

Some new oscillation criteria of second-order nonlinear perturbed delay differential equations

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ABSTRACT In this paper, a class of second-order perturbed delay differential equation of the form

$$(\eta(\kappa)u'(\kappa))' + f_1(\kappa, u(\tau(\kappa))) = f_2(\kappa, u(\kappa), u'(\kappa))$$

is considered. Employing the transform technique, the studied equation changed into a binomial type equation and then using Riccati transform, comparison theorem along with integral averaging method some new oscillation criteria are obtained. Examples are provided to show the importance and novelty of the main results.

KEYWORDS second-order, perturbed differential equations, delay, oscillation.

ACKNOWLEDGEMENTS The authors sincerely thank the anonymous referee for the careful evaluation, insightful remarks, and constructive suggestions, which have greatly enhanced the structure and overall quality of the manuscript.

FOR CITATION Ramya B., Srinivasan R. Some new oscillation criteria of second-order nonlinear perturbed delay differential equations. *Nanosystems: Phys. Chem. Math.*, 2026, **17** (3), 269–279.

1. Introduction

The qualitative analysis of nonlinear differential equations has been an active area of research due to its fundamental importance in mathematics and its wide range of applications in science and engineering. Among various topics, the oscillatory behavior of solutions has received considerable attention because it provides valuable information about the long-term dynamics and stability of mathematical models. In particular, delay differential equations have emerged as an effective tool for describing processes in which the present state depends not only on the current conditions but also on past states. Such equations arise naturally in physics, chemistry, biology, control theory, population dynamics, and many other applied fields.

Recent advances in nanoscience and nanotechnology have further increased interest in the study of delay differential equations. Many nanoscale phenomena involve memory effects, delayed interactions, relaxation processes, and nonlinear responses that cannot be adequately described by ordinary differential equations. Mathematical models incorporating delay terms have therefore become important in the analysis of nanoelectromechanical systems, nanosensors, nanomechanical resonators, quantum devices, and nanostructured materials. The investigation of oscillatory properties of such models is essential for understanding stability, resonance behavior, signal transmission, and dynamic performance in nanoscale systems.

Motivated by these developments, numerous researchers have established oscillation criteria for various classes of second-order nonlinear delay differential equations. Although significant progress has been made, many existing results require restrictive assumptions and do not cover several important nonlinear perturbed models. Consequently, the development of new oscillation conditions remains an important and challenging problem.

In this paper, we establish several new oscillation criteria for a class of second-order nonlinear perturbed delay differential equations. The obtained results extend and improve a number of related results available in literature. The proposed approach provides effective sufficient conditions for oscillation and broadens the applicability of existing oscillation theory. Illustrative examples are presented to demonstrate the usefulness and effectiveness of the theoretical findings. see for example [1–8] and the references therein.

In the last century, oscillation theory of differential equations has developed quickly and played an important role in the qualitative theory of differential equations. The study of oscillation theory plays an important role in physical science and technology; for example, electromagnetic vibration in radio technology and optical science, self-existed vibration in control systems, sound vibration and also in the research of a lossless high-speed computer network and physical sciences. All of this different phenomena can be unified into oscillation theory through an oscillation equation, see the monographs [9–12] and the references quoted therein.

In view of the above facts, in this paper we study the oscillatory properties of solutions of the second-order perturbed delay differential equation of the form

$$(\eta(\kappa)u'(\kappa))' + f_1(\kappa, u(\tau(\kappa))) = f_2(\kappa, u(\kappa), u'(\kappa)), \kappa \in I_0, \quad (1.1)$$

where $I_0 = [\kappa_0, \infty)$, $\kappa_0 > 0$, subject to the following conditions :

(H_1) $\eta \in C'(I_0, (0, \infty))$ and

$$\int_{\kappa_0}^{\infty} \frac{1}{\eta(\kappa)} d\kappa = \infty;$$

(H_2) $f_1 \in C(I_0 \times \mathbb{R}, \mathbb{R})$, $f_2 \in C(I_0 \times \mathbb{R}^2, \mathbb{R})$, $xf_1(\kappa, x) > 0$ and $xf_2(\kappa, x, y) > 0$ for $x \neq 0$;

(H_3) There exist functions $\phi_1, \phi_2 \in C(I_0, \mathbb{R}_+)$ such that

$$\frac{f_1(\kappa, x)}{x^\alpha} \geq \phi_1(\kappa), \quad \frac{f_2(\kappa, x, y)}{x} \leq \phi_2(\kappa),$$

where α is a ratio of odd natural numbers;

(H_4) $\tau \in (I_0, \mathbb{R})$, $\tau'(\kappa) \geq 0$, $\tau(\kappa) \leq \kappa$ and $\lim_{\kappa \rightarrow \infty} \tau(\kappa) = \infty$.

By a solution of (1.1), we denote a function $u \in C'(I_0, \mathbb{R})$ such that $\eta u' \in C'(I_0, \mathbb{R})$ and satisfies (1.1) on I_0 . We consider those solutions u of (1.1) such that $\sup\{|u(\kappa)| : \kappa \geq T\} > 0$ for $T \geq T_u$. Such a solution u of (1.1) is said to be oscillatory if the set $\{\kappa \in [\kappa_0, \infty) : u(\kappa) = 0\}$ is unbounded; otherwise it is said to be nonoscillatory.

