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Original article

Phase transition and thermodynamic properties of the Hard-Core-Potts model

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ABSTRACT We investigate translation-invariant Gibbs measures of the Hard-Core–Potts (HC–Potts) model on the Cayley tree. The model combines Potts-type ferromagnetic interactions with a hard-core exclusion rule, leading to a nontrivial interplay between magnetic ordering and occupancy constraints. For the hinge-type fertile graph, we analyze the cases $k = 2$ and $k = 3$, and determine explicit critical values of θ that mark the transition from uniqueness to multiplicity of Gibbs measures. The model exhibits up to five translation-invariant phases depending on the interaction strength. Thermodynamic quantities such as magnetization and quadrupolar moment are computed, revealing ordered phases at low temperatures and a paramagnetic phase at high temperatures.

KEYWORDS Gibbs distribution, Potts model, Hard-Core model, phase transition, Cayley tree

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1. Introduction

Recent progress in nanoscience and nanotechnology has increased interest in studying phase transitions and ordering phenomena in finite systems, where thermodynamic behavior differs from bulk materials. The theory of Gibbs measures provides a rigorous framework for describing ordered and disordered phases and their coexistence in lattice models. Studying Gibbs measures on Cayley trees is therefore crucial for understanding phase transition mechanisms and the stability of nanoscale systems (see, e.g., [1–3]).

Exactly solvable models play a vital role in statistical mechanics, particularly in elucidating the nature of phase transitions and collective phenomena (see, e.g., [4–10]). Among these, the Potts model serves as one of the fundamental frameworks for describing magnetic ordering, while the Hard-Core (HC) model captures systems with exclusion constraints, such as lattice gases and independent sets on graphs.

The q-state Potts model is a generalization of the Ising model and has been widely used to investigate the long-term behavior of many complex systems. It enables the study of how the internal components of a system interact based on their intrinsic properties [9, 11]. In contrast to the Ising model, the Potts model allows each spin to take any of $q > 2$ possible values. A four-component version of this model was first studied by Ashkin and Teller around 1943 [12]. The model was later named after Renfrey B. Potts, who developed its theoretical foundations in his 1952 doctoral thesis [13], following a suggestion by his advisor, C. Domb. A similar problem was independently considered by Kihara et al. two years later [14].

The Potts model on the Cayley tree (also known as the Bethe lattice) was first studied by N. Ganikhodjaev [15, 16]. Since then, many works have investigated limiting Gibbs measures of finite-state Potts models on Cayley trees (see [17–21]).

In contrast, the HC model imposes strict exclusion constraints on spin values and has found applications in combinatorics, statistical mechanics, and queueing theory. HC models are particularly relevant for studying random independent sets on graphs [22, 23] and gas molecules on lattices [4]. In [24], A. Mazel and Yu. Suhov introduced the HC model on the d-dimensional lattice $\mathbb{Z}^d$. Many studies have since analyzed the limiting Gibbs measures of finite-state HC models (see [8, 25–29]).

The combination of these two models, known as the Hard-Core-Potts (HC-Potts) model, provides a unified framework for describing both magnetic and structural ordering. This hybrid model enables the investigation of the interplay between spin alignment and particle exclusion, an important aspect of complex systems exhibiting competing interactions.

Hybrid lattice models combining different interaction types have attracted considerable attention in recent years. One important example is the Potts-SOS model, where each spin interacts through both Potts-type ferromagnetic coupling and a solid-on-solid (SOS) height constraint (see, e.g., [30]). A related extension is the mixed-type Potts-SOS model, which incorporates different coupling mechanisms at different layers or spin components, leading to complex dynamical behavior and multiple competing phases (see [31–34]).

The present work differs fundamentally from these SOS-based hybrid systems. Unlike the height-dependent SOS constraint, the HC-Potts model couples classical Potts ferromagnetism with a hard-core exclusion rule that restricts admissible configurations through local occupancy constraints. As a result, while Potts-SOS models describe a competition between magnetic order and surface fluctuations, the HC-Potts model captures the interplay between magnetic alignment and particle exclusion. This leads to a qualitatively different structure of admissible states and produces distinct critical phenomena, particularly in the presence of fertile graphs such as the hinge-type graph considered here.

Cayley trees provide a convenient mathematical framework for analyzing such models, since their recursive structure allows exact characterization of Gibbs measures and critical phenomena. In this work, we focus on translation-invariant Gibbs measures for the HC-Potts model defined on Cayley trees of order $k \geq 2$.

2. Definitions and known facts

The Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite tree, i.e., a cycle-free graph such that from each vertex of which issues exactly $k+1$ edges. We denote by V the set of the vertices of tree and by L the set of edges. The distance on the Cayley tree, denoted by $d(x, y)$, is defined as the number of nearest neighbor pairs of the shortest path between the vertices x and y (where a path is a collection of nearest neighbor pairs, two consecutive pairs sharing at least a given vertex)

For a fixed $x^0 \in V$, called the root, let

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m$$

be respectively the ball and the sphere of radius n with center at x^0. For $x \in W_n$ let

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \ldots, k\},$$

be the set of direct successors of x. Note that in $\Im^k$ any vertex $x \neq x^0$ has k direct successors, and root x^0 has $k+1$ direct successors.

We consider Hard-Core-Potts (HC-Potts) model on a Cayley tree, where the spin takes values in the set $\Phi := \{0, 1, 2, \ldots, q\}$, and is assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; the set of all configurations is Φ^V.

