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Weakly periodic measure and phase transition: q-state p-adic Potts model on the Cayley tree of order k

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ABSTRACT In this paper, we investigate weakly periodic p-adic quasi Gibbs measures for the q-state Potts model on the Cayley tree of order k. Furthermore, we demonstrate that for all $q \geq 3$ and $k \geq 2$, there exist a prime number p and a parameter θ that guarantee the occurrence of a phase transition.

KEYWORDS p-adic numbers; Potts model; p-adic quasi Gibbs measure; translation-invariant, periodic, weakly periodic Gibbs measure; phase transition.

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1. Introduction

The application of p-adic analysis in the study of complex physical systems, particularly nanosystems, is motivated by the non-Archimedean nature of hierarchical structures [1, 2]. Nanomaterials often exhibit fragmented, fractal, or hierarchical energy landscapes where traditional Euclidean geometry fails to capture the underlying dynamics [3]. In this context, the p-adic metric provides a more natural framework for describing the ultrametricity inherent in disordered systems, such as spin glasses and polymers at the nanoscale [4]. This aligns with the fundamental work of G. Parisi and his collaborators, who established that the state space of complex disordered systems possesses a hierarchical, ultrametric structure [5, 6].

In nanoscience, the relaxation processes in complex molecules (e.g., proteins or nanoclusters) often occur through a hierarchy of energy barriers. As noted in the work of Avetisov (et al.), the transition dynamics between states in these systems can be modeled using p-adic diffusion equations [7]. The p-adic Gibbs measure, as explored in this paper, provides the statistical foundation for understanding equilibrium states in such hierarchical models, offering a more accurate representation of thermodynamic stability in non-Archimedean state spaces [8].

The p-adic Potts model on a Cayley tree serves as a powerful mathematical abstraction for analyzing interactions within branched polymers and dendrimers [9]. Dendrimers are nanoscopic, highly branched macromolecules whose structural symmetry is perfectly mirrored by the topology of the Cayley tree, also known as the Bethe lattice. The topological congruence between these nanostructures and the Cayley tree makes the study of phase transitions-specifically the existence of multiple Gibbs measures-essential for predicting how “information” or “energy” is distributed across these molecular networks under non-Archimedean temperature constraints. Such theoretical insights are increasingly crucial for the development of molecular electronics and the design of functional nanomaterials [10]. By investigating the q-state p-adic Potts model, this research provides a rigorous mathematical framework to evaluate the stability of thermodynamic phases in hierarchical molecular assemblies, offering a predictive toolset for future simulations of nanostructured systems.

While traditional statistical mechanics uses the field of real numbers $\mathbb{R}$, p-adic mathematical physics, as pioneered by Volovich and Khrennikov, suggests that at the Planck scale or within specific disordered media, the geometry of space-time or state-space may be non-Archimedean [1, 11]. For Nanosystems, this is particularly relevant in:
Quantum dots and tunneling: Where discrete energy levels and hierarchical transitions dominate [12].
Molecular dynamics: Where the “basins of attraction” in a protein’s folding landscape form an ultrametric space [7, 13].

By analyzing the q-state p-adic Potts model, this research contributes to the theoretical understanding of phase transitions in systems where interactions follow a hierarchical logic. This provides a rigorous mathematical toolset for future simulations of nanostructured materials and hierarchical molecular assemblies [14].

The Potts model is used in many areas, including magnetism, physics, image processing, medicine, sociology, biology and social dynamics (see e.g., [15]).

The p-adic Potts model was initially studied on the integers $\mathbb{Z}$ (see [16]), and the existence of a phase transition was proven when $\mid q \mid_p < 1$. This model was first studied on a Cayley tree in [17]. In that paper, it has been proven that there is a unique Gibbs measure when $\mid q \mid_p = 1$, and at least two Gibbs measures when $\mid q \mid_p < 1$. Subsequently, the research moved forward with [18], which introduced the notion of a p-adic quasi Gibbs measure. Furthermore, this paper achieved two significant goals: it classified the p-adic interpretation of phase transition into three separate categories, and it provided a rigorous demonstration of a strong phase transition for the model when situated on the binary tree.

Subsequent academic contributions have concentrated on various specialized aspects of the model, including: the examination of the chaotic behavior associated with the p-adic Potts-Bethe mapping ([19–21]), the analysis of translation-invariant p-adic Gibbs measures ([22, 23]), studies related to periodic p-adic quasi Gibbs measures ([24–26]), the construction and study of non-periodic, constructive measures ([25, 27–29]). Related investigations also addressed weakly periodic Gibbs measures defined over the real numbers ([30, 31]) and explored weakly periodic p-adic quasi Gibbs measures as applied to the Ising model ([32, 33]). Most recently, for the 3-state Potts model on the binary tree, [34] successfully demonstrated that a phase transition occurs when $p > 3$, and a quasi phase transition is present when $p = 3$.

Several recent studies [35–37] have extensively investigated various lattice models, such as the mixed spin-1/2 and spin-1 Ising model, the three-state SOS model with one-level competing interactions, and the Hard-Core-Potts model. These investigations were primarily conducted within the framework of the real number field $\mathbb{R}$, where the authors established specific conditions for the occurrence of phase transitions on the Cayley tree.

In [38], the authors investigated translation-invariant p-adic quasi Gibbs measures for the q-state Potts model with an external field on a Cayley tree of order two and established the conditions for phase transitions. In contrast, the present paper explores a broader class of weakly periodic p-adic quasi Gibbs measures for the q-state Potts model without an external field on a Cayley tree of an arbitrary order $k \geq 2$, rigorously demonstrating that a phase transition always occurs under certain parameter regimes.

This paper extends the foundational work of [34] by focusing on weakly periodic p-adic quasi Gibbs measures for the q-state Potts model defined on the Cayley tree of order k. Detailed in Sections 4 and 5, a significant discovery is the proof establishing the occurrence of a phase transition phenomenon for all configurations where $q \geq 3$ and $k \geq 2$, provided certain values of p and θ are present.

2. Preliminaries

2.1. p-adic numbers and measures

For the field of rational numbers, $\mathbb{Q}$, and a predetermined prime p, any $x \in \mathbb{Q} \setminus \{0\}$ possesses a representation $x = p^r \dfrac{n}{m}$, where $r \in \mathbb{Z}$, $m \in \mathbb{Z}^+$, and n, m are integers such that $\gcd(n, p) = \gcd(m, p) = 1$.

