NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Irgashev B.Yu., et al. *Nanosystems: Phys. Chem. Math.*, 2026, **17** (2), 172–178.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2026-17-2-172-178

Original article

The Cauchy problem for a fourth-order equation involving a fractional derivative in the Caputo sense

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ABSTRACT In this article, the Cauchy problem in a half-plane is studied for a fourth-order inhomogeneous equation with a fractional derivative in the Caputo sense. The uniqueness of the solution is demonstrated using the Laplace transform. In constructing the solution, partial solutions expressed in terms of Wright functions are first found. Green's functions are then constructed using these partial solutions. The solution is constructed explicitly using the Green function. An explicit form of the fundamental solution is also obtained.

KEYWORDS Fourth-order equation, Cauchy problem, Caputo derivative, Wright function, asymptotics, fundamental solution, existence, uniqueness.

FOR CITATION Irgashev B.Yu. The Cauchy problem for a fourth-order equation involving a fractional derivative in the Caputo sense. *Nanosystems: Phys. Chem. Math.*, 2026, **17** (2), 172–178.

1. Introduction and statement of the problem

In recent decades, fractional calculus has evolved from a field of purely mathematical research into a powerful tool for modeling real physical processes. Fractional derivatives become particularly significant when describing phenomena in media with complex internal structures, memory, and nonlocal properties. As noted in [1], fractional dynamics investigates the behavior of objects and systems characterized by power-law nonlocality, power-law long-term memory, or fractal properties. Classical continuum models based on integer-order derivatives often prove inadequate when transitioning to the nanoscale, where effects of structural heterogeneity, anomalous transport, and viscoelastic relaxation begin to dominate. Work [2] provides a rigorous stochastic justification for this fact: the non-exponential relaxation of a single nanoobject naturally leads to the description of nanoparticle kinetics using differential equations with fractional derivatives.

It is noteworthy that the equation studied in this work is of particular interest for nanomechanics. Fourth-order equations traditionally describe the bending vibrations of beams, membranes, and plates [3]. In the nanoscale range, such models are applied to analyze the dynamics of nanobeams, nanofilms, and filamentary nanostructures situated in complex viscoelastic media. Study [4] investigates waves in magneto-elastic nanobeams using a fractional derivative to model internal damping, while nonlocal elasticity theory allows for the consideration of nanoscale size effects. In [5], a new mathematical model of thermoelasticity is proposed for analyzing coupled thermomechanical waves in functionally graded viscoelastic nanobeams. Works [6–8] also demonstrate that introducing fractional derivatives effectively accounts for the rheological properties of materials: [6] investigates nonlinear vibrations of beams made of fractional-order Zener viscoelastic material; [7] examines the vibrations of a nanobeam on a fractional-order viscoelastic Winkler-Pasternak foundation using nonlocal elasticity theory. In study [9], dedicated to an inverse problem for the heat conduction equation with a load, variable coefficients, and fractional integral operators, a mathematical method for analyzing heat transfer processes at the nanoscale is developed. Thus, the investigation of equations of type (1) holds not only theoretical but also direct applied significance for nanotechnologies.

This paper investigates the initial value problem for a fourth-order equation involving the Caputo fractional derivative. The Cauchy problem for second-order equations with a fractional derivative with respect to spatial variables (diffusion-wave equations) has been studied in considerable detail in works [10–14]. For equations with the Laplace operator and the Dzhrbashyan-Nersesyan fractional derivative of order $(0, 2)$, analogous results were obtained in [15], and for equations of odd order – in [16]. In [17], a solution to the Cauchy problem for an even-order equation with the Riemann-Liouville derivative of order $(0, 1)$ was obtained. A boundary value problem with the Caputo fractional derivative with applications in nanotechnologies was also investigated in [18].

Closely related to the topic of this work is the study [19], where the Cauchy problem for a homogeneous fourth-order equation with the Caputo fractional derivative was reduced to a Schrödinger equation by lowering the order of the fractional derivative. However, [19] did not consider the inhomogeneous equation, and the question of solution uniqueness remained open.

In contrast to the previous works, this paper proves a uniqueness theorem for the solution of the Cauchy problem, constructs particular solutions using the Wright function and, based on them, formulates the Green function, obtains an explicit integral representation of the solution for both the homogeneous and inhomogeneous equations, and provides the explicit form of the fundamental solution.