To describe the hard-core interactions between neighboring spins, let $G = (\Phi, K)$ be a graph with vertex set Φ, and edge set K. A configuration σ is called *G-admissible* on a Cayley tree if $\{\sigma(x), \sigma(y)\} \in K$ is an edge of G for any pair of nearest neighbors $\langle x, y\rangle \in L$. The set of all G-admissible configurations is denoted by Ω^G. For a subset $A \subset V$, σ_A denotes the restriction of σ to A, and Ω^G_A denotes the set of all G-admissible configurations on A. We also introduce a matrix of activities λ on the edges of G,

$$\lambda \colon \{i, j\} \in K \to \lambda_{i,j} \in \mathbb{R}_+,$$

where $\lambda_{i,j}$ is called the activity of the pair $\{i, j\}$.

For each pair $\langle x, y\rangle \in L$, the spins interact through two mechanisms:

- **Potts ferromagnetic interaction.** The term

$$-J\, \delta_{\sigma(x),\sigma(y)}$$

  contributes energy J if the spins coincide, and 0 otherwise. For $J > 0$, this term favors alignment.
- **Hard-Core (activity) interaction.** Each pair $i, j \in \Phi$ has a nonnegative activity weight $\lambda_{i,j}$.
  - If $\lambda_{i,j} = 0$, the pair $\{i, j\}$ is forbidden.
  - If $\lambda_{i,j} > 0$, the pair is allowed and contributes a term $-\frac{1}{\beta} \log \lambda_{i,j}$ to the energy.

Combining both interactions, the Hamiltonian of the HC-Potts model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x),\sigma(y)} - \frac{1}{\beta} \sum_{\langle x,y\rangle \in L} \log \lambda_{\sigma(x),\sigma(y)}. \tag{1}$$

The first term promotes magnetic ordering; the second encodes exclusion constraints and activity effects. Special cases include:

(i) Pure Potts model: $\lambda_{i,j} = 1$, $\qquad (ii)$ Pure Hard-Core model: $J = 0$.

The finite-volume Gibbs distribution on V_n defined as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right\}, \tag{2}$$

where $\beta = 1/T, T > 0$ is temperature, Z_n^{-1} is the normalizing factor and $\{h_x = (h_{0,x}, h_{1,x}, \ldots, h_{q,x}) \in R^{q+1}, x \in V\}$ is a collection of vectors and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle\in L_n} \delta_{\sigma(x)\sigma(y)} - \frac{1}{\beta} \sum_{\langle x,y\rangle\in L_n} \log \lambda_{\sigma(x),\sigma(y)}.$$

We say that the probability distributions (2) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n\in\Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}). \tag{3}$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. In this case, there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$

$$\mu(\{\sigma_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such a measure is called a splitting Gibbs measure corresponding to the Hamiltonian (1) and vector-valued function $h_x, x \in V$.

The following statement specifies the conditions on h_x that guarantee the compatibility of the distributions.

Theorem 1. Probability distributions $\mu_n(\sigma_n)$, $n \in \mathbb{N}$ in (2) are compatible if for any $x \in V$ the following equation holds:

$$h_x = \sum_{y\in S(x)} F(h_y, \theta), \tag{4}$$

where $F : h = (h_1, \ldots, h_q) \in R^q \to F(h, \theta) = (F_1, \ldots, F_q) \in R^q$ is defined as

$$F_i = \ln\left(\frac{\lambda_{i,i}(\theta-1)e^{h_i} + \sum_{j=1}^q \lambda_{i,j}e^{h_j} + \lambda_{i0}}{\lambda_{0,0}\theta + \sum_{j=1}^q \lambda_{0,j}e^{h_j}}\right),$$

and $\theta = \exp(J\beta)$, $S(x)$ is the set of direct successors of x.

Proof. **Necessity.** Suppose that (3) holds, we want to prove (4). Substituting (2) into (3), obtain that for any configurations σ_{n-1}: $x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in \{0, 1, \ldots, q\}$:

$$\frac{Z_{n-1}}{Z_n} \sum_{\omega_n\in\Omega_{W_n}} \exp\left(\sum_{x\in W_{n-1}} \sum_{y\in S(x)} (J\beta\delta_{\sigma_{n-1}(x)\omega_n(y)} + \log \lambda_{\sigma_{n-1}(x),\omega_n(y)} + h_{\omega_n(y),y})\right) =$$

$$= \exp\left(\sum_{x\in W_{n-1}} h_{\sigma_{n-1}(x),x}\right), \tag{5}$$

where ω_n: $x \in W_n \mapsto \omega_n(x)$.

From (5) we get:

$$\frac{Z_{n-1}}{Z_n} \sum_{\omega_n\in\Omega_{W_n}} \prod_{x\in W_{n-1}} \prod_{y\in S(x)} \lambda_{\sigma_{n-1}(x),\omega_n(y)} \exp\left(J\beta\delta_{\sigma_{n-1}(x)\omega_n(y)} + h_{\omega_n(y),y}\right) =$$

$$= \prod_{x\in W_{n-1}} \exp\left(h_{\sigma_{n-1}(x),x}\right).$$

Fix $x \in W_{n-1}$ and rewrite the last equality for $\sigma_{n-1}(x) = i$, $i = 1, \ldots, q$ and $\sigma_{n-1}(x) = 0$, then dividing each of them to the last one we get

$$\prod_{y\in S(x)} \frac{\sum_{u=0}^q \lambda_{i,u} \exp(J\beta\delta_{iu} + h_{u,y})}{\sum_{u=0}^q \lambda_{0,u} \exp(J\beta\delta_{0u} + h_{u,y})} = \exp\left(h_{i,x} - h_{0,x}\right),\ i = 1, \ldots, q-1. \tag{6}$$

Now we change the variables as follows

$$h_{i,x} - h_{0,x} \to h_{i,x},\ \ i = 2, \ldots, q-1.$$

Then (6) implies (4).