The p-adic norm on $\mathbb{Q}$, defined as $\mid x \mid_p = p^{-r}$ for non-zero $x = p^r \dfrac{n}{m}$ and $\mid 0 \mid_p = 0$, is a non-Archimedean valuation. Consequently, it adheres to the strong triangle inequality, which states that $\mid x + y \mid_p \leq \max\{\mid x \mid_p, \mid y \mid_p\}$.

Later, we will use the following important properties of the p-adic norm:
1) if $\mid x \mid_p \neq \mid y \mid_p$, then $\mid x \pm y \mid_p = \max\{\mid x \mid_p, \mid y \mid_p\}$;
2) if $\mid x \mid_p = \mid y \mid_p$, then $\mid x \pm y \mid_p \leq \mid x \mid_p$.

The field of rational numbers, $\mathbb{Q}$, is incomplete with respect to the p-adic norm. The completion of $\mathbb{Q}$ under this norm results in the construction of the field of p-adic numbers, $\mathbb{Q}_p$. Every non-zero p-adic number y possesses a unique canonical expansion:

$$y = p^{\gamma(y)}(y_0 + y_1 p + y_2 p^2 + ...),$$

where the p-adic valuation $\gamma(y) \in \mathbb{Z}$ and the coefficients y_j satisfy $0 < y_0 \leq p - 1$ and $0 \leq y_j \leq p - 1$ for $j \geq 1$. The p-adic norm is extended to $\mathbb{Q}_p$ as $\mid y \mid_p = p^{-\gamma(y)}$, where $\gamma(y)$ is also denoted as $\mathrm{ord}_p(y)$.

Let a be an element of the p-adic field $\mathbb{Q}_p$ and let $r > 0$. We define the open ball $B(a, r)$, the closed ball $\overline{B(a, r)}$, and the sphere $S(a, r)$ centered at a with radius r as the sets:

$$B(a, r) = \{x \in \mathbb{Q}_p : \mid x - a \mid_p < r\},$$
$$\overline{B(a, r)} = \{x \in \mathbb{Q}_p : \mid x - a \mid_p \leq r\},$$
$$S(a, r) = \{x \in \mathbb{Q}_p : \mid x - a \mid_p = r\}.$$

The sets $\mathbb{Z}_p := \overline{B(0, 1)}$ and $\mathbb{Z}_p^* = S(0, 1)$ are respectively called *p-adic integers* and *p-adic units*.

p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Put

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : \mid x - 1 \mid_p < p^{-1/(p-1)} \right\}.$$

The set $\mathcal{E}_p$ is a range of exponential. A more detailed see [39], [1].

Lemma 1. *(Hensel's lemma) [40,41] Let $F(x)$ be a polynomial whose coefficients are p-adic integers. Let x^* be a p-adic integer such that for some $i \geq 0$ one has*

$$F(x^*) \equiv 0 (\mathrm{mod}\, p^{2i+1}), \quad F'(x^*) \equiv 0 (\mathrm{mod}\, p^i), \quad F'(x^*) \not\equiv 0 (\mathrm{mod}\, p^{i+1}).$$

Then $F(x)$ has a p-adic integer root x_* such that $x_* \equiv x^* (\mathrm{mod}\, p^{i+1})$.

The symbol $o[x]$ was adopted in the work [42] to facilitate simplified computations. By definition, $z = o[x]$ means that $\mid z \mid_p < \mid x \mid_p$.

Let $\mathbb{F}_p = \{1, 2, ...p-1\}$. It is known [43] that, $\mathbb{F}_p$ is a group under the multiplication. Let $a \in \mathbb{Z}_p^*$, $k = mp^s$ and

$$Sol_p(x^k - a) = \{\xi \in \mathbb{F}_p : \xi^k - a = o[1]\}, \quad \kappa_p = card(Sol_p(x^k - a)) = (m, p-1),$$

here (c, d) is the greatest common divisor of c and d.

Theorem 1. *[42, 44] Let $p \geq 3$ and let $k = mp^s$, where $(p, m) = 1$, $s \geq 0$. Suppose that $a \in Z_p^*$ and it has the following canonical form:*

$$a = a_0 + a_1 p + a_2 p^2 + ...$$

with $Sol_p(x^k - a) \neq \emptyset$. Then the followings conditions are equivalent:

i) *$x^k = a$ has a solution;*

ii) *$a = a_0^{p^s} + o[p^s]$;*

iii) *for any $\xi \in Sol_p(x^k - a)$ the equation $x^k = a$ has a unique solution in $B(\xi, 1)$.*

Moreover, if the equation $x^k = a$ has a solution, then its number of solutions equals $\kappa_p = (m, p-1)$.

Note that, if $a \in \mathbb{Q}_p \setminus \mathbb{Z}_p^*$, after substituting $x = \dfrac{x_*}{\mid x \mid_p}$, $a = \dfrac{a_*}{\mid a \mid_p}$, we obtain the equation $x_*^k = a_*$ under condition $k \mid ord_p(a)$ (k divides $ord_p(a)$). Clear that here $a_* \in \mathbb{Z}_p^*$.

In [45], the equation $x^k = a$ was studied in $\mathbb{Q}_2$.

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ represents an algebra of subsets of X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is termed a p-adic measure if it satisfies the property of finite additivity: for any finite collection of pairwise disjoint sets $A_1, A_2, \ldots, A_n \in \mathcal{B}$, the following equality holds:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Furthermore, if the total measure is unity ($\mu(X) = 1$), the p-adic measure is specifically referred to as a p-adic probability measure. A p-adic measure μ is defined as bounded if the supremum of the p-adic norm of the measures of all sets in the algebra is finite:

$$\sup\{\mid \mu(A) \mid_p : A \in \mathcal{B}\} < \infty.$$

2.2. Cayley tree

The Cayley tree Γ^k (where $k \geq 1$) is an infinite $(k+1)$-regular graph, meaning every vertex has a degree of $k+1$. Let V and L denote the vertex and edge sets, respectively. An edge $l = \langle x, y \rangle$ signifies that x and y are nearest neighbors. Given a fixed root $x_0 \in \Gamma^k$, $d(x, y)$ represents the length of the shortest path between x and y. We define the shells $W_n = \{x \in V : d(x, x_0) = n\}$ and the finite subgraph $V_n = \{x \in V : d(x, x_0) \leq n\}$, with L_n being the corresponding set of edges in V_n. For any vertex $x \in W_n$, $S(x)$ is the set of its direct successors (neighbors in W_{n+1}): $S(x) = \{y \in W_{n+1} : d(y, x) = 1\}$. The set of all neighbors of x is $S_1(x) = \{y \in V : d(y, x) = 1\}$, and the unique parent of x (the neighbor closer to the root) is denoted $x \downarrow = S_1(x) \setminus S(x)$ (see [46]).