The proposed approach can be extended to equations of arbitrary even order, which opens up prospects for further research in the field of mathematical modeling of nanosystems, including problems concerning the dynamics of nanobeams, nanofilms, membranes, and other nanostructures in complex media with memory and nonlocal properties.

Let us proceed directly to the formulation of the problem. Consider the equation in the domain $\Omega = \{(x,y) : -\infty < x < \infty, 0 < y \leq T\}$

$$L[u] \equiv \left({}_C D_{0y}^{\alpha} + \frac{\partial^4}{\partial x^4} \right) u = f\left(x,y\right), \tag{1}$$

where ${}_C D_{0y}^{\alpha}$ is the operator of fractional (in the Caputo sense) integro-differentiation of order α, $1 < \alpha < 2$, defined by the relation

$${}_C D_{0y}^{\alpha} u = D_{0y}^{\alpha-2} u_{yy} = \frac{1}{\Gamma\left(2-\alpha\right)} \int\limits_0^y \frac{\frac{\partial^2}{\partial \tau^2} u\left(x,\tau\right)}{\left(y-\tau\right)^{\alpha-1}} d\tau,$$

here $D_{0y}^{\alpha-2}$ is the operator of fractional differentiation in the sense of Riemann-Liouville [20].

A regular solution of equation (1) in the domain Ω is defined as a function $u = u(x,y)$ satisfying the conditions: $\dfrac{\partial^k u}{\partial x^k} \in C\left(\overline{\Omega}\right),\ k = \overline{0,3};\ \dfrac{\partial^4 u}{\partial x^4} \in C\left(\Omega\right);\ \dfrac{\partial^m u}{\partial y^m} \in C\left(\overline{\Omega}\right),\ m = \overline{0,1};\ {}_C D_{0y}^{\alpha} u \in C\left(\Omega\right).$

Cauchy Problem. Find a regular solution $u(x,y)$ of equation (1) in the domain Ω satisfying the initial conditions

$$\lim_{y \to +0} u\left(x,y\right) = \varphi\left(x\right), \lim_{y \to +0} \frac{\partial u\left(x,y\right)}{\partial y} = \psi\left(x\right), \tag{2}$$

where $\varphi\left(x\right), \psi\left(x\right)$ are given functions.

2. Uniqueness of solution

Theorem 1. The Cauchy problem with homogeneous initial conditions, in the class of functions satisfying the conditions $\left| \dfrac{\partial^m u}{\partial y^m} \right| \leq M e^{\sigma y},\ 0 < M = const.,\ m = 0,1$, where σ is a fixed constant, has only the trivial solution.

Proof. Let $u(x,y)$ be a solution of the Cauchy problem with zero initial conditions. Consider the Laplace transform of this function

$$v\left(x,p\right) = \int\limits_0^{+\infty} e^{-py} u\left(x,y\right) dy,$$

hence

$$\left| v\left(x,p\right) \right| \leq M \int\limits_0^{+\infty} e^{-(p-\sigma)y} dy = \frac{M}{p-\sigma}.$$

Let us find the fourth-order partial derivative with respect to the variable x of the transform

$$\frac{\partial^4 v\left(x,p\right)}{\partial x^4} = \int\limits_0^{+\infty} e^{-py} \frac{\partial^4 u\left(x,y\right)}{\partial x^4} dy = -\int\limits_0^{+\infty} e^{-py} {}_C D_{0y}^{\alpha} u\, dy = -p^{\alpha} v\left(x,p\right).$$

From this, we obtain the problem

$$\begin{cases} \dfrac{\partial^4 v\left(x,p\right)}{\partial x^4} = -p^{\alpha} v\left(x,p\right), \\ \left| v\left(x,p\right) \right| \leq \dfrac{M}{p-\sigma}. \end{cases} \tag{3}$$

The general solution of problem (3) has the form

$$v\left(x,p\right) = e^{Bx}\left(c_1 \cos Bx + c_2 \sin Bx\right) + e^{-Bx}\left(c_3 \cos Bx + c_4 \sin Bx\right), \quad B = \frac{\sqrt{2}}{2} p^{\frac{\alpha}{4}}.$$

By virtue of the condition of problem (3), we obtain

$$v\left(x,p\right) \equiv 0,$$

and consequently

$$u(x,y) \equiv 0.$$

Theorem 1 is proved.

3. Construction of the Solution

First, we construct some particular solutions. The following lemma holds.