Sufficiency. Suppose that (4) holds. It is equivalent to the representations

$$\prod_{y\in S(x)} \sum_{u=0}^q \lambda_{i,u} \exp(J\beta\delta_{iu} + h_{u,y}) = a(x) \exp\left(h_{i,x}\right),\ \ i = 1, \ldots, q. \tag{7}$$

for some function $a(x) > 0, x \in V$. We have

$$\text{LHS of (3)} = \frac{1}{Z_n}\exp(-\beta H(\sigma_{n-1}))\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{u=1}^{q}\lambda_{\sigma_{n-1}(x),u}\exp(J\beta\delta_{\sigma_{n-1}(x)u}+h_{u,y}). \tag{8}$$

Substituting (7) into (8) and denoting $A_n(x) = \prod_{x\in W_{n-1}} a(x)$, we get

$$\text{RHS of (3)} = \frac{A_{n-1}}{Z_n}\exp(-\beta H(\sigma_{n-1}))\prod_{x\in W_{n-1}}\exp(h_{\sigma_{n-1}(x),x}). \tag{9}$$

Since $\mu^{(n)}$, $n \geq 1$ is a probability, we should have

$$\sum_{\sigma_{n-1}\in\Omega_{V_{n-1}}}\sum_{\omega_n\in\Omega_{W_n}}\mu^{(n)}(\sigma_{n-1},\omega_n) = 1.$$

Hence from (9), we get $Z_{n-1}A_{n-1} = Z_n$, and (3) holds. □

Hence, Theorem 1 follows: for any collection $h = h_x,\ x \in V$ satisfying the functional equation above there exists a unique splitting Gibbs measure μ, and conversely every splitting Gibbs measure corresponds to such a solution.

3. Translation-invariant Gibbs measures

In this section, we consider Gibbs measures which are translation-invariant, i.e., we assume $h_x = h = (h_1,\dots,h_q) \in R^q$ for all $x \in V$. Then from equation (4) we get $h = kF(h,\theta)$, i.e.,

$$h_i = k\ln\left(\frac{\lambda_{i,i}(\theta-1)e^{h_i}+\sum_{j=1}^{q}\lambda_{i,j}e^{h_j}+\lambda_{i,0}}{\lambda_{0,0}\theta+\sum_{j=1}^{q}\lambda_{0,j}e^{h_j}}\right),\quad i = 1,\dots,q. \tag{10}$$

Denoting $z_i = \exp(h_i), i = 1,\dots,q-1$, we get from (10)

$$z_i = \left(\frac{\lambda_{i,i}(\theta-1)z_i+\sum_{j=1}^{q}\lambda_{i,j}z_j+\lambda_{i,0}}{\lambda_{0,0}\theta+\sum_{j=1}^{q}\lambda_{0,j}z_j}\right)^k,\quad i = 1,\dots,q. \tag{11}$$

Remark 1. Note that
- If in the (11) we set $\lambda = 1$, then we obtain the q-state Potts model;
- If $\theta = 1$, the Hard-Core model is obtained.

It is important to note that if the system (11) admits more than one positive solution, then each solution corresponds to a distinct translation-invariant limiting Gibbs measure (TILGM). We say that a phase transition occurs for the model (1), if the system (11) has more than one positive solution.

Definition 1. The graph is called fertile if there is an activity set λ such that the corresponding Hamiltonian has at least two TILGMs (see [25]).

Let $q = 2$. Here, we consider four types of fertile graphs [25] with three vertices $0, 1, 2$ (the set of values of $\sigma(x)$), which have the forms

$$\begin{aligned}
&wand: && \{0,0\},\{0,1\},\{1,2\},\{2,2\};\\
&hinge: && \{0,0\},\{0,1\},\{1,1\},\{1,2\},\{2,2\};\\
&wrench: && \{0,0\},\{0,1\},\{1,1\},\{1,2\};\\
&pipe: && \{0,0\},\{0,1\},\{1,2\}.
\end{aligned}$$

In this paper we consider hinge-type graphs. More precisely, we consider the activity $\lambda = (\lambda_{i,j})_{\{i,j\}\in K}$ defined as (see Fig. 1)

$$\lambda_{i,j} = \begin{cases} 1, & \text{if}\quad i = j \quad\text{or}\quad |\, i-j \,| = 1;\\ 0, & \text{otherwise.}\end{cases}$$

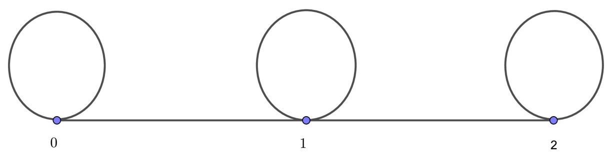

FIG. 1. The hinge-type graph with three vertices.