3. p-adic quasi Gibbs measure for the Potts model

The model is defined over the p-adic numbers ($\mathbb{Q}_p$). States (Φ): Each vertex x on the graph V can be in one of q possible states, $\Phi = \{1, 2, \ldots, q\}$. Configuration ($\sigma$): A configuration is an assignment of states to the vertices. Ω_A is the set of all possible configurations on a subset A of vertices. Configuration union: The operation $(\sigma_{n-1} \vee \varphi^{(n)})$ provides a way to define a configuration on a larger set V_n by combining an existing configuration σ_{n-1} on the smaller set V_{n-1} with a configuration $\varphi^{(n)}$ on the boundary or complement $W_n = V_n \setminus V_{n-1}$. This is crucial for building the measure step-by-step toward the infinite system.

The (formal) Hamiltonian of the p-adic Potts model is defined by

$$H(\sigma) = J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)} \tag{1}$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\langle x, y\rangle$ stands for nearest neighbor vertices and δ_{ij} is the Kronecker's symbol, i.e.

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Assume that $\mathbf{h} : V \to \mathbb{Q}_p^q$ is a mapping, i.e. $\mathbf{h}_x = (h_{1,x}, h_{2,x}, ..., h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$, $i \in \Phi$ and $x \in V$. Given $n \in \mathbb{N}$, we consider a sequence of p-adic probability measures $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}, \tag{2}$$

where $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor

$$Z_n^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}. \tag{3}$$

We say that p-adic probability measures (2) are compatibly if all $n \in \mathbb{N}$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\sum_{\varphi^{(n)} \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \varphi^{(n)}) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}). \tag{4}$$

We notice that a non-Archimedean analogue of the Kolmogorov's extension theorem was proved in [16]. According to this theorem if (4) holds, then there exists a unique splitting p-adic measure μ_h on Ω such that for all $n \in \mathbb{N}$ and $\sigma \in \Omega_{V_{n-1}}$,

$$\mu(\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \text{ for all } \sigma_n \in \Omega_{V_n},\ n \in \mathbb{N}.$$

Such measure is called a *p-adic quasi Gibbs measure* corresponding to the Hamiltonian (1) and vector-valued function $\mathbf{h}_x$, $x \in V$. By $QG(H)$ we denote the set of all p-adic quasi Gibbs measure associated with function $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$. If all values of h_x belong to the set $\mathcal{E}_p$ then corresponding measure be *p-adic Gibbs measure* (see [18]).

The following statement is equivalent to compatibility of $\mu_{\mathbf{h}}^{(n)}$.

Theorem 2. *[18] The p-adic probability measures $\mu_h^{(n)}, n \in \mathbb{N}$ are defined by* (2) *satisfy the compatibility condition* (4) *if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\widehat{\boldsymbol{h}_x} = \prod_{y \in S(x)} F(\widehat{\boldsymbol{h}_y}, \theta), \tag{5}$$

here a vector $\widehat{\boldsymbol{h}} = \left(\widehat{h}_1, \widehat{h}_2, ..., \widehat{h}_{q-1}\right) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\boldsymbol{h} = (h_1, h_1, ..., h_q) \in \mathbb{Q}_p^q$ as follows

$$\widehat{h}_i = \frac{h_i}{h_q},\ i = 1, 2, ..., q-1 \tag{6}$$

and mapping
$F : \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \to \mathbb{Q}_p^{q-1}$ is defined by $F(x;\theta) = (F_1(x;\theta), ..., F_{q-1}(x;\theta), 1)$ with

$$F_i(x;\theta) = \frac{(\theta-1)x_i + \sum\limits_{j=1}^{q-1} x_j + 1}{\sum\limits_{j=1}^{q-1} x_j + \theta},\quad x = (x_1, x_2, ..., x_{q-1}) \in \mathbb{Q}_p^{q-1}, i = \overline{1, q-1}. \tag{7}$$

It is easy to check that the set of vectors $(\underbrace{h, ..., h, h}_{m}, 1, ..., 1), (m = 1, ..., q-1)$ is invariant for the right side of (5) as a mapping. Therefore, in what follows, we restrict ourselves to ones of such lines, let us say $(h, h, ..., h, 1)$.

4. Weakly periodic p-adic quasi Gibbs measure

Let G_k be a group of the free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$, respectively. There exists a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k (see [47]). Let G_k^* be a normal subgroup of the group G_k and $G_k/G_k^* = \{H_1, H_2, ..., H_r\}$, $r \geq 1$ is a quotient group of G_k by G_k^*.

Definition 1. *A set $h = \{h_x, x \in G_k\}$ of quantities is called G_k^*-periodic if $h_x = h_i$, for all $x \in H_i$. A G_k-periodic quantities are called translation-invariant.*

Definition 2. *A set of quantities* $h = \{h_x, x \in G_k\}$ *is called* G_k^**-weakly periodic if* $h_x = h_{ij}$*, for any* $x \in H_i$ *and* $y_\downarrow \in H_j$.

Definition 3. *A* p*-adic quasi Gibbs measure* μ *is said to be* G_k^**-(weakly) periodic if it corresponds to a* G_k^**-(weakly) periodic* h*. A* G_k*-periodic measure is called a translation-invariant measure.*

Let

$$H_A = \{x \in G_k : \sum_{i\in A} \omega_x(a_i) \ \text{is an even number}\},$$

where $\emptyset \neq A \subseteq N_k = \{1, 2, 3, ..., k+1\}$, and $\omega_x(a_i)$ is the number of letters a_i in a word $x \in G_k$. Note that H_A is a normal subgroup of the G_k (see [46]). It can be checked that weakly periodic Gibbs measure depends on choosing the normal subgroup of G_k. In real case, different weakly periodic Gibbs measures are found (see [30], [31]). The set of weakly periodic Gibbs measures also includes the set of periodic (in particular translation-invariant) Gibbs measures.