Theorem 2. Functions of the form

$$u_j\left(x,y\right)=y^b\phi\left(-\frac{\alpha}{4},b+1;\lambda_j xy^{-\frac{\alpha}{4}}\right),\quad j=1,2,$$

where

$$\phi\left(-\delta,\varepsilon;z\right)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma\left(-\delta k+\varepsilon\right)}$$

is the Wright function;

$$x,y>0,b\in R,\lambda_1=e^{\frac{3\pi}{4}i},\lambda_2=e^{-\frac{3\pi}{4}i},$$

satisfy equation (1).

Proof. By direct calculation, we obtain

$$\frac{\partial^4 u_j\left(x,y\right)}{\partial x^4}=-y^{b-\alpha}\phi\left(-\frac{\alpha}{4},b+1-\alpha;\lambda_j xy^{-\frac{\alpha}{4}}\right), \tag{4}$$

$$\frac{\partial^2 u_j\left(x,y\right)}{\partial y^2}=y^{b-2}\phi\left(-\frac{\alpha}{4},b-1;\lambda_j xy^{-\frac{\alpha}{4}}\right).$$

Furthermore, since

$$\frac{1+\frac{\alpha}{4}}{2}\pi<\left|\arg\lambda_j\right|<\pi,$$

using formula [16]

$$D_{0y}^{\nu}y^{\varepsilon-1}\phi\left(-\delta,\varepsilon;\lambda y^{-\delta}\right)=y^{\varepsilon-\nu-1}\phi\left(-\delta,\varepsilon-\nu;\lambda y^{-\delta}\right), \tag{5}$$

for

$$0<\delta<1,\quad \frac{1+\delta}{2}\pi<\left|\arg\lambda\right|\le\pi;\quad \varepsilon,\nu\in R,$$

we have

$$ {}_C D_{0y}^{\alpha}u_j=D_{0y}^{\alpha-2}\left\{y^{b-2}\phi\left(-\frac{\alpha}{4},b-1;\lambda_j xy^{-\frac{\alpha}{4}}\right)\right\}=$$

$$y^{b-\alpha}\phi\left(-\frac{\alpha}{4},b+1-\alpha;\lambda_j xy^{-\frac{\alpha}{4}}\right). \tag{6}$$

From (4) and (6) the validity of Theorem 2 follows.

Theorem 2 is proved.

Let us recall some known properties (see [16]) of the Wright function:

$$\frac{d}{dz}\phi\left(\delta,\varepsilon;z\right)=\phi\left(\delta,\varepsilon+\delta;z\right),\quad (\delta>-1). \tag{7}$$

$$\left|\phi\left(-\delta,\varepsilon;z\right)\right|\le C\exp\left(-\nu|z|^{\frac{1}{1-\delta}}\right),\quad |z|\to\infty, \tag{8}$$

where

$$C=C\left(\delta,\varepsilon,\nu\right),\quad \delta\in\left(0,1\right),\quad \varepsilon\in R,$$

$$\nu<\left(1-\delta\right)^{\frac{\delta}{1-\delta}}\cos\frac{\pi-\left|\arg z\right|}{1-\delta};\quad \frac{1+\delta}{2}\pi<\left|\arg z\right|\le\pi.$$

$$\int_0^{+\infty}\phi\left(-\delta,\varepsilon;\lambda x\right)dx=-\frac{1}{\lambda\Gamma\left(\delta+\varepsilon\right)},\quad \frac{1+\delta}{2}\pi<\left|\arg\lambda\right|\le\pi. \tag{9}$$

Following works [16],[17], consider the following function

$$G_b\left(x-\xi,y-\eta\right)=-\frac{\lambda_1(y-\eta)^b}{4}\phi\left(-\frac{\alpha}{4},b+1;\lambda_1\frac{|x-\xi|}{(y-\eta)^{\frac{\alpha}{4}}}\right)-$$

$$-\frac{\lambda_2(y-\eta)^b}{4}\phi\left(-\frac{\alpha}{4},b+1;\lambda_2\frac{|x-\xi|}{(y-\eta)^{\frac{\alpha}{4}}}\right),\quad b\in R. \tag{10}$$

The following theorem holds.