Oscillatory properties of perturbed differential equations was first initiated by Kartsatos [13–15] and proved that the perturbed equation

$$u^{(n)}(\kappa) + p(\kappa)f_1(u(\kappa)) = q(\kappa), \quad (1.2)$$

is oscillatory if the unforced equation is oscillatory and there exists an oscillatory function $R(\kappa)$ vanishing at infinity such that $R^{(n)}(\kappa) = q(\kappa)$. For $n = 2$, the authors in [16] obtained oscillation criteria for (1.2) without using the above assumption.

Studying oscillation of the general perturbed differential equation

$$(\eta(\kappa)u')' + f_1(\kappa, u) = f_2(\kappa, u, u'), \quad (1.3)$$

the authors in [17–20] assumed certain monotonicity properties of the functions f_1 and f_2 . Similar conditions have been also imposed in [21–24] for obtaining oscillation criteria for the perturbed equation with damping term

$$(\eta(\kappa)u')' + p(\kappa)u' + f_1(\kappa, u) = f_2(\kappa, u, u'). \quad (1.4)$$

In [25–30] the authors studied the oscillatory properties of the following second-order perturbed differential equation

$$(\eta(\kappa)\psi(u(\kappa))|f(u'(\kappa))|^{\alpha-1}f(u'(\kappa)))' + q(\kappa)f_1(u(\kappa)) = f_2(\kappa, u(\kappa), u'(\kappa)), \quad (1.5)$$

or its particular cases using generalized Riccati transform technique along with other well-known methods.

Recently, in [31], the authors obtained conditions for the oscillation of solutions of the perturbed functional differential equation

$$\delta u^{(n)}(\kappa) + f_1(\kappa, u(\tau(\kappa))) = f_2(\kappa, u(\tau(\kappa))), \quad (1.6)$$

where $n \geq 1$ and $\delta = \pm 1$.

From the review of literature, we see that most of the previous studies focussed on the oscillatory behavior of solutions of perturbed differential equations without delay except [31]. Therefore, in this work, we investigate the effect of introducing a delay on the oscillatory behavior of solutions of the equation (1.1). We use different technique to obtain the oscillation criteria for the equation (1.1). Hence the results obtained here are new and complement to the oscillation theory of perturbed differential equations. Some numerical examples are provided to illustrate the importance and novelty of the main results. Because α is a ratio of odd positive integers, considering nonoscillatory solutions of (1.1), we can restrict our attention only to positive ones.

2. Auxiliary results

In this section, we present some auxiliary results that will be needed to get our main theorems. First, we transform the perturbed equation (1.1) into a binomial-type equation using a positive solution of the related auxiliary second-order linear ordinary differential equation

$$(\eta(\kappa)x'(\kappa))' = \phi_2(\kappa)x(\kappa), \kappa \geq \kappa_0. \quad (2.1)$$

Lemma 2.1. *Let (H_1) and (H_3) hold. Then differential equation (2.1) possesses a positive decreasing solution on I_0 .*

Proof. The proof is similar to Theorem 2.46 of [32] and so the details are skipped. $\square$

Lemma 2.2. Let $(H_1) - (H_4)$ hold and let $u(\kappa)$ be a nonoscillatory solution of (1.1). Then (1.1) can be transformed as

$$(\eta(\kappa)u'(\kappa))' - \phi_2(\kappa)u(\kappa) + \phi_1(\kappa)u^\alpha(\tau(\kappa)) \leq 0, \quad \kappa \geq \kappa_1, \quad (2.2)$$

for some $\kappa_1 \in I_0$.

Proof. Let u be a nonoscillatory solution of (1.1) on I_0 . Since α is a ratio of odd positive integer, it is enough to assume that $u(\kappa)$ is positive for all $\kappa \geq \kappa_1$ for some $\kappa_1 \in I_0$ (the case $u(\kappa) < 0$ can be treated similarly and so omitted). Now, using the hypotheses, we see that equation (1.1) can be transformed as

$$(\eta(\kappa)u'(\kappa))' + \phi_1(\kappa)u^\alpha(\tau(\kappa)) \leq \phi_2(\kappa)u(\kappa), \quad \kappa \geq \kappa_1.$$

□

Our next result is based on an equivalent representation of the differential operator

$$L(u(\kappa)) = (\eta(\kappa)u'(\kappa))' - \phi_2(\kappa)u(\kappa) \quad (2.3)$$

in terms of a positive solution $x(\kappa)$ of (2.1).

Lemma 2.3. Let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . Then the operator (2.3) can be represented as

$$L(u(\kappa)) = \frac{1}{x(\kappa)} \left(\eta(\kappa)x^2(\kappa) \left(\frac{u(\kappa)}{x(\kappa)} \right)' \right)'. \quad (2.4)$$

Proof. By direct calculation, we find that

$$\begin{aligned} L(u(\kappa)) &= \frac{1}{x(\kappa)} (\eta(\kappa)x(\kappa)u'(\kappa) - \eta(\kappa)u(\kappa)x'(\kappa))' \\ &= (\eta(\kappa)u'(\kappa))' - (\eta(x'(\kappa)))' \frac{u(\kappa)}{x(\kappa)} \\ &= (\eta(\kappa)u'(\kappa))' - \phi_2(\kappa)u(\kappa), \end{aligned}$$

where we have used the fact that $x(\kappa)$ is a positive solution of (2.1).

□

Define

$$w(\kappa) = \eta(\kappa)x^2(\kappa), \mu(\kappa) = \frac{u(\kappa)}{x(\kappa)} \quad \text{and} \quad Q(\kappa) = x(\kappa)\phi_1(\kappa)x^\alpha(\tau(\kappa)).$$

Lemma 2.4. Let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . If the equation

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa)\mu^\alpha(\tau(\kappa)) = 0, \quad \kappa \geq \kappa_1, \quad (2.5)$$

is oscillatory, then (1.1) is oscillatory.