H_A is a normal divisor of index 2 in G_k, i.e. $G_k/H_A = \{H_0, H_1\}$ (see [46]). If $\mid A \mid = k+1$ (where $\mid A \mid$ is the number of elements of the set A), i.e. $A = N_k$, then weakly periodic measure coincides with $G_k^{(2)}$-periodic measure. Therefore, we consider $A \subset N_k$ such that $A \neq N_k$. H_A-weakly periodic collection $\mathbf{h}_x$ has the following form

$$\mathbf{h}_x = \begin{cases} \mathbf{h}_1, & \text{if } x \in H_0, \ x_\downarrow \in H_0, \\ \mathbf{h}_2, & \text{if } x \in H_0, \ x_\downarrow \in H_1, \\ \mathbf{h}_3, & \text{if } x \in H_1, \ x_\downarrow \in H_0, \\ \mathbf{h}_4, & \text{if } x \in H_1, \ x_\downarrow \in H_1, \end{cases} \tag{8}$$

where $H_0 = H_A$ and $H_1 = G_k \setminus H_A$ [46].
We assume that $\mathbf{h}_i = \left(h_i^{(1)}, h_i^{(2)}, ..., h_i^{(q-1)}, 1\right)$. Let $h_i^{(j)} = h_i$, $j = \overline{1, q-1}$, $i = \overline{1, 4}$. Using (5) and (8), we can obtain following system of the equations

$$\begin{cases} h_1 = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k-|A|} \cdot \left(\dfrac{(\theta+q-2)h_2+1}{(q-1)h_2+\theta}\right)^{|A|}, \\ h_2 = \left(\dfrac{(\theta+q-2)h_3+1}{(q-1)h_3+\theta}\right)^{|A|-1} \cdot \left(\dfrac{(\theta+q-2)h_4+1}{(q-1)h_4+\theta}\right)^{k+1-|A|}, \\ h_3 = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k+1-|A|} \cdot \left(\dfrac{(\theta+q-2)h_2+1}{(q-1)h_2+\theta}\right)^{|A|-1}, \\ h_4 = \left(\dfrac{(\theta+q-2)h_3+1}{(q-1)h_3+\theta}\right)^{|A|} \cdot \left(\dfrac{(\theta+q-2)h_4+1}{(q-1)h_4+\theta}\right)^{k-|A|}. \end{cases} \tag{9}$$

For simplicity, let $\mid A \mid = 1$. In this case, we denote by H_{A_1} the set

$$H_{A_1} = \{x \in G_k : \omega_x(a_1) \ \text{is an even number}\}.$$

Then (9) gives one

$$\begin{cases} h_1 = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k-1} \cdot \dfrac{(\theta+q-2)h_2+1}{(q-1)h_2+\theta}, \\ h_2 = \left(\dfrac{(\theta+q-2)h_4+1}{(q-1)h_4+\theta}\right)^{k}, \\ h_3 = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k} \\ h_4 = \dfrac{(\theta+q-2)h_3+1}{(q-1)h_3+\theta} \cdot \left(\dfrac{(\theta+q-2)h_4+1}{(q-1)h_4+\theta}\right)^{k-1}. \end{cases} \tag{10}$$

We consider the following operator $\mathcal{A}:\mathbb{Q}_p^4\to\mathbb{Q}_p^4$ defined as

$$\begin{cases} h_1' = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k-1}\cdot\dfrac{(\theta+q-2)h_2+1}{(q-1)h_2+\theta}, \\ h_2' = \left(\dfrac{(\theta+q-2)h_4+1}{(q-1)h_4+\theta}\right)^{k}, \\ h_3' = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k} \\ h_4' = \dfrac{(\theta+q-2)h_3+1}{(q-1)h_3+\theta}\cdot\left(\dfrac{(\theta+q-2)h_4+1}{(q-1)h_4+\theta}\right)^{k-1}. \end{cases} \tag{11}$$

It is obvious that the sets of vectors

$$I_1=\{\mathbf{h}=(h,h,h,h)\mid h\in\mathbb{Q}_p\},\ \ I_2=\{\mathbf{h}=(h_1,h_2,h_2,h_1)\mid h_1,h_2\in\mathbb{Q}_p\}$$

are invariant in respect to the operator $\mathcal{A}$.

Clearly, the fixed points of the operator $\mathcal{A}$ on invariant set I_1 are translation-invariant solutions of (5), i.e.

$$h=\left(\frac{(\theta+q-2)h+1}{(q-1)h+\theta}\right)^{k}. \tag{12}$$

To find the fixed points of the operator $\mathcal{A}$ on the invariant set I_2, one should solve the following system of equations

$$\begin{cases} h_1 = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k-1}\cdot\dfrac{(\theta+q-2)h_2+1}{(q-1)h_2+\theta}, \\ h_2 = \left(\dfrac{(\theta+q-2)h_1+1}{(q-1)h_1+\theta}\right)^{k}. \end{cases} \tag{13}$$

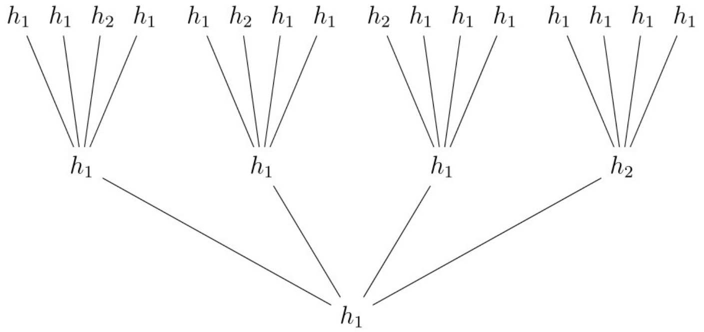

FIG. 1. This figure depicts the placement of values h_1 and h_2 associated with the invariant set I_2 on the Cayley tree of order four.

