$\lim_{z\to-\infty}\Delta^{\rm ees}_{00\mu}(z)=1,\quad \lim_{z\nearrow0}\Delta^{\rm ees}_{00\mu}(z)=+\infty. \tag{13}$$

The relations (12) and (13) imply that

$$\lim_{z\to-\infty}\Delta^{\rm ees}_{00\mu}(z)=1,\quad \Delta^{\rm ees}_{00\mu}(z_{11})<0\quad\text{and}\quad \lim_{z\nearrow0}\Delta^{\rm ees}_{00\mu}(z)=+\infty.$$

This means that there exist real numbers z_{21} and z_{22} such that

$$z_{21}<z_{11}<z_{22}<0 \tag{14}$$

and

$$\Delta^{\rm ees}_{00\mu}(z_{21})=\Delta^{\rm ees}_{00\mu}(z_{22})=0. \tag{15}$$

The equality (8) implies that the operator $V^{\rm ees}_{00\mu}$ has rank at most two. Therefore, by the minimax principle, the operator $H^{\rm ees}_{00\mu}$ has at most two eigenvalues below zero. By the first statement in Lemma 3, the function $\Delta^{\rm ees}_{00\mu}(z)$ has at most two zeros in $\mathbb{R}\setminus[0,8]$. Hence, and by (15) the function $\Delta^{\rm ees}_{00\mu}(z)$ has precisely two zeros (z_{21} and z_{22}), lying in $(-\infty,0)$. Therefore

$$\begin{aligned}&\Delta^{\rm ees}_{00\mu}(z)>0\quad\text{if}\quad z<z_{21};\\ &\Delta^{\rm ees}_{00\mu}(z)<0\quad\text{if}\quad z_{21}<z<z_{22};\\ &\Delta^{\rm ees}_{00\mu}(z)>0\quad\text{if}\quad z_{22}<z<0.\end{aligned} \tag{16}$$

The equalities $\Delta^{\rm ees}_{00\mu}(z_{21})=\Delta^{\rm ees}_{00\mu}(z_{22})=0$ yield the following relations

$$\begin{aligned}(1+\mu a_{33}(z_{21}))(1+\mu a_{44}(z_{21})) &= \mu^2(a_{34}(z_{21}))^2>0,\\ (1+\mu a_{33}(z_{22}))(1+\mu a_{44}(z_{22})) &= \mu^2(a_{34}(z_{22}))^2>0.\end{aligned} \tag{17}$$

Hence $1+\mu a_{33}(z_{21})$ and $1+\mu a_{44}(z_{21})$ (resp. $1+\mu a_{33}(z_{22})$ and $1+\mu a_{44}(z_{22})$) have the same signs. Combining this with (11) and (14) yields

$$\begin{aligned}&1+\mu a_{33}(z_{21})>0\quad\text{and}\quad 1+\mu a_{44}(z_{21})>0;\\ &1+\mu a_{33}(z_{22})<0\quad\text{and}\quad 1+\mu a_{44}(z_{22})<0.\end{aligned} \tag{18}$$

For the roots z_{21} and z_{22} of

$$\Delta^{\rm ees}_{00\mu}(z)=(1+\mu a_{33}(z))\,(1+\mu a_{44}(z))-\mu^2a_{34}^2(z)=0$$

we then have

$$\sqrt{1+\mu a_{33}(z_{21})}\sqrt{1+\mu a_{44}(z_{21})}=-\mu a_{34}(z_{21}) \tag{19}$$

and

$$\sqrt{-[1+\mu a_{33}(z_{22})]}\sqrt{-[1+\mu a_{44}(z_{22})]}=-\mu a_{34}(z_{22}). \tag{20}$$

Using the explicit representation (10) for $\Delta^{\rm ees}_{0\lambda\mu}(z)$ and (19) one arrives with the following equality:

$$\Delta^{\rm ees}_{0\lambda\mu}(z_{21})=-\lambda\mu\big[\sqrt{1+\mu a_{44}(z_{21})}a_{23}(z_{21})+\sqrt{1+\mu a_{33}(z_{21})}a_{24}(z_{21})\big]^2. \tag{21}$$