3.1. The case “Hinge“

In the following we restrict our attention to the case where $q = 2$. In this case, denoting $x = \sqrt[k]{z_1}$ and $y = \sqrt[k]{z_2}$, from (11), we get

$$x = \frac{\theta x^k + y^k + 1}{\theta + x^k} \qquad \text{and} \qquad y = \frac{\theta y^k + x^k}{\theta + x^k}. \tag{12}$$

In particular, considering only the second equation of this system, we find the solutions $y = 1$ and

$$x^k = \theta \left(y^{k-1} + y^{k-2} + \cdots + y\right). \tag{13}$$

We begin by analyzing the case $y = 1$.

3.2. Case $y = 1$

In this case, from the first equation in (12), we get that

$$\theta x^{k+1} - \theta x^k + \theta x - 2 = 0 \tag{14}$$

and hence, the following statement follows.

Lemma 1. For all $k \geq 2$, there exist a unique positive root for (14).

Proof. To do this, it is sufficient to show $f'(x) > 0$ for any $x > 0$. Indeed,

$$f'(x) = (k+1)x^k - kx^{k-1} + 1.$$

It is easy to see that, if $x \geq 1$ then $f'(x) > 0$.

Let $0 < x < 1$. Then

$$f'(t) = t^k - kt + k + 1,$$

where $t = 1/x$ $(t > 1)$. Due to Bernoulli's inequality we get

$$t^k + k + 1 = (1 + t - 1)^k + k + 1 \geq 1 + k(t-1) + k + 1 = 2 + kt.$$

From here

$$f'(t) = t^k - kt + k + 1 > 0.$$

□

3.3. Case $y \neq 1$

Let us consider the second situation as presented in (12).

3.3.1. Case $k = 2$. In this case, using (13) and the first equation in (12), we get

$$y^4 - (\theta^3 - 2\theta^2)y^3 + (\theta^4 - 2\theta^3 + 2)y^2 - (\theta^3 - 2\theta^2)y + 1 = 0, \tag{15}$$

which is a polynomial with symmetric coefficients and hence, denoting $\xi = y + 1/y$, (15) can be rewritten as

$$\xi^2 - \theta^2(\theta - 2)\xi + \theta^4 - 2\theta^3 = 0. \tag{16}$$

If $0 < \theta < 2$ or $\theta > 1 + \sqrt{5}$, this equation has two solutions

$$\xi_{1,2} = \frac{\theta^2}{2}\left(\theta - 2 \pm \sqrt{\theta(\theta-2)(\theta^2 - 2\theta - 4)}\right).$$

Note that $\xi > 2$ for $\theta > 1 + \sqrt{5}$ and $\xi_{1,2}(1 + \sqrt{5}) = 2 + 2\sqrt{5}$.

We find $y_i,\ i = 1, 2, 3, 4$ the following equation

$$y^2 - \xi_1 y + 1 = 0, \text{ and } y^2 - \xi_2 y + 1 = 0.$$

Summarizing, we obtain the following result.

Theorem 2. For the HC-Potts model with $k = 2$, there exists a critical value $\theta_1 = 1 + \sqrt{5}$ such that

1. If $\theta < \theta_1$, then there is a unique TILGM, i.e., phase transition does not occur;
2. If $\theta = \theta_1$, then there are three TILGMs, i.e., the phase transition occurs;
3. If $\theta > \theta_1$, then there are five TILGMs, i.e., the phase transition occurs;

Remark 2. Our results differ from classical Hard Core model and classical Potts model, i.e.,

• The existence of three TILGMs for the Hard Core model in the case “Hinge“ and $\lambda > \frac{9}{4}$ was proved in [27].

• In [22], it is shown that for the Potts model, when $\theta > 1 + 2\sqrt{2}$, there exist five TILGMs.

3.3.2. Case $k=3$. In this case, our main result is

Theorem 3. For the HC-Potts model with $k=3$, there exist critical value $\theta_2 \approx 2.078$ such that

1. If $\theta < \theta_2$, then there is a unique TILGM, i.e., the phase transition does not occur;
2. If $\theta = \theta_2$, then there are three TILGMs, i.e., the phase transition occurs;
3. If $\theta > \theta_2$, then there are five TILGMs, i.e., the phase transition occurs.

Proof. For $k=3$, by (13) and the first equation in (12) we obtain

$$y^8-(\theta^4-3\theta^2+1)y^7+y^6+(\theta^6-3\theta^4+2)y^5+(2\theta^6-7\theta^4+6\theta^2-2)y^4+ \\ +(\theta^6-3\theta^4+2)y^3+y^2-(\theta^4-3\theta^2+1)y+1=0, \quad (17)$$

which is a polynomial with symmetric coefficients and hence, denoting $\xi = y+1/y$, (17) can be rewritten as

$$\xi^4-(\theta^4-3\theta^2+1)\xi^3-3\xi^2+(\theta^6-9\theta^2+5)\xi+2\theta^6-7\theta^4+6\theta^2-2=0. \quad (18)$$