Theorem 3. Let $\varphi(x) \in C^4(\mathbb{R})$, $\psi(x) \in C(\mathbb{R})$ and $\left|\varphi^{(k)}(x)\right|, |\psi(x)| \le M = \text{const.}$, $k = 0, \ldots, 4$. Then the representation

$$u(x,y) = \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}}(x-\xi, y)\,\varphi(\xi)\,d\xi + \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}+1}(x-\xi, y)\,\psi(\xi)\,d\xi, \tag{11}$$

is a regular solution of the following Cauchy problem

$$\begin{cases} {}_C D_{0y}^{\alpha} u + u_{xxxx} = 0, \\ \lim\limits_{y\to+0} u(x,y) = \varphi(x), \quad \lim\limits_{y\to+0} u_y(x,y) = \psi(x). \end{cases}$$

Proof. First, we show that (11) satisfies the initial conditions. We have

$$\int\limits_{-\infty}^{+\infty} y^{-\frac{\alpha}{4}} \phi\left(-\frac{\alpha}{4}, -\frac{\alpha}{4}+1, \lambda_j \frac{|x-\xi|}{y^{\frac{\alpha}{4}}}\right) \varphi(\xi)\,d\xi = \left(j = 1,2;\ t = \frac{|x-\xi|}{y^{\frac{\alpha}{4}}}\right) =$$

$$= \int\limits_{0}^{+\infty} \phi\left(-\frac{\alpha}{4}, -\frac{\alpha}{4}+1, \lambda_j t\right)\left\{\varphi\left(x + y^{\frac{\alpha}{4}} t\right) + \varphi\left(x - y^{\frac{\alpha}{4}} t\right)\right\} dt,$$

now, taking into account (8), (9) and the restrictions on the function φ from the conditions of Theorem 2, we obtain

$$\lim_{y\to+0} \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}}(x-\xi, y)\,\varphi(\xi)\,d\xi = \varphi(x).$$

Furthermore, considering (5) and using (7), we have

$$\frac{\partial}{\partial y} \int\limits_{-\infty}^{+\infty} y^{-\frac{\alpha}{4}} \phi\left(-\frac{\alpha}{4}, -\frac{\alpha}{4}+1, \lambda_j \frac{|x-\xi|}{y^{\frac{\alpha}{4}}}\right) \varphi(\xi)\,d\xi =$$

$$= \frac{-2\phi\left(-\frac{\alpha}{4}, \frac{\alpha}{2}, 0\right) \varphi_{tt}(x)\, y^{-1+\frac{\alpha}{2}}}{\lambda_j^3}$$

$$+ \frac{y^{-1+\alpha}}{\lambda_j^4} \int\limits_{0}^{+\infty} \left\{\varphi_{tttt}\left(x + y^{\frac{\alpha}{4}} t\right) + \varphi_{tttt}\left(x - y^{\frac{\alpha}{4}} t\right)\right\} \phi\left(-\frac{\alpha}{4}, \frac{3\alpha}{4}, \lambda_j t\right) dt,$$

now, taking into account (8)-(10), the restrictions on the function φ from the conditions of Theorem 2, and $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 0$, we obtain

$$\lim_{y\to+0} \frac{\partial}{\partial y} \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}}(x-\xi, y)\,\varphi(\xi)\,d\xi = 0.$$

After performing similar transformations and calculations for the second term in representation (11), we obtain

$$\lim_{y\to+0} \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}+1}(x-\xi, y)\,\psi(\xi)\,d\xi = 0,$$

$$\lim_{y\to+0} \frac{\partial}{\partial y} \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}+1}(x-\xi, y)\,\psi(\xi)\,d\xi = \psi(x).$$

Thus, representation (11) satisfies the initial conditions of the Cauchy problem.

Now we show that representation (11) satisfies the homogeneous equation (1). Taking into account (5) and formula [20]

$${}_C D_{0y}^{\alpha} f = D_{0y}^{\alpha} f - \frac{f(0)}{\Gamma(1-\alpha)} y^{-\alpha} - \frac{f'(0)}{\Gamma(2-\alpha)} y^{1-\alpha},$$

we have

$${}_C D_{0y}^{\alpha} u(x,y) = D_{0y}^{\alpha} \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}}(x-\xi, y)\,\varphi(\xi)\,d\xi - \frac{y^{-\alpha}}{\Gamma(1-\alpha)} \lim_{y\to+0} \int\limits_{-\infty}^{+\infty} G_{-\frac{\alpha}{4}}(x-\xi, y)\,\varphi(\xi)\,d\xi -$$

$$-\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}\lim_{y\to+0}\frac{\partial}{\partial y}\int\limits_{-\infty}^{+\infty}G_{-\frac{\alpha}{4}}(x-\xi,y)\,\varphi(\xi)\,d\xi=$$

$$=\int\limits_{-\infty}^{+\infty}G_{-\frac{5\alpha}{4}}(x-\xi,y)\,\varphi(\xi)\,d\xi-\frac{y^{-\alpha}\varphi(x)}{\Gamma(1-\alpha)}. \tag{12}$$