Clearly, (21) implies that

$$\Delta^{\rm ees}_{0\lambda\mu}(z_{21})<0. \tag{22}$$

Analogously, the identity (20) gives that

$$\Delta^{\rm ees}_{0\lambda\mu}(z_{22})=\lambda\mu\big[\sqrt{-[1+\mu a_{44}(z_{22})]}a_{23}(z_{22})+\sqrt{-[1+2\mu a_{33}(z_{22})]}a_{24}(z_{22})\big]^2>0. \tag{23}$$

Meanwhile, for $\lambda<-12$ va $\mu<-12$ we have

$$Q_0^-(\mu)=\tfrac{16-5\pi}{4\pi}\mu^2+\tfrac{4(10-3\pi)}{3\pi}\mu+\tfrac12>0$$

and

$$4\Big(Q_1^-(\mu)-12Q_0^-(\mu)\Big)=\tfrac{(16-5\pi)}{\pi}\mu^2+\tfrac{400-123\pi}{3\pi}\mu+5>0.$$

The above inequalities obeys that

$$Q^-(0,\lambda,\mu) = 4Q_0^-(\mu)(\lambda+10) - 4\Big(Q_1^-(\mu) - 12Q_0^-(\mu)\Big) < 0.$$

Lemma 4 and inequality $Q^-(0,\lambda,\mu) < 0$ give

$$\lim_{z\to-\infty} \Delta_{0\lambda\mu}^{\mathrm{ees}}(z) = 1 \quad \text{and} \quad \lim_{z\nearrow 0} \Delta_{0\lambda\mu}^{\mathrm{ees}}(z) = -\infty. \tag{24}$$

Taking into account (22), (23) and (24) this implies there existence of real numbers z_{31}, z_{32} and z_{33} such that

$$z_{31} < z_{21} < z_{32} < z_{22} < z_{33} < 0 \tag{25}$$

and

$$\Delta_{0\lambda\mu}^{\mathrm{ees}}(z_{31}) = \Delta_{0\lambda\mu}^{\mathrm{ees}}(z_{32}) = \Delta_{0\lambda\mu}^{\mathrm{ees}}(z_{33}) = 0. \tag{26}$$

The equality (8) implies that the operator $V_{0\lambda\mu}^{\mathrm{ees}}$ has rank at most three. Therefore, again by the minimax principle [42, Theorem XIII.1], the operator $H_{0\lambda\mu}^{\mathrm{ees}}$ has at most three discrete eigenvalues. Then, Lemma 3 guarantees that the function $\Delta_{0\lambda\mu}^{\mathrm{ees}}(z)$ has at most three zeros in $\mathbb{R}\setminus[0,8]$. Given these bounds and established results, the function $\Delta_{0\lambda\mu}^{\mathrm{ees}}(z)$ is found to have precisely three zeros (z_{31}, z_{32}, and z_{33}), all of which lie in the interval $(-\infty,0)$.

Let

$$\begin{aligned} A_{11}(z) :=& \ \det\begin{pmatrix} 1+\mu a_{33}(z) & \mu a_{34}(z) \\ \mu a_{34}(z) & 1+\mu a_{44}(z)\end{pmatrix}, \\ A_{22}(z) :=& \ \det\begin{pmatrix} 1+\lambda a_{22}(z) & \mu a_{24}(z) \\ \lambda a_{24}(z) & 1+\mu a_{44}(z)\end{pmatrix}, \\ A_{33}(z) :=& \ \det\begin{pmatrix} 1+\lambda a_{22}(z) & \mu a_{23}(z) \\ \lambda a_{23}(z) & 1+\mu a_{33}(z)\end{pmatrix}. \end{aligned} \tag{27}$$

and

$$\begin{aligned} A_{12}(z) &:= \det\begin{pmatrix} \lambda a_{23}(z) & \mu a_{34}(z) \\ \lambda a_{24}(z) & 1+\mu a_{44}(z)\end{pmatrix}, & A_{21}(z) &:= \det\begin{pmatrix} \mu a_{23}(z) & \mu a_{24}(z) \\ \mu a_{34}(z) & 1+\mu a_{44}(z)\end{pmatrix}, \\ A_{13}(z) &:= \det\begin{pmatrix} \lambda a_{23}(z) & 1+\mu a_{33}(z) \\ \lambda a_{24}(z) & \mu a_{34}(z)\end{pmatrix}, & A_{31}(z) &:= \det\begin{pmatrix} \mu a_{23}(z) & \mu a_{24}(z) \\ 1+\mu a_{33}(z) & \mu c_{34}(z)\end{pmatrix}, \\ A_{23}(z) &:= \det\begin{pmatrix} 1+\lambda a_{22}(z) & \mu a_{23}(z) \\ \lambda a_{24}(z) & \mu a_{34}(z)\end{pmatrix}, & A_{32}(z) &:= \det\begin{pmatrix} 1+\lambda a_{22}(z) & \mu a_{24}(z) \\ \lambda a_{23}(z) & \mu a_{34}(z)\end{pmatrix}. \end{aligned} \tag{28}$$