Using group representation of Cayley tree Γ^k, the values $h_x, x\in V$ corresponding to the invariant set I_2 are defined as follows:

(A). If at vertex x, we have $h_x=h_1$ then

$$h_y=\begin{cases} h_1, \text{ on } k-1 \text{ vertices of S(x);} \\ h_2, \text{ on } 1 \text{ vertex of S(x),} \end{cases} \tag{14}$$

(B). If at vertex x, we have $h_x=h_2$ then $h_y=h_1$, where $y\in S(x)$ (see Figure 1).

By the notation $\sqrt[k]{h_2}=x$, we get $h_1=\dfrac{1-\theta x}{(q-1)x-(\theta+q-2)}$. It follows from the first equation of (13) that

$$\frac{1-\theta x}{(q-1)x-(\theta+q-2)}=x^{k-1}\cdot\frac{(\theta+q-2)x^k+1}{(q-1)x^k+\theta},$$

or

$$(q-1)(\theta+q-2)x^{2k}-(\theta+q-2)^2x^{2k-1}+\theta(q-1)x^{k+1}-(\theta+q-2)x^{k-1}+\theta^2x-\theta=0. \tag{15}$$

Using notation $\sqrt[k]{h_2} = x$, one obtains from (12) that

$$(q-1)x^{k+1} - (\theta + q - 2)x^k + \theta x - 1 = 0. \tag{16}$$

Factoring the left hand side of (15), we have

$$\big((q-1)x^{k+1} - (\theta + q - 2)x^k + \theta x - 1\big) \cdot \big((\theta + q - 2)x^{k-1} + \theta\big) = 0. \tag{17}$$

Finding non-translation weakly periodic solutions of (15) is equivalent to solving the following equation

$$x^{k-1} = -\frac{\theta}{\theta + q - 2}, \tag{18}$$

where $\theta + q - 2 \neq 0$.

Remark 1. *It is easy to see that if k is odd and x^* is a solution of* (18) *then $-x^*$ is also a solution of* (18).

Let $\theta = 1 + p^l(\theta_l + \theta_{l+1}p + ...), l \geq 1$, $a = -\dfrac{\theta}{\theta + q - 2} = p^{\gamma(a)}(a_0 + a_1 p + ...), a_0 \neq 0$, (b, c) be the greatest common divisors of b, c and $k - 1 = mp^s$, where $(m, p) = 1$, $s \in \{0, 1, 2, ...\}$.

Theorem 3. *Let $\mid q-1 \mid_p < 1$. Then the following assertions hold*

- *if $\mid \theta - 1 \mid_p \neq \mid q - 1 \mid_p$ then the equation* (18) *is solvable iff*
  (1) $k-1$ *divides* $\gamma(a)$*;*
  (2) $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1 (\operatorname{mod} p)$*;*
  (3) $a_0^{p^s} \equiv a \mid a \mid_p (\operatorname{mod} p^{s+1})$.
  Moreover, (18) *has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$. Here, $\gamma(a) = -\min\{ord_p(\theta - 1), ord_p(q-1)\}$ and $a_0 = (1-\theta) \mid \theta - 1 \mid_p (\operatorname{mod} p)$ if $\mid \theta - 1 \mid_p > \mid q-1 \mid_p$; $a_0 = (1-q) \mid q - 1 \mid_p (\operatorname{mod} p)$ if $\mid \theta - 1 \mid_p < \mid q - 1 \mid_p$.*
- *if $\mid \theta - 1 \mid_p = \mid q-1 \mid_p$, $\theta_l + q_l \neq p$ or $\mid \theta - 1 \mid_p = \mid q - 1 \mid_p$, $\theta_l + q_l = p$, $1 + \theta_{l+1} + q_{l+1} \neq p$ then* (18) *is solvable iff*
  (1) $k-1$ *divides* $\gamma(a)$*;*
  (2) $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1 (\operatorname{mod} p)$*;*
  (3) $a_0^{p^s} \equiv a \mid a \mid_p (\operatorname{mod} p^{s+1})$.
  Moreover, (18) *has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$. Here, $q = 1 + p^l(q_l + q_{l+1}p + ...)$, $\gamma(a) = -ord_p(\theta - 1)$, $a_0 = \dfrac{-1}{\theta_l + q_l}(\operatorname{mod} p)$ if $\theta_l + q_l \neq p$; $\gamma(a) = -ord_p(\theta - 1) - 1$, $a_0 = \dfrac{-1}{1 + \theta_{l+1} + q_{l+1}}(\operatorname{mod} p)$ if $1 + \theta_{l+1} + q_{l+1} \neq p$.*
- *if $\mid \theta - 1 \mid_p = \mid q - 1 \mid_p$, $\theta_l + q_l = p$, $1 + \theta_{l+j} + q_{l+j} = p, j = 1, 2, 3, ...r-1$, $r \geq 2$, $1 + \theta_{l+r} + q_{l+r} \neq p$ then equation* (18) *is solvable iff*
  (1) $k-1$ *divides* $\gamma(a)$*;*
  (2) $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1 (\operatorname{mod} p)$*;*
  (3) $a_0^{p^s} \equiv a \mid a \mid_p (\operatorname{mod} p^{s+1})$.
  Moreover, (18) *has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$. Here, $q = 1 + p^l(q_l + q_{l+1}p + ...)$, $\gamma(a) = -ord_p(\theta - 1) - r$, $a_0 = \dfrac{-1}{1 + \theta_{l+r} + q_{l+r}}(\operatorname{mod} p)$.*
- *if $\mid \theta - 1 \mid_p = \mid q - 1 \mid_p$, $\theta_l + q_l = p$, $1 + \theta_{l+j} + q_{l+j} = p, j \in \mathbb{N}$ then the equation* (18) *does not have any solution.*

Proof. **Case 1.** Let $\mid \theta - 1 \mid_p \neq \mid q - 1 \mid_p$. We rewrite a as follows

$$a = \max\{\mid \theta - 1 \mid_p, \mid q - 1 \mid_p\} \cdot \begin{cases} \dfrac{1}{(1-\theta) \mid \theta - 1 \mid_p}(\operatorname{mod} p) + o[1], & \text{if } \mid \theta - 1 \mid_p > \mid q - 1 \mid_p, \\ \dfrac{1}{(1-q) \mid q - 1 \mid_p}(\operatorname{mod} p) + o[1], & \text{if } \mid \theta - 1 \mid_p < \mid q - 1 \mid_p \,. \end{cases} \tag{19}$$

(19) gives one that $\gamma(a) = -\min\{ord_p(\theta - 1), ord_p(q-1)\}$ and $a_0 = (1-\theta) \mid \theta - 1 \mid_p (\operatorname{mod} p)$ if $\mid \theta - 1 \mid_p > \mid q - 1 \mid_p$; $a_0 = (1-q) \mid q - 1 \mid_p (\operatorname{mod} p)$ if $\mid \theta - 1 \mid_p < \mid q - 1 \mid_p$.