Proof. Let $u(\kappa)$ be a nonoscillatory solution of (1.1) on I_0 . Since α is a ratio of odd positive integer, it is enough to take $u(\kappa) > 0$ and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$ for some $\kappa_1 \geq \kappa_0$. Using Lemma 2.2 and Lemma 2.3, we have

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa)\mu^\alpha(\tau(\kappa)) \leq 0.$$

But, in view of a result in [33], it leads to a contradiction.

□

Note that from (H_1) and decreasing nature of $x(\kappa)$, we see that

$$\int_{\kappa_1}^{\infty} \frac{1}{w(\kappa)} d\kappa = \int_{\kappa_1}^{\infty} \frac{1}{\eta(\kappa)x^2(\kappa)} d\kappa \geq \frac{1}{x^2(\kappa_1)} \int_{\kappa_1}^{\infty} \frac{1}{\eta(\kappa)} d\kappa = \infty. \quad (2.6)$$

Thus, equation (2.5) is in the canonical form.

Next, we present the structure of nonoscillatory solutions of (2.5).

Lemma 2.5. Let $x(\kappa)$ be a positive decreasing solution of (2.1), then the corresponding function $\mu(\kappa)$ satisfies

$$\mu(\kappa) > 0, \quad w(\kappa)\mu'(\kappa) > 0, \quad (w(\kappa)\mu'(\kappa))' \leq 0 \quad (2.7)$$

for all $\kappa \geq \kappa_1$.

Proof. Let $u(\kappa)$ be an eventually positive solution of (1.1), that is, $u(\kappa) > 0$, $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$, for some $\kappa_1 \geq \kappa_0$. Since $x(\kappa) > 0$ for all $\kappa \geq \kappa_0$, it follows that $\mu(\kappa) = \frac{u(\kappa)}{x(\kappa)} > 0$ for all $\kappa \geq \kappa_1$. From Lemma 2.4, we see that (2.5) holds for all $\kappa \geq \kappa_1$. Hence, $(w(\kappa)\mu'(\kappa))' \leq 0$ for all $\kappa \geq \kappa_1$. Moreover, by condition (2.6), it follows that $w(\kappa)\mu'(\kappa) > 0$ for all $\kappa \geq \kappa_1$. This completes the proof. □

3. Main results

In this section, we derive oscillation results for equation (1.1) with the help of equation (2.5).

Theorem 3.1. *Let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . If*

$$\int_{\kappa_1}^{\infty} Q(\kappa) d\kappa = \infty, \quad (3.1)$$

for all $\kappa_1 \in I_0$, then equation (1.1) is oscillatory.

Proof. Assume for the sake of contradiction that $u(\kappa)$ is an eventually positive (nonoscillatory) solution of (1.1), that is $u(\kappa) > 0$, and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$ for some $\kappa_1 \geq \kappa_0$. Then the corresponding function $\mu(\kappa) > 0$ and $\mu(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$.

From Lemma 2.5, the function $\mu(\kappa)$ satisfies condition (2.7) for all $\kappa \geq \kappa_1$. Since $\mu(\kappa)$ is positive and increasing, there exists a constant $M > 0$ such that $\mu(\kappa) \geq M$ for all $\kappa \geq \kappa_2 \geq \kappa_1$. Using this in (2.5) and then integrating it from κ_2 to κ , we obtain

$$M^\alpha \int_{\kappa_2}^{\kappa} Q(s) ds \leq w(\kappa_2) \mu'(\kappa_2) - w(\kappa) \mu(\kappa).$$

As $\kappa \rightarrow \infty$, it follows that

$$\int_{\kappa_2}^{\infty} Q(\kappa) d\kappa < \infty,$$

which contradicts (3.1). □

Remark 3.1. *Note that Theorem 3.1 is independent of α and the delay argument and hence, it is applicable to linear, sub-linear and super-linear equations as well as both delay and advanced type differential equations.*

Next, we present several oscillation criteria for equation (1.1) when condition (3.1) fails to hold. Our first result is Myshkis-type oscillation criterion for (1.1).

Theorem 3.2. *Assume $\alpha = 1$ and let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . If*

$$\liminf_{\kappa \rightarrow \infty} \int_{\tau(\kappa)}^{\kappa} Q(s) \Omega(\tau(s)) ds > \frac{1}{e}, \quad (3.2)$$

where

$$\Omega(\kappa) = \int_{\kappa_*}^{\kappa} \frac{1}{w(s)} ds, \quad \text{for any } \kappa_* \in I_0,$$

then equation (1.1) is oscillatory.

Proof. Assume the contrary that, $u(\kappa)$ is an eventually positive (non oscillatory) solution of (1.1), that is $\mu(\kappa) > 0$, and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1 \geq \kappa_0$. Then the corresponding function satisfies $\mu(\kappa) > 0$, $\mu(\tau(\kappa)) > 0$ and condition (2.7) holds for all $\kappa \geq \kappa_1$, by Lemma 2.5.

From the monotonicity of $w(\kappa)\mu'(\kappa)$, we have

$$\mu(\kappa) \geq \int_{\kappa_1}^{\kappa} \frac{w(s)\mu'(s)}{w(s)} ds \geq \Omega(\kappa)w(\kappa)\mu'(\kappa), \quad (3.3)$$

which yields

$$\frac{\mu(\kappa)}{\Omega(\kappa)} \text{ is decreasing.} \quad (3.4)$$

Applying (3.3) in (2.5), and letting $z(\kappa) = w(\kappa)\mu'(\kappa)$, we see that $z(\kappa)$ is a positive solution of the inequality

$$z'(\kappa) + Q(\kappa)\Omega(\tau(\kappa))z(\tau(\kappa)) \leq 0. \quad (3.5)$$

But, by Lemma 1.4.1 of [11], condition (3.2) implies that inequality (3.5) has no eventually positive solution, a contradiction. This ends the proof. □

Our next result is of Hille-Wintner type oscillation criterion for equation (1.1).