The definition (28) implies that

$$\mu A_{12}(z) = \lambda A_{21}(z), \quad \mu A_{13}(z) = \lambda A_{31}(z), \quad A_{23}(z) = A_{32}(z) \tag{29}$$

From the definitions (27) and (28) one derives that

$$\begin{aligned} A_{11}(z)A_{33}(z) - A_{13}(z)A_{31}(z) &= \Delta_{0\lambda\mu}^{\mathrm{ees}}(z)\cdot[1+\mu a_{33}(z)], \\ A_{11}(z)A_{22}(z) - A_{12}(z)A_{21}(z) &= \Delta_{0\lambda\mu}^{\mathrm{ees}}(z)\cdot[1+\mu a_{44}(z)], \\ A_{22}(z)A_{33}(z) - A_{23}(z)A_{32}(z) &= \Delta_{0\lambda\mu}^{\mathrm{ees}}(z)\cdot[1+\lambda a_{22}(z)] \end{aligned} \tag{30}$$

and

$$A_{11}(z)A_{22}(z)A_{33}(z) - A_{12}(z)A_{23}(z)A_{31}(z) = \Delta_{0\lambda\mu}^{\mathrm{ees}}(z)R_{\lambda\mu}(z), \tag{31}$$

where

$$R_{\lambda\mu}(z) = [1+\lambda a_{22}(z)]\cdot[1+\mu a_{33}(z)]\cdot[1+\mu a_{44}(z)] - \lambda\mu^2 a_{13}(z)a_{14}(z)a_{24}(z).$$

Then the equality $\Delta_{0\lambda\mu}^{\mathrm{ees}}(z_{31}) = 0$ and identity (30) resp. (31) imply that

$$\begin{aligned} A_{11}(z_{31})A_{33}(z_{31}) &= A_{13}(z_{31})A_{31}(z_{31}), \\ A_{11}(z_{31})A_{22}(z_{31}) &= A_{12}(z_{31})A_{21}(z_{31}), \\ A_{22}(z_{31})A_{33}(z_{31}) &= A_{23}(z_{31})A_{32}(z_{31}) \end{aligned} \tag{32}$$

resp.

$$A_{11}(z)A_{22}(z)A_{33}(z) = A_{12}(z)A_{23}(z)A_{31}(z). \tag{33}$$

It is worth noting that the functions $1+\mu a_{33}(z)$ and $1+\mu a_{44}(z)$ both exhibit strict decrease on the interval $(-\infty, 0)$. Then the relations $z_{31} < z_{21}$ and (18) yield that

$$1+\mu a_{33}(z_{31}) > 1+\mu a_{33}(z_{21}) > 0 \quad \text{and} \quad 1+\mu a_{44}(z_{31}) > 1+\mu a_{44}(z_{21}) > 0. \tag{34}$$

The inequality (34), the negativity of λ, μ and positivity of the functions a_{23}, a_{24}, a_{34} (See Proposition 3) yield that

$$\begin{aligned} A_{12}(z_{31}) &= \lambda a_{23}(z_{31})\big[1+\mu a_{44}(z_{31})\big] - \lambda\mu a_{24}(z_{31})a_{34}(z_{31}) < 0, \\ A_{13}(z_{31}) &= \lambda\mu a_{23}(z_{31})a_{34}(z_{31}) - \lambda a_{24}(z_{31})\big[1+\mu a_{33}(z_{31})\big] > 0. \end{aligned} \tag{35}$$

The relations (29) and (32) yield that

$$\begin{aligned} A_{11}(z_{31})A_{33}(z_{31}) &= A_{13}(z_{31})A_{31}(z_{31}) = \frac{\lambda A_{31}^2(z_{31})}{\mu} > 0, \\ A_{11}(z_{31})A_{22}(z_{31}) &= A_{12}(z_{31})A_{21}(z_{31}) = \frac{\mu A_{12}^2(z_{31})}{\lambda} > 0. \end{aligned} \tag{36}$$