(18) has solutions of the form:

$$\xi_1(\theta)=\frac{\theta^4-3\theta^2+1-2\sqrt{2k(\theta)}+2\sqrt{d_1}}{4}, \quad \xi_2(\theta)=\frac{\theta^4-3\theta^2+1-2\sqrt{2k(\theta)}-2\sqrt{d_1}}{4},$$

$$\xi_3(\theta)=\frac{\theta^4-3\theta^2+1-2\sqrt{2k(\theta)}+2\sqrt{d_2}}{4}, \quad \xi_4(\theta)=\frac{\theta^4-3\theta^2+1-2\sqrt{2k(\theta)}-2\sqrt{d_2}}{4},$$

where $d_1(\theta)=-2p(\theta)-2k(\theta)+\sqrt{\frac{2}{k(\theta)}}\cdot q(\theta),\ d_2(\theta)=-2p(\theta)-2k(\theta)-\sqrt{\frac{2}{k(\theta)}}\cdot q(\theta)$,

$$q(\theta)=-\frac{1}{8}(\theta^4-5\theta^2+3)(\theta^8-4\theta^6+7\theta^4-6\theta^2-9),\ p(\theta):=-\frac{1}{8}(3\theta^8-18\theta^6+33\theta^4-18\theta^2+27),$$

$$k(\theta)=\frac{\theta(\theta^2-2)\alpha}{12}+\frac{\theta^3(\theta^2+1)(\theta^2-2)}{\alpha}+\frac{1}{8}(\theta^8-6\theta^6+11\theta^4-6\theta^2+9),$$

$$\alpha=\sqrt[3]{108\theta(2\theta^4-6\theta^2+9)+12\theta\sqrt{-12\theta^{10}+288\theta^8-1980\theta^6+5820\theta^4-8748\theta^2+6561}}.$$

A graphical analysis reveals that $\xi_1(\theta)<0$ and $\xi_2(\theta)<0$ for any $\theta>0$. $\xi_3(\theta)>2$ and $\xi_4(\theta)>2$ for any $\theta>\theta_2\approx 2.078$ (see Fig. 2). Note that $\xi_3(\theta_2)=\xi_4(\theta_2)$.

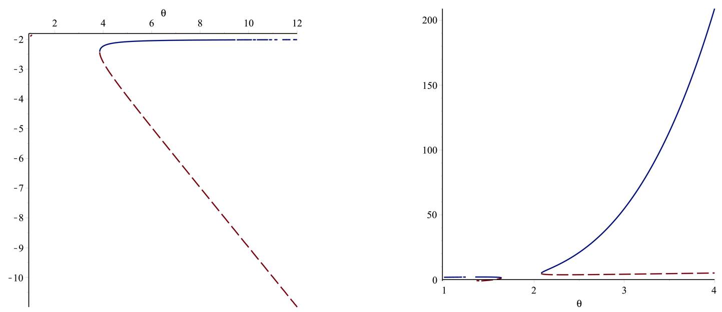

FIG. 2. On the left, the graphs of the functions $\xi_1(\theta)$ (solid line) and $\xi_2(\theta)$ (dashed line) and on the right the graphs of the functions $\xi_3(\theta)$ (solid line) and $\xi_4(\theta)$ (dashed line).

We find $y_i,\ i=1,2,3,4$ the following equation

$$y^2-\xi_3 y+1=0, \text{ and } y^2-\xi_4 y+1=0.$$

□

3.4. Construction of the finite-volume Gibbs measures for the case "Hinge"

Let $\theta = e^{\beta J}$ and denote

$$z_0:=1, \qquad z_1:=e^{h_1}, \qquad z_2:=e^{h_2}.$$

For $q=2$ the local sums entering Theorem 1 are

$$S_0=\theta+z_1, \qquad S_1=\theta z_1+1+z_2, \qquad S_2=\theta z_2+z_1.$$

For translation–invariant boundary fields the fixed–point system reads

$$z_i=\left(\frac{S_i}{S_0}\right)^k, \qquad i=1,2,$$

equivalently

$$S_i^{\,k} = S_0^{\,k}\, z_i, \qquad i = 0, 1, 2. \tag{19}$$

For $n \geq 1$ the finite-volume Gibbs distribution on V_n with boundary field h is

$$\mu_n(\sigma_{V_n}) = \frac{1}{Z_n(h)} \exp\Big(\beta J \sum_{\langle x,y\rangle \subset V_n} \delta_{\sigma(x),\sigma(y)}\Big) \prod_{\langle x,y\rangle \subset V_n} \lambda_{\sigma(x),\sigma(y)} \exp\Big(\sum_{x \in W_n} h_{\sigma(x)}\Big). \tag{20}$$

Fix a configuration $\sigma_{V_{n-1}}$ on V_{n-1}. Summing (20) over all boundary configurations $\omega \in \Phi^{W_n}$ and using the tree factorization, one obtains

$$\sum_{\omega \in \Phi^{W_n}} \mu_n(\sigma_{V_{n-1}} \vee \omega) = \frac{1}{Z_n(h)} \exp\big(\beta J H_{V_{n-1}}(\sigma_{V_{n-1}})\big) \prod_{x \in W_{n-1}} \Big(\sum_{u=0}^{2} \lambda_{\sigma(x),u} e^{\beta J \delta_{\sigma(x),u}} z_u\Big)^k. \tag{21}$$

Observe that the inner sums equal $S_{\sigma(x)}$, hence,

$$\sum_{\omega} \mu_n(\sigma_{V_{n-1}} \vee \omega) = \frac{1}{Z_n(h)} \exp\big(\beta J H_{V_{n-1}}(\sigma_{V_{n-1}})\big) \prod_{x \in W_{n-1}} S_{\sigma(x)}^{\,k}. \tag{22}$$

Now apply (19). Since $S^k_{\sigma(x)} = S^k_0 z_{\sigma(x)}$ for every $\sigma(x) \in \{0, 1, 2\}$, we get

$$\sum_{\omega} \mu_n(\sigma_{V_{n-1}} \vee \omega) = \frac{S_0^{\,k|W_{n-1}|}}{Z_n(h)} \exp\big(\beta J H_{V_{n-1}}(\sigma_{V_{n-1}})\big) \prod_{x \in W_{n-1}} z_{\sigma(x)}. \tag{23}$$

To recover $\mu_{n-1}(\sigma_{V_{n-1}})$ on the right-hand side, the normalizing constants must satisfy the recursion

$$Z_n(h) = S_0^{\,k|W_{n-1}|}\, Z_{n-1}(h). \tag{24}$$

Indeed, substituting (24) into (23) yields exactly

$$\sum_{\omega} \mu_n(\sigma_{V_{n-1}} \vee \omega) = \mu_{n-1}(\sigma_{V_{n-1}}),$$

which is the Kolmogorov compatibility condition.