Theorem 3.3. Assume that $\alpha = 1$ and let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . If,

$$\liminf_{t \rightarrow \infty} \Omega(\tau(\kappa)) \int_{\kappa}^{\infty} Q(s) ds > \frac{1}{4}, \quad (3.6)$$

then equation (1.1) is oscillatory.

Proof. Suppose, on the contrary that $u(\kappa)$ is an eventually positive (nonoscillation) solution of (1.1), that is, $u(\kappa) > 0$ and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1 \geq \kappa_0$. Proceeding as in the proof of Theorem 3.2, we see that $\mu(\kappa)$ satisfies the equation (2.5) for all $\kappa \geq \kappa_1$. That is,

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa)\mu(\tau(\kappa)) \leq 0, \quad \kappa \geq \kappa_1. \quad (3.7)$$

However, by Corollary 2 of [34], condition (3.6) implies $\mu(\kappa)$ is oscillatory, which is a contradiction. $\square$

Next, we present Koplatadze-type oscillation criterion for equation (1.1).

Theorem 3.4. Assume that $\alpha = 1$ and let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . If

$$\limsup_{\kappa \rightarrow \infty} \left\{ \frac{1}{\Omega(\tau(\kappa))} \int_{\kappa_1}^{\tau(\kappa)} Q(s)\Omega(s)\Omega(\tau(s))ds + \int_{\tau(\kappa)}^{\kappa} Q(s)\Omega(\tau(s))ds + \Omega(\tau(\kappa)) \int_{\kappa}^{\infty} Q(s)ds \right\} > 1 \quad (3.8)$$

for some $\kappa \in I_0$, then equation (1.1) is oscillatory.

Proof. Assume, on the contrary, that $u(\kappa)$ is an eventually positive (nonoscillation) solution of (1.1), that is, $u(\kappa) > 0$ and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$, for some $\kappa_1 \in I_0$. Proceeding as in the proof of Theorem 3.2, we see that $\mu(\kappa) = \frac{u(\kappa)}{x(\kappa)} > 0$ satisfies condition (2.7) and inequality (3.7) for all $\kappa \geq \kappa_1$. Integrating (3.7) yields

$$\mu'(\kappa) \geq \frac{1}{w(\kappa)} \int_{\kappa}^{\infty} Q(s)\mu(\tau(s))ds.$$

Integrating again from κ_1 to ∞ gives one

$$\begin{aligned} \mu(\kappa) &\geq \int_{\kappa_1}^{\kappa} \frac{1}{w(s)} \int_s^{\infty} Q(s_1)\mu(\tau(s_1))ds_1 ds \\ &= \int_{\kappa_1}^{\kappa} \frac{1}{w(s)} \int_s^{\kappa} Q(s_1)\mu(\tau(s_1))ds_1 ds + \int_{\kappa_1}^{\kappa} \frac{1}{w(s)} \int_s^{\infty} Q(s_1)\mu(\tau(s_1))ds_1 ds. \end{aligned}$$

Using the change of order of integration, we get

$$\mu(\kappa) \geq \int_{\kappa_1}^{\kappa} Q(s)\Omega(s)\mu(\tau(s))ds + \Omega(\kappa) \int_{\kappa}^{\infty} Q(s)\mu(\tau(s))ds.$$

Thus,

$$\mu(\tau(\kappa)) \geq \int_{\kappa_1}^{\tau(\kappa)} Q(s)\Omega(s)\mu(\tau(s))ds + \Omega(\tau(\kappa)) \int_{\tau(\kappa)}^{\kappa} Q(s)\mu(\tau(s))ds + \Omega(\tau(\kappa)) \int_{\kappa}^{\infty} Q(s)\mu(\tau(s))ds. \quad (3.9)$$

Since $\mu(\kappa)$ is increasing and by (3.4), $\frac{\mu(\kappa)}{\Omega(\kappa)}$ is decreasing, inequality (3.9) becomes

$$\begin{aligned} \mu(\tau(\kappa)) &\geq \frac{\mu(\tau(\kappa))}{\Omega(\tau(\kappa))} \int_{\kappa_1}^{\tau(\kappa)} Q(s)\Omega(s)\Omega(\tau(s))ds \\ &\quad + \mu(\tau(\kappa)) \int_{\tau(\kappa)}^{\kappa} Q(s)\Omega(\tau(s))ds + \Omega(\tau(\kappa))\mu(\tau(\kappa)) \int_{\tau(\kappa)}^{\infty} Q(s)ds. \end{aligned}$$

Dividing the last inequality by $\mu(\tau(\kappa))$ yields

$$1 \geq \frac{1}{\Omega(\tau(\kappa))} \int_{\kappa_1}^{\tau(\kappa)} Q(s)\Omega(s)\Omega(\tau(s))ds + \int_{\tau(\kappa)}^{\kappa} Q(s)\Omega(\tau(s))ds + \Omega(\tau(\kappa)) \int_{\tau(\kappa)}^{\infty} Q(s)ds,$$

which contradicts (3.8). $\square$

Remark 3.2. Note that by using the results in [35–37], one can obtain some more criteria for all oscillation of equation (1.1) in the case $\alpha = 1$ and the details are left to the reader.