According to Theorem 1, (18) is solvable iff

- $k-1$ divides $\gamma(a)$;
- $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1 (\operatorname{mod} p)$;
- $a_0^{p^s} \equiv a \mid a \mid_p (\operatorname{mod} p^{s+1})$.

Moreover, (18) has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$.

Case 2. $\mid \theta - 1 \mid_p = \mid q - 1 \mid_p$, i.e. $\theta = 1 + p^l(\theta_l + \theta_{l+1}p + ...)$ and $q = 1 + p^l(q_l + q_{l+1}p + ...)$. We rewrite a as follows

$$a = p^{-l}\left(-\frac{1 + p^l(\theta_l + \theta_{l+1}p + ...)}{(\theta_l + q_l) + (\theta_{l+1} + q_{l+1})p + ...}\right). \tag{20}$$

Case 2.1. Let $\theta_l + q_l \neq p$. In this case, we get $\gamma(a) = -ord_p(\theta - 1),\ a_0 = \dfrac{-1}{\theta_l + q_l}(\bmod\, p)$. Due to Theorem 1, equation (18) is solvable iff

- $k - 1$ divides $\gamma(a)$;
- $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1(\bmod\, p)$;
- $a_0^{p^s} \equiv a \mid a \mid_p (\bmod\, p^{s+1})$.

Moreover, (18) has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$.

Case 2.2. Let $\theta_l + q_l = p,\ 1 + \theta_{l+1} + q_{l+1} \neq p$. In this case, we get $\gamma(a) = -ord_p(\theta - 1) - 1,\ a_0 = \dfrac{-1}{1 + \theta_l + q_l}(\bmod\, p)$. Due to Theorem 1, equation (18) is solvable iff

- $k - 1$ divides $\gamma(a)$;
- $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1(\bmod\, p)$;
- $a_0^{p^s} \equiv a|a|_p(\bmod\, p^{s+1})$.

Moreover, (18) has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$.

Case 2.3. Let $\theta_l + q_l = p,\ 1 + \theta_{l+j} + q_{l+j} = p, j = 1, 2, 3, ...r - 1,\ r \geq 2,\ 1 + \theta_{l+r} + q_{l+r} \neq p$. In this case, we get $\gamma(a) = -l - r,\ a_0 = \dfrac{-1}{1 + \theta_{l+r} + q_{l+r}}(\bmod\, p)$.

Again, due to Theorem 1, equation (18) is solvable iff

- $k - 1$ divides $\gamma(a)$;
- $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1(\bmod\, p)$;
- $a_0^{p^s} \equiv a|a|_p(\bmod\, p^{s+1})$.

Moreover, (18) has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $p^{\frac{\gamma(a)}{k-1}}\mathbb{Z}_p^*$.

Case 2.4. Let $\theta_l + q_l = p,\ 1 + \theta_{l+j} + q_{l+j} = p, j \in \mathbb{N}$. In this case, we have $\theta = 2 - q$ and equation (18) does not have any solution. □

The following example ensures that the set satisfying the conditions in Theorem 3 is nonempty.

Example 1. *a) Let* $p = 3,\ q = 2 \cdot 3^6 + 1,\ \theta = 3^{12} + 1,\ k = 7$. *Then we have* $x^6 = 3^{-6}(1 + o[1])$. *Therefore, in this case according to Theorem 1, equation* (18) *has two solutions in* $3^{-1}\mathbb{Z}_3^*$.

b) Let $p = 5,\ q = 1 + 2 \cdot 5^{20},\ \theta = 1 + 2 \cdot 5^{20} + 5^{21},\ k = 11$. *Then we have* $x^{10} = 5^{-20}(1 + o[1])$. *Therefore, in this case according to Theorem 1, equation* (18) *has two solutions in* $5^{-2}\mathbb{Z}_5^*$.

Theorem 4. *Let* $\mid q - 1 \mid_p = 1$. *Then equation* (18) *is solvable iff*

(1) $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1(\bmod\, p)$*;*
(2) $a_0^{p^s} \equiv a(\bmod\, p^{s+1})$.

Moreover, equation (18) *has* $\kappa_p = (m, p-1)$ *solutions and these solutions belong to* $\mathbb{Z}_p^*$. *Here* $a_0 = \dfrac{1}{1-q}(\bmod\, p)$.

Proof. Let $\mid q - 1 \mid_p = 1$. We rewrite a as follows

$$a = \frac{1}{1-q} + o[1], \tag{21}$$

where $\gamma(a) = 0$, $a_0 = \dfrac{1}{1-q}(\bmod\, p)$. Again, due to Theorem 1, equation (21) gives one that equation (18) is solvable iff

- $a_0^{\frac{p-1}{(m,p-1)}} \equiv 1(\bmod\, p)$;
- $a_0^{p^s} \equiv a(\bmod\, p^{s+1})$.

Moreover, equation (18) has $\kappa_p = (m, p-1)$ solutions and these solutions belong to $\mathbb{Z}_p^*$. □

The following example ensures that the set satisfying the conditions in Theorem 4 is nonempty.