The inequalities (36) give that the numbers $A_{11}(z_{31}),\ A_{22}(z_{31}),\ A_{33}(z_{31})$ has the same signs. The relations (16) and (25) yield that

$$A_{11}(z_{31}) = \Delta_{00\mu}^{\mathrm{ees}}(z_{31}) > 0, \ \text{therefore} \quad A_{22}(z_{31}) > 0, \quad A_{33}(z_{31}) > 0. \tag{37}$$

Then the relations (33), (35) and (37) obeys that

$$A_{23}(z_{31}) < 0.$$

The equality (29) and inequalities $A_{23}(z_{31}) < 0$, (35) imply that

$$\begin{aligned} &A_{12}(z_{31}) < 0, \quad A_{21}(z_{31}) < 0, \\ &A_{23}(z_{31}) < 0, \quad A_{32}(z_{31}) < 0, \\ &A_{31}(z_{31}) > 0, \quad A_{13}(z_{31}) > 0. \end{aligned} \tag{38}$$

The equalities (29) and (30) give that

$$\begin{aligned} A_{12}^2(z_{31}) &= \tfrac{\lambda A_{11}(z_{31})A_{22}(z_{31})}{\mu}, & A_{21}^2(z_{31}) &= \tfrac{\mu A_{11}(z_{31})A_{22}(z_{31})}{\lambda} \\ A_{23}^2(z_{31}) &= A_{22}(z_{31})A_{33}(z_{31}), & A_{32}^2(z_{31}) &= A_{22}(z_{31})A_{33}(z_{31}) \\ A_{31}^2(z_{31}) &= \tfrac{\mu A_{11}(z_{31})A_{33}(z_{31})}{\lambda}, & A_{13}^2(z_{31}) &= \tfrac{\lambda A_{11}(z_{31})A_{33}(z_{31})}{\mu}. \end{aligned} \tag{39}$$

Taking into account the signs of the numbers $A_{ij}(z_{31})$ in (38) and using the equality (39) we arrive that

$$\begin{aligned} A_{12}(z_{31}) &= -\sqrt{\tfrac{\lambda A_{11}(z_{31})A_{22}(z_{31})}{\mu}}, & A_{21}(z_{31}) &= -\sqrt{\tfrac{\mu A_{11}(z_{31})A_{22}(z_{31})}{\lambda}}, \\ A_{23}(z_{31}) &= -\sqrt{A_{22}(z_{31})A_{33}(z_{31})}, & A_{32}(z_{31}) &= -\sqrt{A_{22}(z_{31})A_{33}(z_{31})}, \\ A_{31}(z_{31}) &= \sqrt{\tfrac{\mu A_{11}(z_{31})A_{33}(z_{31})}{\lambda}}, & A_{13}(z_{31}) &= \sqrt{\tfrac{\lambda A_{11}(z_{31})A_{33}(z_{31})}{\mu}}. \end{aligned} \tag{40}$$

We can represent the determinant $\Delta_{\gamma\lambda\mu}^{\mathrm{ees}}(z)$ in (10) as follow:

$$\begin{aligned} \Delta_{\gamma\lambda\mu}^{\mathrm{ees}}(z) =& [1+2\gamma a_{11}(z)]\Delta_{0\lambda\mu}^{\mathrm{ees}}(z) - 2\gamma\lambda a_{12}^2(z)A_{11}(z) + 2\gamma\mu a_{12}(z)a_{13}(z)A_{12}(z) - 2\gamma\mu a_{12}(z)a_{14}(z)A_{13}(z) \\ &+ 2\gamma\lambda a_{13}(z)a_{12}(z)A_{21}(z) - 2\gamma\mu a_{13}^2(z)A_{22}(z) + 2\gamma\mu a_{13}(z)a_{14}(z)A_{23}(z) \\ &- 2\gamma\lambda a_{14}(z)a_{12}(z)A_{31}(z) + 2\gamma\mu a_{14}(z)a_{13}(z)A_{32}(z) - 2\gamma\mu a_{14}^2(z)A_{33}(z). \end{aligned}$$