Furthermore, by direct calculation we obtain

$$\frac{\partial^4}{\partial x^4}\left(\int\limits_{-\infty}^{+\infty}G_{-\frac{\alpha}{4}}(x-\xi,y)\,\varphi(\xi)\,d\xi\right)=$$

$$=\frac{y^{-\alpha}\varphi(x)}{\Gamma(1-\alpha)}-\int\limits_{-\infty}^{+\infty}y^{-\frac{5\alpha}{4}}\phi\left(-\frac{\alpha}{4},-\frac{5\alpha}{4}+1,\lambda_j\frac{|x-\xi|}{y^{\frac{\alpha}{4}}}\right)\varphi(\xi)\,d\xi. \tag{13}$$

From (12) and (13) we obtain

$${}_CD_{0y}^{\alpha}\left(\int\limits_{-\infty}^{+\infty}G_{-\frac{\alpha}{4}}(x-\xi,y)\,\varphi(\xi)\,d\xi\right)+\frac{\partial^4}{\partial x^4}\left(\int\limits_{-\infty}^{+\infty}G_{-\frac{\alpha}{4}}(x-\xi,y)\,\varphi(\xi)\,d\xi\right)=0.$$

It is shown similarly that

$${}_CD_{0y}^{\alpha}\left(\int\limits_{-\infty}^{+\infty}G_{-\frac{\alpha}{4}+1}(x-\xi,y)\,\psi(\xi)\,d\xi\right)+\frac{\partial^4}{\partial x^4}\left(\int\limits_{-\infty}^{+\infty}G_{-\frac{\alpha}{4}+1}(x-\xi,y)\,\psi(\xi)\,d\xi\right)=0.$$

Theorem 3 is proved.

Theorem 4. Let $f(x,y)=D_{0y}^{-\gamma}g(x,y)$, $\gamma>0$, $g(x,y)\in C\left(\overline{\Omega}\right)$, $|g(x,y)|\le M=\text{const}$. Then the representation

$$u(x,y)=\int\limits_{0}^{y}d\eta\int\limits_{-\infty}^{+\infty}G_{\frac{3\alpha}{4}-1}(x-\xi,y-\eta)\,f(\xi,\eta)\,d\xi, \tag{14}$$

is a solution of the problem

$$\begin{cases} {}_CD_{0y}^{\alpha}u(x,y)+u_{xxxx}(x,y)=f(x,y), \\ \lim\limits_{y\to+0}u(x,y)=\lim\limits_{y\to+0}u_y(x,y)=0. \end{cases}$$

Proof. First, we show that representation (14) satisfies equation (1). We have

$$\frac{\partial}{\partial x^3}\int\limits_{0}^{y}d\eta\int\limits_{-\infty}^{+\infty}G_{\frac{3\alpha}{4}-1}(x-\xi,y-\eta)\,f(\xi,\eta)\,d\xi=$$

$$=\int\limits_{0}^{y}d\eta\int\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x^3}G_{\frac{3\alpha}{4}-1}(x-\xi,y-\eta)\,f(\xi,\eta)\,d\xi,$$

taking into account the integration by parts formula [16]

$$\int\limits_{a}^{b}g(s)\,D_{as}^{\alpha}h(s)\,ds=\int\limits_{a}^{b}h(s)\,D_{bs}^{\alpha}g(s)\,ds,\quad \alpha\le 0,$$

we obtain

$$\frac{\partial}{\partial x}\int\limits_{0}^{y}d\eta\int\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x^3}G_{\frac{3\alpha}{4}-1}(x-\xi,y-\eta)\,f(\xi,\eta)\,d\xi=$$

$$\frac{\partial}{\partial x}\int\limits_{0}^{y}d\eta\int\limits_{-\infty}^{+\infty}g(\xi,\eta)\,D_{y\eta}^{-\gamma}\left\{\frac{\partial}{\partial x^3}G_{\frac{3\alpha}{4}-1}(x-\xi,y-\eta)\right\}d\xi=$$

$$=f(x,y)-\int\limits_{0}^{y}d\eta\int\limits_{-\infty}^{+\infty}G_{\gamma-\frac{\alpha}{4}-1}(x-\xi,y-\eta)\,g(\xi,\eta)\,d\xi. \tag{15}$$