For the Cayley tree of order k one has

$$|W_m| = (k+1)k^{m-1}, \qquad m \geq 1,$$

so that

$$\sum_{m=1}^{n-1} |W_m| = (k+1) \sum_{m=1}^{n-1} k^{m-1} = \frac{k+1}{k-1}\,(k^{\,n-1} - 1).$$

Iterating (24) from $Z_1(h)$ gives one

$$Z_n(h) = S_0^{\,k \sum_{m=1}^{n-1} |W_m|}\; Z_1(h) = S_0^{\frac{k(k+1)}{k-1}\,(k^{n-1}-1)}\; \Big(\sum_{i=0}^{2} z_i S_i^{\,k+1}\Big),$$

where $Z_1(h) = \sum_{i=0}^{2} z_i S_i^{\,k+1}$.

Let

$$N_1 = \sum_{x \in W_n} \mathbf{1}_{\{\sigma(x)=1\}}, \qquad N_2 = \sum_{x \in W_n} \mathbf{1}_{\{\sigma(x)=2\}}.$$

Then

$$\prod_{x \in W_n} z_{\sigma(x)} = z_1^{\sum_{x \in W_n} \mathbf{1}_{\{\sigma(x)=1\}}}\; z_2^{\sum_{x \in W_n} \mathbf{1}_{\{\sigma(x)=2\}}} = z_1^{N_1} z_2^{N_2}.$$

Using the expression for $Z_n(h)$ above, from (20), we obtain

$$\mu_n(\sigma_{V_n}) = \frac{\exp\big(\beta J H_{V_n}(\sigma_{V_n})\big)}{S_0^{\frac{k(k+1)}{k-1}(k^{n-1}-1)} \sum_{i=0}^{2} z_i S_i^{\,k+1}}\, z_1^{N_1} z_2^{N_2}.$$

This proves that for any translation–invariant fixed point z the family $\{\mu_n\}$ is Kolmogorov–consistent and hence extends to a splitting Gibbs measure on the infinite tree.

4. Thermodynamic properties of the solutions for $k = 2$.

Now we study the five resulting quantities from the thermodynamic point of view. For this purpose, we use the partition function.

Let Λ be a finite subset of V. We will denote by $\sigma(\Lambda)$ the restriction of σ to Λ. Let $\overline{\sigma}(V\backslash\Lambda)$ be a fixed boundary configuration. The total energy of $\sigma(\Lambda)$ under condition $\overline{\sigma}(V\backslash\Lambda)$ is defined as

$$H(\sigma(\Lambda) \mid \overline{\sigma}(V\backslash\Lambda)) = -J \sum_{\langle x,y\rangle: x,y\in\Lambda} \delta_{\sigma(x)\sigma(y)} - \beta^{-1} \sum_{\langle x,y\rangle: x,y\in\Lambda} \log\lambda_{\sigma(x),\sigma(y)} - \\ - J \sum_{\langle x,y\rangle: x\in\Lambda, y\notin\Lambda} \delta_{\sigma(x)\sigma(y)} - \beta^{-1} \sum_{\langle x,y\rangle: x\in\Lambda, y\notin\Lambda} \log\lambda_{\sigma(x),\sigma(y)}.$$

Then partition function $Z_\Lambda(\overline{\sigma}(V\backslash\Lambda))$ in volume Λ with boundary condition $\overline{\sigma}(V\backslash\Lambda)$ is defined as

$$Z_\Lambda(\overline{\sigma}(V\backslash\Lambda)) = \sum_{\sigma(\Lambda)\in\Omega(\Lambda)} \exp(-\beta H_\Lambda(\sigma(\Lambda) \mid \overline{\sigma}(V\backslash\Lambda))),$$

where $\Omega(\Lambda)$ is the set of all configurations in volume Λ and $\beta = \dfrac{1}{T}$ is the inverse temperature. Then the conditional Gibbs measure μ_Λ of a configuration $\sigma(\Lambda)$ is defined as

$$\mu_\Lambda(\sigma(\Lambda) \mid \overline{\sigma}(V\backslash\Lambda)) = \frac{\exp(-\beta\, H(\sigma(\Lambda) \mid \overline{\sigma}(V\backslash\Lambda)))}{Z_\Lambda(\overline{\sigma}(V\backslash\Lambda))}.$$