In the following, we derive Atkinson type oscillation conditions for equation (1.1) in the case $\alpha > 1$ or $\alpha < 1$.

Theorem 3.5. Assume that $\alpha > 1$ and let $x(\kappa)$ be a positive decreasing solution of (2.1) on I_0 . If

$$\int_{\kappa_1}^{\infty} \frac{1}{w(\kappa)} \int_{\kappa}^{\infty} Q(s) \frac{\Omega^{\alpha}(\tau(s))}{\Omega^{\alpha}(s)} ds d\kappa = \infty, \quad (3.10)$$

then equation (1.1) is oscillatory.

Proof. Assume the contrary that condition (3.10) holds, but $u(\kappa)$ is an eventually positive (nonoscillatory) solution of (1.1). That is, $u(\kappa) > 0$ and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$ for some $\kappa_1 \geq \kappa_0$. Then the corresponding function $\mu(\kappa) = \frac{x(\kappa)}{u(\kappa)} > 0$ satisfies condition (2.7) and equation (2.5) for all $\kappa \geq \kappa_1$. That is,

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa)\mu^{\alpha}(\tau(\kappa)) \leq 0, \quad \kappa \geq \kappa_1. \quad (3.11)$$

Using that $\frac{\mu(\kappa)}{\Omega(\kappa)}$ is decreasing in (3.11), we get

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa) \frac{\Omega^{\alpha}(\tau(\kappa))}{\Omega^{\alpha}(\kappa)} \mu^{\alpha}(\kappa) \leq 0, \quad \kappa \geq \kappa_1. \quad (3.12)$$

Integrating (3.12) from κ to ∞ gives one

$$\int_{\kappa}^{\infty} Q(s) \frac{\Omega^{\alpha}(\tau(s))}{\Omega^{\alpha}(s)} \mu^{\alpha}(s) ds \leq w(\kappa)\mu'(\kappa),$$

or equivalently,

$$\frac{1}{w(\kappa)} \int_{\kappa}^{\infty} Q(s) \frac{\Omega^{\alpha}(\tau(s))}{\Omega^{\alpha}(s)} ds \leq \frac{\mu'(\kappa)}{\mu^{\alpha}(\kappa)}.$$

Integrating the previous inequality from κ_1 to κ , we have

$$\begin{aligned} \int_{\kappa_1}^{\kappa} \frac{1}{w(s)} \int_s^{\infty} Q(s_1) \frac{\Omega^{\alpha}(\tau(s_1))}{\Omega^{\alpha}(s_1)} ds_1 ds &\leq \int_{\kappa_1}^{\kappa} \frac{\mu'(s)}{\mu^{\alpha}(s)} ds \\ &\leq \frac{\mu^{\alpha-1}(\kappa_1)}{\alpha-1} - \frac{\mu^{1-\alpha}(\kappa)}{\alpha-1}. \end{aligned}$$

As $\kappa \rightarrow \infty$, we obtain

$$\int_{\kappa_1}^{\kappa} \frac{1}{w(s)} \int_s^{\infty} Q(s_1) \frac{\Omega^{\alpha}(\tau(s_1))}{\Omega^{\alpha}(s_1)} ds_1 ds \leq \frac{\mu^{1-\alpha}(\kappa_1)}{\alpha-1} < \infty,$$

a contradiction to (3.10). This ends the proof. $\square$

We conclude this section with the following theorem for the case $\alpha < 1$.

Theorem 3.6. Let $\alpha < 1$ and assume $x(\kappa)$ is a positive decreasing solution of (2.1) on I_0 . If

$$\int_{\kappa_1}^{\infty} Q(\kappa) \Omega^{\alpha}(\tau(\kappa)) dt = \infty, \quad (3.13)$$

then equation (1.1) is oscillatory.

Proof. For the sake of contradiction, let condition (3.13) holds but $u(\kappa)$ is an eventually positive (non-oscillatory) solution of (1.1). That is, $u(\kappa) > 0$ and $u(\tau(\kappa)) > 0$ for all $\kappa \geq \kappa_1$, for some $\kappa_1 \in I_0$. Then the corresponding function $\mu(\kappa) = \frac{u(\kappa)}{x(\kappa)} > 0$ satisfies condition (2.7) and equation (2.5) for all $\kappa \geq \kappa_1$.

Now, using (3.3) in (2.5), we find

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa)\Omega^{\alpha}(\tau(\kappa))(w(\tau(\kappa))\mu'(\tau(\kappa)))^{\alpha} \leq 0, \quad \kappa \geq \kappa_1.$$

Since $\tau(\kappa) \leq t$ and $w(\kappa)\mu'(\kappa)$ is decreasing, we further obtain

$$(w(\kappa)\mu'(\kappa))' + Q(\kappa)\Omega^{\alpha}(\tau(\kappa))(w(\kappa)\mu'(\kappa))^{\alpha} \leq 0, \quad \kappa \geq \kappa_1.$$

Dividing the last inequality by $(w(\kappa)\mu'(\kappa))^\alpha$ and then integrating from κ_1 to κ , we have

$$\begin{aligned} \int_{\kappa_1}^{\kappa} Q(s) \Omega^\alpha(\tau(s)) ds &\leq - \int_{\kappa_1}^{\kappa} \frac{(w(s)\mu'(s))'}{(w(s)\mu'(s))^\alpha} ds \\ &\leq \frac{(w(\kappa_1)\mu'(\kappa_1))^{1-\alpha}}{1-\alpha} - \frac{(w(\kappa)\mu'(\kappa))^{1-\alpha}}{1-\alpha}. \end{aligned} \quad (3.14)$$

Letting $t \rightarrow \infty$ in (3.14), we get a contradiction with (3.13). $\square$

Remark 3.3. Note that by using the results in [10, 11], one can derive several oscillation criteria for equation (1.1) in the case of $\alpha > 1$ or $\alpha < 1$. The details are left to the reader.