Example 2. *a) Let* $q = 5,\ p = 11,\ k = 6$. *Then we have* $x^5 = \dfrac{-\theta}{\theta+3}$ *and* $a_0 = 8$, $m = 5$, $s = 0$,

$$a_0^{\frac{p-1}{(m,p-1)}} \equiv 8^{\frac{10}{(5,10)}}(\operatorname{mod} 11) \equiv 8^2(\operatorname{mod} 11) \not\equiv 1(\operatorname{mod} 11).$$

Therefore, according to Theorem 1, the equation $x^5 = \dfrac{-\theta}{\theta+3}$ *does not have any solution in* $\mathbb{Q}_{11}$.

b) Let $q = 5,\ p = 7,\ k = 6$. *Then we have* $x^5 = \dfrac{-\theta}{\theta+3}$ *and* $a_0 = 5$, $m = 5$, $s = 0$,

$$a_0^{\frac{p-1}{(m,p-1)}} \equiv 5^{\frac{6}{(5,6)}}(\operatorname{mod} 7) \equiv 5^6(\operatorname{mod} 7) \equiv 1(\operatorname{mod} 7),$$

$a_0^{p^s} \equiv \dfrac{-\theta}{\theta+3}(\operatorname{mod} 7)$ *or* $5 \equiv \dfrac{-\theta}{\theta+3}(\operatorname{mod} 7)$, $1 + 6\theta \equiv 0(\operatorname{mod} 7)$. *Clearly, the last congruence holds. By Theorem 1, the equation* $x^5 = \dfrac{-\theta}{\theta+3}$ *has a unique solution in* $\mathbb{Q}_7$.

c) Let $q = 9,\ p = 7,\ k = 4$. *Then we have* $x^3 = \dfrac{-\theta}{\theta+7}$ *and* $a_0 = 6$, $m = 3$, $s = 0$,

$$a_0^{\frac{p-1}{(m,p-1)}} \equiv 6^{\frac{6}{(6,3)}}(\operatorname{mod} 7) \equiv 6^2(\operatorname{mod} 7) \equiv 1(\operatorname{mod} 7),$$

$a_0^{p^s} \equiv \dfrac{-\theta}{\theta+7}(\operatorname{mod} 7)$ *or* $6 \equiv \dfrac{-\theta}{\theta+7}(\operatorname{mod} 7)$, $42 + 6\theta \equiv 0(\operatorname{mod} 7)$. *Clearly, the last congruence holds. By Theorem 1, the equation* $x^3 = \dfrac{-\theta}{\theta+7}$ *has tree solutions in* $\mathbb{Q}_7$.

Corollary 1. *Let* $p \geq 3$ *and* $\mid q \mid_p < 1$. *Then the following assertions hold*

- *if* $\mid k-1 \mid_p \leq \mid 2(\theta-1)+q \mid_p$, *then* (18) *does not have any solution;*
- *if* $\mid k-1 \mid_p > \mid 2(\theta-1)+q \mid_p$, *then* (18) *has exactly* $\kappa_p = (m, p-1)$ *solutions* $x_{\xi_i} \in B(\xi_i, 1)$, $i = 1, 2, ..., \kappa_p$, $\xi_i \in Sol_p(x^{k-1} - a)$. *Moreover, one of solutions of* (18) *belongs* $\mathcal{E}_p$.

Proof. The proof is straightforward by Corollary 3.5 in [42]. □

The following example ensures that the set satisfying the conditions in Corollary 1 is nonempty.

Example 3.

- *Let* $k = 10$, $p = 3$, $q = 6$, $\theta = 4$. *Then* $\mid k-1 \mid_p = \mid 9 \mid_3 = \dfrac{1}{9} < \dfrac{1}{3} = \mid 12 \mid_3 = \mid 2(\theta-1)+q \mid_p$ *and* (18) *does not have any solution.*
- *Let* $k = 10$, $p = 3$, $q = 6$, $\theta = 25$. *Then* $\mid k-1 \mid_p = \mid 9 \mid_3 = \dfrac{1}{9} > \dfrac{1}{27} = \mid 54 \mid_3 = \mid 2(\theta-1)+q \mid_p$ *and* (18) *has a unique (* $\kappa_3 = (m, p-1) = (1, 2) = 1$*) solution in* $\mathcal{E}_3$.
- *Let* $k = 19$, $p = 3$, $q = 6$, $\theta = 25$. *Then* $\mid k-1 \mid_p = \mid 18 \mid_3 = \dfrac{1}{9} > \dfrac{1}{27} = \mid 54 \mid_3 = \mid 2(\theta-1)+q \mid_p$ *and* (18) *has two (* $\kappa_3 = (m, p-1) = (2, 2) = 2$*) solutions. Moreover, one of them belongs to* $\mathcal{E}_3$, *another has the form* $-1 + o[1]$.

Theorem 5. *Let* $k \geq 2$, $q \geq 3$ *be fixed natural numbers. Then there exist prime* p *and* $\theta \in \mathcal{E}_p$, *such that equation* (18) *has at least one solution such that* $x^* \in Z_p^* \setminus \mathcal{E}_p$.

Proof. Let p be any prime such that

$$\mid q - i \mid_p = 1,$$

where $i = 0, 1, 2$. Note that there exists such a prime p, for instance, any prime greater than q may be such p ($p > q$). For this prime p, $k - 1 = mp^s$, where $s = 0, 1, 2, ...$

We can choose θ as follows

$$\theta = \frac{2 - q + (2-q)(1-q)^{p^s} p^{s+1}}{1 + (1-q)^{p^s} + (1-q)^{p^s} p^{s+1}}.$$

Since $(1-q)^{p^s} \equiv (1-q)(\operatorname{mod} p)$, we have $\theta \in \mathcal{E}_p$. Then

$$a = -\frac{\theta}{\theta + q - 2} = \frac{1}{(1-q)^{p^s}} + p^{s+1},$$

i.e.

$$a = a_0^{p^s} + o[p^s],$$

where $a_0 = \dfrac{1}{1-q}(\operatorname{mod} p)$. Since $a_0^{p^s} \equiv a_0(\operatorname{mod} p) \equiv \dfrac{1}{1-q}(\operatorname{mod} p)$, we get $a \in \mathbb{Z}_p^* \setminus \mathcal{E}_p$. Therefore, in this case, according to Theorem 1, equation (18) has $\kappa_p = (m, p-1) \geq 1$ solutions in $\mathbb{Z}_p^* \setminus \mathcal{E}_p$. Because if the solution $x^* \in \mathcal{E}_p$ then ${x^*}^{k-1} = a \in \mathcal{E}_p$. However, $a \in \mathbb{Z}_p^* \setminus \mathcal{E}_p$. □

The following example demonstrates the practical application of Theorem 5.