We consider the configuration $\sigma(V_n)$, the partition functions Z_{V_n} and the conditional Gibbs measure $\mu_\Lambda(\sigma(\Lambda) \mid \overline{\sigma}(V\backslash\Lambda))$ in volume V_n and simplicity, we denote them by σ_n, $Z^{(n)}$ and μ_n, respectively. For $q = 2$, the partition function $Z^{(n)}$ can be decomposed into the following components:

$$Z^{(n)} = Z_0^{(n)} + Z_1^{(n)} + Z_2^{(n)},$$

where

$$Z_i^{(n)} = \sum_{\sigma_n\in\Omega: \sigma(x^0)=i} \exp(-\beta H_{V_n}(\sigma \mid \overline{\sigma}(V\backslash V_n))),\ i = 0, 1, 2.$$

Introducing notation $\theta = \exp(J\beta)$ and we get

$$Z_0^{(n)} = \left(\theta Z_0^{(n-1)} + Z_1^{(n-1)}\right)^2;$$
$$Z_1^{(n)} = \left(Z_0^{(n-1)} + \theta Z_1^{(n-1)} + Z_2^{(n-1)}\right)^2;$$
$$Z_2^{(n)} = \left(Z_1^{(n-1)} + \theta Z_2^{(n-1)}\right)^2.$$

Introducing the following notations

$$u_n(x^0) = \frac{Z_1^{(n)}(x^0)}{Z_0^{(n)}(x^0)}, \qquad v_n(x^0) = \frac{Z_2^{(n)}(x^0)}{Z_0^{(n)}(x^0)}, \tag{25}$$

we obtain the following system of recurrent equations:

$$\begin{cases} u_n = \left(\dfrac{1+\theta u_{n-1}+v_{n-1}}{\theta+u_{n-1}}\right)^2; \\ v_n = \left(\dfrac{u_{n-1}+\theta v_{n-1}}{\theta+u_{n-1}}\right)^2. \end{cases}$$

If $u = \lim u_n$ and $v = \lim v_n$ then

$$\begin{cases} u = \left(\dfrac{1+\theta u+v}{\theta+u}\right)^2; \\ v = \left(\dfrac{u+\theta v}{\theta+u}\right)^2. \end{cases} \tag{26}$$

Remark 3. Note that, if we introduce the substitution $\sqrt{u} = x$ and $\sqrt{v} = y$ into a system of (26) equations, it coincides with a system of (12) equations. That is, the fixed points of the system of (12) equations will also be the fixed points of the system of (26).

4.1. Spontaneous Magnetization

If the spin values are given as $\sigma \in \{0, 1, 2\}$, the magnetization is defined by

$$\langle\sigma\rangle = \frac{\sum\limits_{\sigma\in\Phi} \sigma Z_\sigma^{(n)}}{\sum\limits_{\sigma\in\Phi} Z_\sigma^{(n)}} = \frac{Z_1^{(n)} + 2Z_2^{(n)}}{Z_0^{(n)} + Z_1^{(n)} + Z_2^{(n)}}.$$

This is not the physical magnetization yet, but only the mean value in $\{0, 1, 2\}$. From the physical point of view, spins are usually considered to take the symmetric values $\{-1, 0, +1\}$. This can be achieved by a simple linear transformation $\tilde{\sigma} = \sigma - 1$, which yields the true physical magnetization.

In this representation the definition of the magnetization becomes symmetric:

$$\langle\tilde{\sigma}\rangle = \frac{\sum\limits_{\sigma\in\Phi} (\sigma - 1)\, Z_\sigma^{(n)}}{\sum\limits_{\sigma\in\Phi} Z_\sigma^{(n)}} = \frac{Z_2^{(n)} - Z_0^{(n)}}{Z_0^{(n)} + Z_1^{(n)} + Z_2^{(n)}}.$$

Using the representation of the recursion relations given in Eqs. (25), the spontaneous magnetization per site is obtained as

$$M = \frac{v - 1}{1 + u + v},$$

where $(u; v)$ is a solution of the fixed point equations (26) or (12).

We denote by M_i the magnetization corresponding to the fixed point $(u_i; v_i)$, $i = \overline{0, 4}$, i.e.,

$$M_i = \frac{v_i - 1}{1 + u_i + v_i}.$$

Note that for $i = 0$, we have $y = 1$, and it follows that $M_0 = 0$.

In Fig. 3, plots of magnetizations M_i, $i = \overline{1, 4}$ for the HC-Potts model on the Cayley tree. Here $M_0 = 0$ corresponds to the non-magnetized (paramagnetic) phase. At low temperatures ($T < T_c$), the system exhibits four magnetized (ferromagnetic) phases, which appear as symmetric pairs with positive and negative magnetizations. As the temperature increases, these magnetized phases gradually approach $M_0 = 0$ and merge at the critical temperature T_{cr}. This behavior indicates a phase transition from the ferromagnetic to the paramagnetic phase, showing that the model possesses ordered phases at low temperatures and a disordered phase at high temperatures.

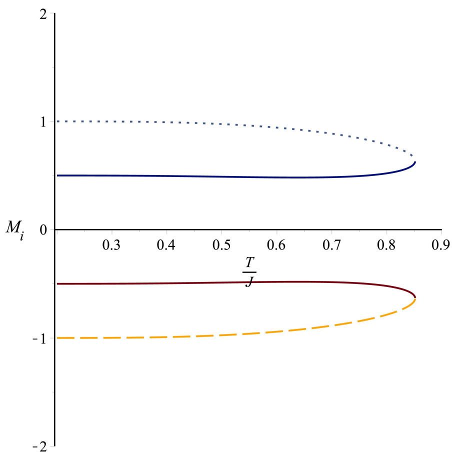

FIG. 3. Plots of spontaneous magnetizations M_i for $i = \overline{1, 4}$ for the case “Hinge“.