4. Examples

In this section, three examples are provided to illustrate the importance and novelty of the main results. In the first example, we provide an explicit solution for the considered equation.

Example 4.1. Consider the second-order nonlinear perturbed delay differential equation

$$u''(\kappa) + \left(1 + \frac{1}{\kappa^2}\right) u(\kappa - 2\pi) = \frac{(1 + \cos^2 \kappa)}{\kappa^2} \frac{u(\kappa)}{(1 + (u'(\kappa))^2)}, \quad \kappa \geq 2\pi. \quad (4.1)$$

Here, $\eta(\kappa) = 1$, $f_1(\kappa, u) = \left(1 + \frac{1}{\kappa^2}\right) u$, $f_2(\kappa, u, v) = \frac{(1 + \cos^2 \kappa)}{\kappa^2} \frac{u(\kappa)}{(1 + v^2)}$;
 $\alpha = 1$ and $\tau(\kappa) = \kappa - 2\pi$. By a simple computation, we find that

$$\frac{f_1(\kappa, u)}{u} \geq \phi_1(\kappa) = \left(1 + \frac{1}{\kappa^2}\right) \text{ and } \frac{f_2(\kappa, u, v)}{u} \leq \phi_2(\kappa) = \frac{2}{\kappa^2}.$$

The auxiliary equation (2.1) becomes

$$x''(\kappa) - \frac{2}{\kappa^2} x(\kappa) = 0, \quad \kappa \geq 2\pi,$$

and $x(\kappa) = \frac{1}{\kappa}$ is a positive decreasing solution. Using this, we have

$$w(\kappa) = \frac{1}{\kappa^2}, \quad Q(\kappa) = \left(1 + \frac{1}{\kappa^2}\right) \frac{1}{\kappa(\kappa - 2\pi)}, \quad \kappa \geq 2\pi,$$

and $\Omega(\kappa) = \frac{\kappa^3}{3}$. The condition (3.6) becomes

$$\liminf_{\kappa \rightarrow \infty} \frac{(\kappa - 2\pi)^3}{3} \int_{\kappa}^{\infty} \left(1 + \frac{1}{s^2}\right) \frac{1}{s(s - 2\pi)} ds \geq \lim_{\kappa \rightarrow \infty} \frac{(\kappa - 2\pi)^3}{3} \left(\frac{1}{\kappa} + \frac{1}{3\kappa^3}\right) = \infty > \frac{1}{4},$$

that is, condition (3.6) holds. Hence by Theorem 3.3, equation (4.1) is oscillatory. In fact $u(\kappa) = \sin \kappa$ is one such oscillatory solution of equation (4.1).

Figure 1 illustrates the oscillatory solution $u(\kappa) = \sin \kappa$ of equation (4.1). The periodic oscillations and repeated sign changes confirm the oscillatory behavior predicted by Theorem 3.3, while the delayed term $u(\kappa - 2\pi)$ overlaps with $u(\kappa)$ due to the periodicity of the sine function.

Example 4.2. Consider the second-order perturbed nonlinear delay differential equation

$$(\kappa u'(\kappa))' + d\kappa^3 u^3(\kappa/2) = \frac{(3 + \sin^2 \kappa)}{\kappa} \frac{u(\kappa)}{(1 + |u'(\kappa)|)}, \quad \kappa \geq 1, \quad (4.2)$$

where $d > 0$ is a constant.

Here,

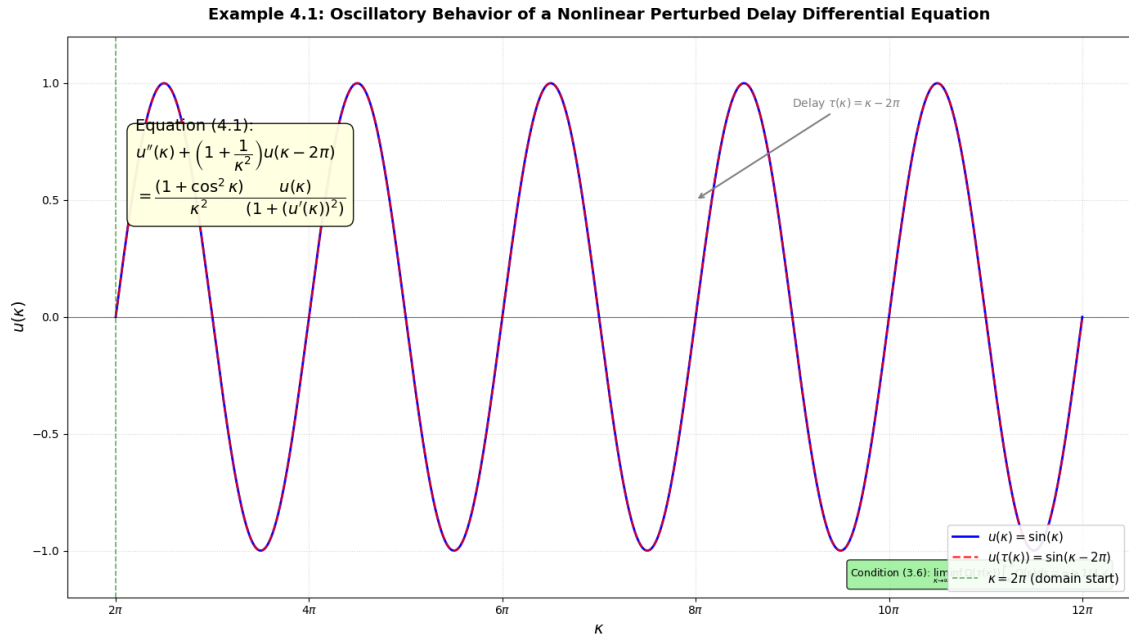
$$\eta(\kappa) = \kappa, \quad f_1(\kappa, u) = d\kappa^3 u^3, \quad f_2(\kappa, u, v) = \frac{(3 + \sin^2 \kappa)}{\kappa} \frac{u}{(1 + |v|)}, \quad \tau(\kappa) = \kappa/2, \quad \alpha = 3.$$