Example 4. *Let $k = 16$, $q = 9$. We can choose prime $p = 5$. Then we get*

$$x^{15} = 268.$$

Let $w(x) := x^{15} - 268$, then $w(2) \equiv 0(\mathrm{mod}\, 5^4)$, $w'(2) \equiv 0(\mathrm{mod}\, 5) \not\equiv 0(\mathrm{mod}\, 5^2)$. Thanks to Hensel's lemma, the equation $x^{15} = 268$ has a unique solution such that $x = 2 + o[5] \notin \mathcal{E}_5$.

5. A phase transition

If there are at least two distinct p-adic quasi Gibbs measures $\mu, \nu \in QG(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition* occurs. Moreover, if there is a sequence of sets A_n such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \to 0$ and $|\nu(A_n)|_p \to \infty$ as $n \to \infty$, then we say that there occurs *a strong phase transition*. If all p-adic quasi Gibbs measures are bounded (or unbounded) it is said that *a quasi phase transition* occurs (see [18]).

In the present paper, we only investigate the phase transition for the p-adic Potts model.

The lemma below plays a key role in demonstrating the boundedness of the measure.

Lemma 2. *Let $x^* \in Z_p^* \setminus \mathcal{E}_p$ be the solution of (18) in Theorem 5. Then for h_1 and h_2 corresponding to x^*, the following statements hold*

(1) $|h_1|_p = |h_2|_p = 1$,
(2) $|(\theta + q - 2)h_1 + 1|_p < 1$, $|(\theta + q - 2)h_2 + 1|_p = 1$,

here p and θ are defined as in the proof of Theorem 5.

Proof. Since $x^* \in Z_p^* \setminus \mathcal{E}_p$, one has $|q - 1|_p = 1$.

(1) By the notation given above $h_2 = {x^*}^k$, $h_1 = \dfrac{1 - \theta x^*}{(q-1)x^* - (\theta + q - 2)}$, we have

$$|h_2|_p = |x^*|_p^k = 1,$$

$$|h_1|_p = \frac{|1 - \theta x^*|_p}{|(q-1)(x^* - 1) - (\theta - 1)|_p} = 1,$$

(2)

$$|(\theta + q - 2)h_2 + 1|_p = |(\theta + q - 2){x^*}^k + 1|_p = 1,$$

$$|(\theta + q - 2)h_1 + 1|_p = \left|\frac{(\theta + q - 2)(1 - \theta x^*)}{(q-1)x^* - (\theta + q - 2)} + 1\right|_p = \frac{|(1-\theta)(\theta + q - 1)x^*|_p}{|(q-1)(x^* - 1) - (\theta - 1)|_p} < 1.$$

□

Theorem 6. *Weakly periodic p-adic quasi Gibbs measure H_{A_1} corresponding to pair $\{h_1, h_2\}$ in Lemma 2 is unbounded.*

Proof. Let $\mathcal{A}_1^{(n)}$, $\mathcal{A}_2^{(n)}$ be numbers of h_1, h_2 in V_n, respectively. (A) and (B) rules show that $\mathcal{A}_1^{(n)} \to \infty$, $\mathcal{A}_2^{(n)} \to \infty$ as $n \to \infty$. According to formula (66) in [34] and Lemma 2, we get

$$\lim_{n\to\infty} |\mu^{(n)}(\sigma)|_p = \lim_{n\to\infty} \frac{1}{|Z_1|_p} \cdot \frac{|h_1|_p^{\mathcal{A}_1^{(n)}} |h_2|_p^{\mathcal{A}_2^{(n)}}}{|(\theta + q - 2)h_1 + 1|_p^{\mathcal{A}_1^{(n)}} |(\theta + q - 2)h_2 + 1|_p^{\mathcal{A}_2^{(n)}}}$$
$$= +\infty.$$

□

Remark 2. $h_0 = (1, 1, ...1)$ *is a translation invariant solution of* (5)*. Using formula (66) in [34], it is easy to see that the measure μ_0 (corresponding to h_0) is bounded iff $|q|_p = 1$. For the considering case, due to Theorem* (5)*, one has $|q - i|_p = 1$, $i = 0, 1, 2$. We deduce that μ_0 is bounded.*

Theorem 7. *Let $k \geq 2$, $q \geq 3$ be fixed natural numbers. Then for q-state Potts model on the Cayley tree of order k, there exist a prime p and a parameter $\theta \in \mathcal{E}_p$ that ensure a phase transition phenomenon.*

Proof. The proof is straightforward from Theorem 6 and Remark 2. □

Remark 3. *One should note that in the present paper, we solved the first open problem from [34] not only for binary tree, but also for the Cayley tree of order $k \geq 2$.*

6. Conclusion

In this paper, we have investigated the existence of p-adic Gibbs measures for the q-state p-adic Potts model on a Cayley tree of order k. By analyzing the functional equations associated with the model, we have rigorously proven the existence of weakly periodic p-adic quasi Gibbs measures. Our main result demonstrates a phase transition phenomenon for this model whenever $q \geq 3$ and $k \geq 2$, under specific conditions governed by the prime p and the model parameter θ. Unlike traditional Archimedean models, the p-adic approach allows for the capturing of non-Archimedean symmetries and ultrametricity, which are essential features of disordered complex systems.

From a broader perspective, these findings have significant implications for nanoscience, particularly in the study of nanostructured macromolecules such as dendrimers and branched polymers. Since the topological structure of a Cayley tree perfectly reflects the hierarchical branching of these nanomaterials, our results provide theoretical insights into how energy or information is distributed across such networks. The identification of phase transitions in p-adic models helps in understanding the stability of thermodynamic states in nanoscale systems where interactions follow a hierarchical rather than a purely Euclidean logic. This is crucial for predicting self-assembly processes and critical phenomena in molecular electronics.

Furthermore, the transition dynamics identified in this work provide a mathematical foundation for modeling relaxation processes in complex molecular landscapes, such as protein folding basins. Future research may extend this approach to explore p-adic models with competing interactions, external fields, or more complex lattice structures to further uncover the rich phase behavior of non-Archimedean physical systems.

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