Using the above representation of the determinant $\Delta_{\gamma\lambda\mu}^{\mathrm{ees}}(z)$, equality $\Delta_{0\lambda\mu}^{\mathrm{ees}}(z_{31}) = 0$ and (40) we find that

$$\begin{aligned} \Delta_{\gamma\lambda\mu}^{\mathrm{ees}}(z_{31}) =& -2\gamma\lambda a_{12}^2(z_{31})A_{11}(z_{31}) - 2\gamma\mu a_{13}^2(z_{31})A_{22}(z_{31}) - 2\gamma\mu a_{14}^2(z_{31})A_{33}(z_{31}) \\ &- 4\gamma\sqrt{\lambda\mu A_{11}(z_{31})A_{33}(z_{31})}\,a_{12}(z_{31})a_{14}(z_{31}) \\ &- 4\gamma\mu a_{13}(z_{31})a_{14}(z_{31})\sqrt{A_{22}(z_{31})A_{33}(z_{31})} \\ &- 4\gamma\sqrt{\lambda\mu A_{11}(z_{31})A_{22}(z_{31})}\,a_{12}(z_{31})a_{13}(z_{31}) \\ =& -2\Big(a_{12}(z_{31})\sqrt{\gamma\lambda A_{11}(z_{31})} + a_{13}(z_{31})\sqrt{\gamma\mu A_{22}(z_{31})} + a_{14}(z_{31})\sqrt{\gamma\mu A_{33}(z_{31})}\Big)^2. \end{aligned} \tag{41}$$

The equality (41) and positivity of the function a_{12}, a_{13}, a_{14} gives that

$$\Delta_{\gamma\lambda\mu}^{\mathrm{ees}}(z_{31}) < 0. \tag{42}$$

Combining the relation (16) with inequalities $z_{21} < z_{32} < z_{22}$ and $z_{22} < z_{33}$ we arrive that

$$A_{11}(z_{32}) < 0, \quad A_{22}(z_{32}) < 0, \quad A_{33}(z_{32}) < 0$$

and

$$A_{11}(z_{33}) > 0, \quad A_{22}(z_{33}) > 0, \quad A_{33}(z_{33}) > 0.$$

respectively.

So, in a similar way we show that

$$\Delta^{\rm ees}_{\gamma\lambda\mu}(z_{32}) = 2\Big(a_{12}(z_{32})\sqrt{-\gamma\lambda A_{11}(z_{32})} + a_{13}(z_{32})\sqrt{-\gamma\mu A_{22}(z_{32})} + a_{14}(z_{32})\sqrt{-\gamma\mu A_{33}(z_{32})}\Big)^2 > 0$$

and

$$\Delta^{\rm ees}_{\gamma\lambda\mu}(z_{33}) = -2\Big(a_{12}(z_{33})\sqrt{\gamma\lambda A_{11}(z_{33})} + a_{13}(z_{33})\sqrt{\gamma\mu A_{22}(z_{33})} + a_{14}(z_{33})\sqrt{\gamma\mu A_{33}(z_{33})}\Big)^2 < 0.$$

The assertions $\gamma < -12, \lambda < -12, \mu < -12$ yield that

$$Q^-(\gamma,\lambda,\mu) = (\gamma+4)\Big(Q_0^-(\mu)\lambda + Q_1^-(\mu)\Big) - 8Q_0^-(\mu) = (\gamma+12)\Big(Q_0^-(\mu)\lambda + Q_1^-(\mu)\Big) - 8Q_0^-(\mu)(\lambda+12) + 8\Big(11Q_0^-(\mu) - Q_1^-(\mu)\Big) > 0.$$

The relation $Q^-(\gamma,\lambda,\mu) > 0$ and Lemma 4 obeys that

$$\lim_{z\to-\infty} \Delta^{\rm ees}_{\gamma\lambda\mu}(z) = 1 \quad \text{and} \quad \lim_{z\nearrow 0} \Delta^{\rm ees}_{\gamma\lambda\mu}(z) = +\infty.$$

Therefore

$$\lim_{z\to-\infty} \Delta^{\rm ees}_{\gamma\lambda\mu}(z) = 1, \quad \Delta^{\rm ees}_{\gamma\lambda\mu}(z_{31}) < 0, \quad \Delta^{\rm ees}_{\gamma\lambda\mu}(z_{32}) > 0,$$
$$\Delta^{\rm ees}_{\gamma\lambda\mu}(z_{33}) < 0, \quad \lim_{z\nearrow 0} \Delta^{\rm ees}_{\gamma\lambda\mu}(z) = +\infty.$$