By simple computation, we find that

$$\frac{f_1(\kappa, u)}{u^3} \geq \phi_1(\kappa) = d\kappa^3 \quad \text{and} \quad \frac{f_2(\kappa, u, v)}{u} \leq \phi_2(u) = \frac{4}{\kappa}.$$

The auxiliary equation (2.1) becomes

$$(\kappa x'(\kappa))' - \frac{4}{\kappa} x(\kappa) = 0$$

FIG. 1. Oscillatory solution $\alpha = 1$

and it has $x(\kappa) = \frac{1}{\kappa^2}$ as a positive decreasing solution. Using this, we find

$$w(\kappa) = \frac{1}{\kappa^3}, \quad Q(\kappa) = \frac{8d}{\kappa^5} \quad \text{and} \quad \Omega(\kappa) = \frac{\kappa^4}{4}.$$

The condition (3.10) becomes

$$\frac{d}{29} \int_2^\infty \kappa^3 \int_\kappa^\infty \frac{1}{s^5} ds d\kappa = \frac{d}{2^{11}} \int_2^\infty \frac{1}{\kappa} d\kappa = \infty,$$

that is, condition (3.10) holds if $d > 0$. Hence, by Theorem 3.5, equation (4.2) is oscillatory for all $d > 0$.

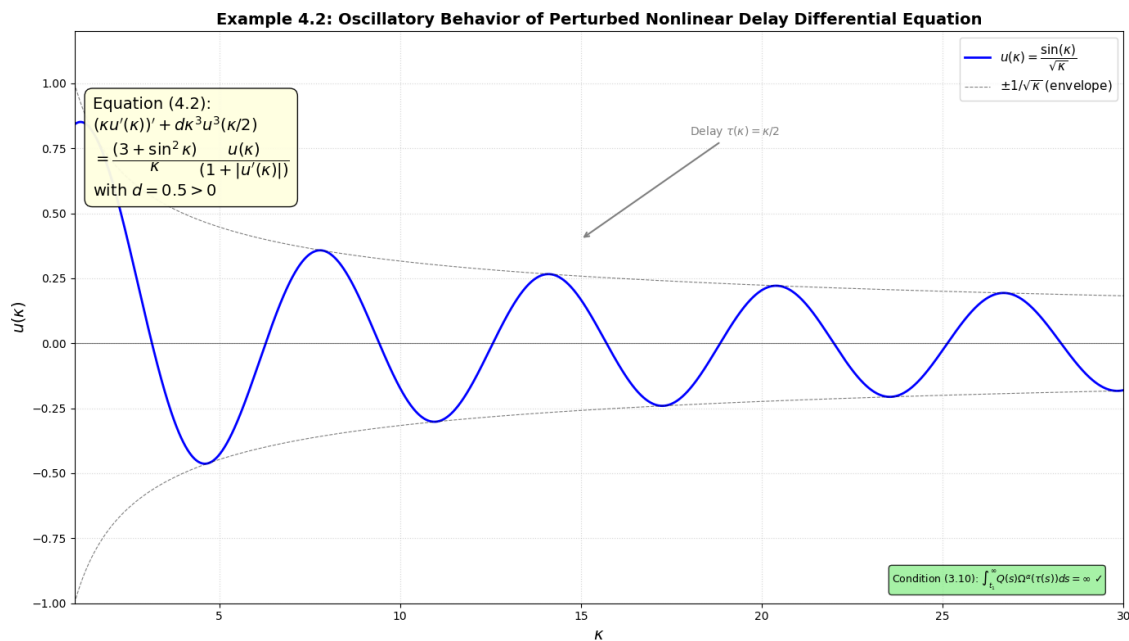
FIG. 2. Oscillatory solution $\alpha = 3$

Figure 2 depicts the oscillatory solution of equation (4.2) with delay $\tau(\kappa) = \kappa/2$. The repeated zero crossings and sustained oscillations verify the conclusion of Theorem 3.5 that equation (4.2) is oscillatory for all $d > 0$.

Example 4.3. Consider the second-order perturbed nonlinear delay differential equation

$$\left(\frac{1}{\kappa}u'(\kappa)\right)' + \frac{d}{\kappa}u^{1/3}(\kappa/2) = \frac{5}{8\kappa^3}(1 + \cos^2\kappa)\frac{u(\kappa)}{(1 + |u'(\kappa)|)}, \quad \kappa \geq 1, \quad (4.3)$$

where $d > 0$ is a constant.