4.2. Quadrupolar Moment

In the three-state HC-Potts model, the quadrupolar moment is introduced to characterize the fraction of “active“ spin states in the system. If the spin variables take values $\sigma \in \{0, 1, 2\}$, the quadrupolar moment is defined as

$$\langle\sigma^2\rangle = \frac{\sum\limits_{\sigma\in\Phi} \sigma^2 Z_\sigma^{(n)}}{\sum\limits_{\sigma\in\Phi} Z_\sigma^{(n)}} = \frac{Z_1^{(n)} + 4Z_2^{(n)}}{Z_0^{(n)} + Z_1^{(n)} + Z_2^{(n)}}.$$

From the physical point of view, it is often more convenient to describe the system in terms of symmetric spin values $\{-1, 0, +1\}$. In the symmetric parametrization, the quadrupolar moment has a clearer interpretation. This can be obtained by applying a simple linear transformation $\tilde{\sigma} = \sigma - 1$.

Hence, the quadrupolar moment becomes as follows

$$\langle \tilde{\sigma}^2 \rangle = \frac{Z_0^{(n)} + Z_2^{(n)}}{Z_0^{(n)} + Z_1^{(n)} + Z_2^{(n)}}.$$

Using the same recursive framework, the quadrupolar moment is given by

$$Q = \frac{1+v}{1+u+v}$$

where $(u; v)$ is a solution of the fixed point equations (9) and (13).

Let Q_i denote the quadrupolar moment corresponding to $(u_i; v_i)$, $i = \overline{0, 4}$, i.e.,

$$Q_i = \frac{1+v_i}{1+u_i+v_i}.$$

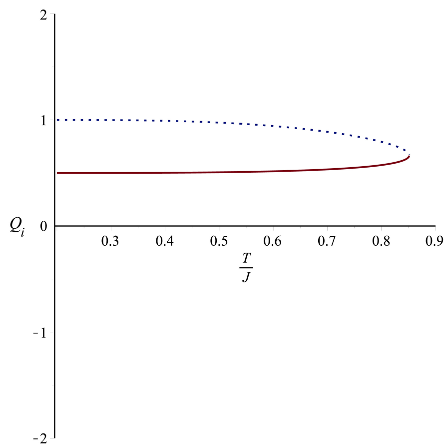

FIG. 4. Plots of Quadrupolar moments Q_i for $i = \overline{1, 4}$ for the cases “Hinge“. Here $Q_1 = Q_2$ (dashed line) and $Q_3 = Q_4$ (solid line).

In Fig. 4, the plots of the quadrupolar moments Q_i for $i = \overline{1, 4}$ are presented. The curves $Q_1 = Q_2$ and $Q_3 = Q_4$ correspond to the symmetric magnetized phases shown in Fig. 4. At low temperatures, the system exhibits an ordered (nematic-like) phase, while all Q_i values merge at the critical temperature T_c, indicating a transition to the paramagnetic (disordered) phase. Both magnetic and quadrupolar orders vanish simultaneously at this point.

5. Conclusion

In this paper, Gibbs measures for the Hard-Core–Potts model on Cayley trees were investigated. For the hinge-type graph and the cases $k = 2$ and $k = 3$, critical values of the parameter θ were found, determining the transition from a unique to multiple translation-invariant Gibbs measures. Thermodynamic analysis showed that the model exhibits ordered phases at low temperatures and a disordered phase at high temperatures, revealing the interplay between magnetic ordering and hard-core exclusion effects.

The thermodynamic analysis of the HC-Potts model on the Cayley tree reveals several important physical features arising from the interplay between ferromagnetic interactions and hard-core exclusion constraints.

The fixed-point solutions of the recursive relations show that at low temperatures (equivalently, for large values of $\theta = e^{\beta J}$), the system admits four distinct magnetized phases in addition to the paramagnetic one. These phases appear in symmetric pairs with positive and negative magnetizations, reflecting the underlying ± 1 symmetry of the transformed spin states. The presence of four magnetized phases is a direct consequence of the hard-core constraints, which restrict admissible local configurations and thereby enrich the phase structure compared to the classical three-state Potts model. As the temperature increases, thermal fluctuations dominate the ferromagnetic ordering, causing the magnetized phases to move continuously toward the paramagnetic fixed point. At the critical temperature T_c, all nonzero magnetizations collapse to zero, marking a transition from an ordered ferromagnetic regime to a disordered paramagnetic phase. This behavior illustrates the spontaneous breaking of symmetry at low temperatures and its restoration at high temperatures.

The quadrupolar order parameter provides additional insight into the distribution of active spin states. For the three-state system, it measures the relative weight of the spin values ± 1 compared to the inactive state. Our analysis shows that each magnetized phase is accompanied by a distinct quadrupolar value, indicating that both magnetic and structural ordering are simultaneously present in the low-temperature regime. As the temperature increases, the quadrupolar moments associated with all phases converge to a common value at the same critical temperature T_c. This demonstrates that the disappearance of magnetic ordering is accompanied by the loss of structural (nematic-like) order. Consequently, the HC-Potts model undergoes a single combined phase transition at which both magnetization and quadrupolar order vanish.

Overall, these results show that the hard-core exclusion mechanism significantly modifies the thermodynamic behavior of the Potts model, producing additional magnetized phases and enforcing a coupled disappearance of magnetic and quadrupolar orders at the critical point.

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