The above relations yield there existence of four zeros $z_{41}, z_{42}, z_{43}, z_{44}$ of function $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$, satisfying the following inequalities

$$z_{41} < z_{31} < z_{42} < z_{32} < z_{43} < z_{33} < z_{44} < 0. \tag{43}$$

The proof for item (ii) follows an analogous procedure. □

Proof of Theorem 1. The result is proven analogously to Theorem 3.1 in [35]. □

Proof of Theorem 2. (i) Assume that $\gamma, \lambda, \mu < -12$.

We first determine the number of bound states for $K = 0$. Combining the results of Lemmas 5, 6, and 7 provides the number of negative zeros for the corresponding determinants:

- $\Delta^{\rm oos}_{\mu}(z)$ has exactly one zero (Lemma 5).
- $\Delta^{\rm ea}_{\lambda\mu}(z)$ has exactly two zeros (Lemma 6).
- $\Delta^{\rm ees}_{\gamma\lambda\mu}(z)$ has exactly four zeros (Lemma 7).

The decomposition (2) and Lemma 3 confirm that the total number of bound states for $H_{\gamma\lambda\mu}(0)$ with negative energy is $1 + 2 + 4 = 7$.

Next, Theorem 1 ensures that for any K, the operator $H_{\gamma\lambda\mu}(K)$ possesses at least seven eigenvalues in $(-\infty, 0)$. Since the rank of the perturbation operator $V_{\gamma\lambda\mu}(K)$ is at most seven, the min-max principle (see [42], page 85) dictates that $H_{\gamma\lambda\mu}(K)$ has at most seven isolated eigenvalues. Therefore, $H_{\gamma\lambda\mu}(K)$ must have *precisely seven* bound states in $(-\infty, 0)$.

(ii) Suppose that $\gamma, \lambda, \mu > 12$. The proof for Item (ii) is entirely analogous, relying on the corresponding assertions for the zeros of the determinants lying in $(8, +\infty)$. □

5. Conclusion

In conclusion, this article provides a comprehensive spectral analysis of the two-boson discrete Schrödinger operator $H_{\gamma\lambda\mu}(K)$ with short-range interactions, including on-site (γ), nearest-neighbor (λ), and next-nearest-neighbor (μ) couplings. The central outcome is the demonstration that the operator's discrete spectrum exhibits a remarkable structural stability under strong coupling conditions. Specifically, we have established sufficient conditions on the interaction parameters such that the operator possesses a total of seven bound states (eigenvalues), located either below $(-\infty, 0)$ or above $(8, +\infty)$ the essential spectrum, irrespective of the quasi-momentum $K \in \mathbb{T}^2$.

Our results significantly advance the understanding of spectral properties in discrete few-body systems, particularly concerning the influence of extended interaction ranges. Previous studies focusing on two-boson systems on a $2D$ lattice with interactions limited to on-site and nearest-neighbor sites [35] showed a maximum of only three eigenvalues. Furthermore, the recent work in [39], which considered a more general interaction profile similar to ours, provided only sufficient conditions for the existence of at least two eigenvalues. In stark contrast, our work explicitly demonstrates that the full inclusion of next-nearest-neighbor coupling (μ) is responsible for stabilizing and increasing the maximum possible number of bound states to seven, providing easily verifiable criteria for this maximum count. This spectral richness highlights the critical role of the interaction range in enhancing localization phenomena.

Despite these advancements, several challenges remain. While we established sufficient and easily verifiable conditions for the existence of seven eigenvalues, we did not provide a definitive count or position of the discrete spectrum for all possible values of the interaction parameters. A complete mapping of the (γ, λ, μ)-space into regions corresponding to exactly $n \in \{0, 1, \dots, 7\}$ eigenvalues is a complex, unsolved problem that requires further computational and analytical methods.

Another significant challenge lies in extending this analysis to the three-boson lattice system, where the increase in the number of degrees of freedom and the complexity of the fiber Hamiltonian make the spectral analysis significantly harder. Future research will focus on employing advanced numerical techniques to fully map the spectral regions and exploring potential applications of these multi-bound states in quantum information processing.

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