Here,

$$\eta(\kappa) = \frac{1}{\kappa}, f_1(\kappa, u) = \frac{d}{\kappa}u^{1/3}, f_2(\kappa, u, v) = \frac{5}{8}\frac{(1 + \cos^2\kappa)}{\kappa^3}\frac{u}{(1 + |v|)}, \tau(\kappa) = \kappa/2, \alpha = 1/3.$$

By simple calculation, we find that

$$\frac{f_1(\kappa, u)}{u^{1/3}} \geq \phi_1(\kappa) = \frac{d}{\kappa} \quad \text{and} \quad \frac{f_2(\kappa, u, v)}{u} \leq \phi_2(\kappa) = \frac{5}{4\kappa^3}.$$

The auxiliary equation (2.1) becomes

$$\left(\frac{1}{\kappa}x'(\kappa)\right)' - \frac{5}{4\kappa^3}x(\kappa) = 0,$$

and it has $x(\kappa) = \frac{1}{\sqrt{\kappa}}$ as a positive decreasing solution. Using this, we find

$$w(\kappa) = \frac{1}{\kappa^2}, Q(\kappa) = \frac{d2^{1/6}}{\kappa^{5/3}} \quad \text{and} \quad \Omega(\kappa) \approx \frac{\kappa^3}{3}.$$

The condition (3.13) becomes

$$\frac{d2^{1/6}}{3^{1/3}} \int_2^\infty \frac{1}{\kappa^{2/3}} d\kappa = \infty,$$

that is, condition (3.13) holds. Hence, by Theorem 3.6, equation (4.3) is oscillatory for all $d > 0$.

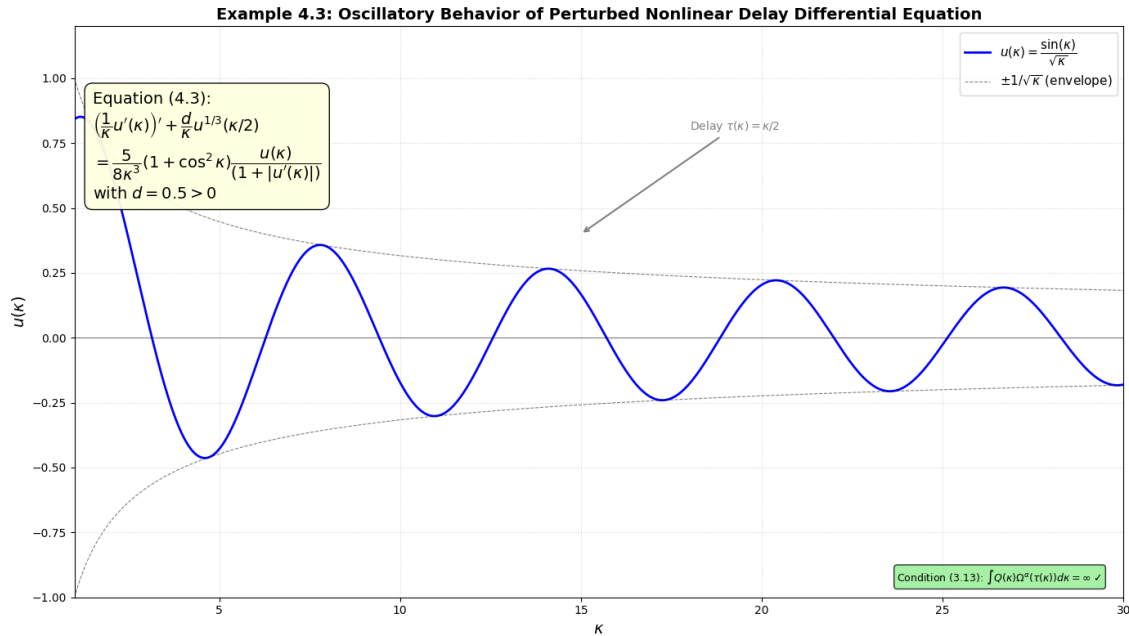


FIG. 3. Oscillatory solution $\alpha = 1/3$

Figure 3 illustrates the oscillatory behavior of a solution of equation (4.3) for $d > 0$. The solution undergoes infinitely many sign changes while remaining bounded by a slowly decaying envelope, thereby confirming the oscillation criterion established in Theorem 3.6.

5. Conclusion

This paper studied the oscillatory properties of solutions of second-order nonlinear perturbed delay differential equation (1.1). The considered equation was first transformed into a simple binomial form by using a positive decreasing solution of the associated second-order linear differential equation. Then the structure of nonoscillatory solutions of the transformed equation easily detected. Then we obtained new oscillation criteria for the transformed equation via Koplatadze method and integral averaging technique. This, in turn, implies the oscillation of the studied equation.

Three specific numerical examples are presented to show the importance and novelty of the main results. Note that we obtained explicit oscillatory solution for one example. Further, none of the results reported in the literature for the second-order perturbed differential equations [13, 16–25, 31] applicable to our equations (4.1)–(4.3) since these equations are either different or contains delay term.

It would be interesting to extend the results of the paper when the condition

$$\int_{\kappa_0}^{\infty} \frac{1}{\eta(\kappa)} d\kappa < \infty$$

holds. Another promising direction for future research is to study the oscillatory behavior of the following equation

$$(\eta(\kappa)z'(\kappa))' + f_1(\kappa, u(\kappa), u(\tau(\kappa))) = f_2(\kappa, u(\kappa), u'(\kappa)),$$

where $z(\kappa) = u(\kappa) + b(\kappa)u(\sigma(\kappa))$.

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Submitted 2 May 2026; revised 2 June 2026; accepted 3 June 2026

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Conflict of interest: the authors declare no conflict of interest.