Next,

$$\frac{\partial^2}{\partial y^2}\int\limits_{-\infty}^{+\infty} d\xi \int\limits_0^y G_{\frac{3\alpha}{4}-1}\left(x-\xi, y-\eta\right) f\left(\xi,\eta\right) d\eta =$$

$$= \int\limits_{-\infty}^{+\infty} d\xi \int\limits_0^y G_{\frac{3\alpha}{4}-3+\gamma}\left(x-\xi, y-\eta\right) g\left(\xi,\eta\right) d\eta,$$

hence

$$D_{0y}^{\alpha-2}\int\limits_{-\infty}^{+\infty} d\xi \int\limits_0^y G_{\frac{3\alpha}{4}-3+\gamma}\left(x-\xi, y-\eta\right) g\left(\xi,\eta\right) d\eta =$$

$$= \int\limits_{-\infty}^{+\infty} d\xi \int\limits_0^y g\left(\xi,\eta\right) G_{-\frac{\alpha}{4}-1+\gamma}\left(x-\xi, t-\eta\right) d\eta. \tag{16}$$

From (15) and (16), it follows that

$$\left({}_CD_{0y}^{\alpha}+\frac{\partial^4}{\partial x^4}\right)\left\{\int\limits_0^y d\eta \int\limits_{-\infty}^{+\infty} G_{\frac{3\alpha}{4}-1}\left(x-\xi, y-\eta\right) f\left(\xi,\eta\right) d\xi\right\} = f\left(x,y\right).$$

We show that representation (14) satisfies the zero initial conditions. Let

$$I\left(x,y\right) = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_0^y \left(y-\eta\right)^{\frac{3\alpha}{4}-1}\phi\left(-\frac{\alpha}{4},\frac{3\alpha}{4};\lambda_j\frac{|x-\xi|}{\left(y-\eta\right)^{\frac{\alpha}{4}}}\right) f\left(\xi,\eta\right) d\eta,$$

make the change of variables

$$z=\frac{|x-\xi|}{\left(y-\eta\right)^{\frac{\alpha}{4}}}, \quad \xi = x \pm z(y-\eta)^{\frac{\alpha}{4}}, \quad d\xi = \pm(y-\eta)^{\frac{\alpha}{4}}dz,$$

we have

$$I\left(x,y\right) = \int\limits_0^y \left(y-\eta\right)^{\alpha-1} d\eta \times$$

$$\int\limits_0^{+\infty} \phi\left(-\frac{\alpha}{4},\frac{3\alpha}{4};\lambda_j z\right)\left(f\left(x+z(y-\eta)^{\frac{\alpha}{4}},\eta\right)+f\left(x-z(y-\eta)^{\frac{\alpha}{4}},\eta\right)\right)dz,$$

hence

$$\lim_{y\to+0} I\left(x,y\right)=0.$$

Similarly, we will have that

$$\lim_{y\to+0}\frac{\partial I\left(x,y\right)}{\partial y}=0.$$

Theorem 4 is proved.

By Theorem 4, the function $G_{\frac{3\alpha}{4}-1}(x-\xi, y-\eta)$ is the fundamental solution of equation (1).

4. Conclusion

In this paper, we study the Cauchy problem for a fourth-order equation involving a fractional derivative in the sense of Caputo. In the case $\alpha = 2$, we have an equation for beam vibrations. The solution is constructed explicitly by finding Green's functions. Sufficient conditions for the existence and uniqueness of the solution are found. It should be noted that the question of the uniqueness of the solution to the Cauchy problem for the beam vibration equation remained open (see [3]). A fundamental solution to equation (1) is also constructed. In the future, the obtained results can be used to study boundary value problems for inhomogeneous linear and nonlinear equations, equations with lower-order terms, inverse problems with an unknown right-hand side, problems with nonlocal conditions, which have various applications in nanosystems. It should be noted that the problem under study has not only theoretical but also applied significance, since fractional integro-differential operators are widely used in mathematical models of nanosystem dynamics. In particular, among mathematical models of transport-diffusion, fractional differential calculus stands out as a tool for describing transport processes in media with a complex structure.

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Submitted 30 January 2026; revised 14 March 2026; accepted 15 March 2026

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Conflict of interest: the authors declare no conflict of interest.