

A Complete geometric classification of spectral configurations for two-particle Schrödinger operators on $\mathbb{Z}$ with a rank-three interaction

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ABSTRACT We study a family of lattice Schrödinger operators $H_{\mu}(K)$, describing two identical bosons on the one-dimensional lattice $\mathbb{Z}$, where $K \in \mathbb{T}$ is the quasi-momentum. The interaction is described by the coupling vector $\mu = (\mu_1, \mu_2, \mu_3)$, where μ_1 acts at the origin and μ_2, μ_3 at the sites $|x| = 1, 2$. In this paper, we focus on the zero quasi-momentum fiber $H_{\mu}(0)$, whose essential spectrum is the interval $[0, 8]$, while discrete eigenvalues may occur both below 0 and above 8.

Using the Fredholm determinant, we derive explicit threshold polynomials whose zero sets:

$$\Gamma^{\pm} := \{\mu \in \mathbb{R}^3 : C^{\pm}(\mu) = 0\}$$

define three critical surfaces in $\mathbb{R}^3$. These surfaces partition the parameter space into regions with constant eigenvalue counts and identify exactly the parameter values at which eigenvalues emerge from, or are absorbed into, the thresholds 0 and 8.

A central result of this work is the complete geometric classification of the spectral configuration sets $\mathcal{S} = (n_-, n_+)$. We show that the (μ_2, μ_3) -plane is partitioned into domains $D_j^{\pm}$, which strictly govern the range of the spectral counting functions. We prove that the parameter μ_3 acts as a global phase controller: the critical values $\mu_3 = \pm 2$ separate qualitatively different spectral regimes, determining the set of all attainable configurations under the global rank-three constraint.

Finally, we extend our analysis to $K \neq 0$ and demonstrate that the discrete spectrum is preserved for all quasi-momenta whenever $|\mu_3| > 2$.

KEYWORDS Two-particle lattice Schrödinger operator, discrete spectrum, rank-three perturbation, Fredholm determinant, threshold asymptotics, critical surfaces, spectral classification.

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1. Introduction

Discrete Schrödinger operators on lattices constitute an important class of models in mathematical physics, providing a natural framework for the study of quantum systems in periodic media such as crystalline solids and optical lattices [1–10]. In contrast to continuous models, lattice Hamiltonians are characterized by bounded dispersion relations, and therefore their essential spectrum forms a compact interval. This feature gives rise to specific spectral effects absent in the continuous case, including the possibility of discrete eigenvalues both below and above the spectral band [11–22].

Recent experimental developments in optical lattice systems have confirmed the existence of lattice-induced bound states, including repulsively bound atom pairs, which have no direct analogue in continuous models [20, 23–25]. These phenomena have stimulated intensive studies of the spectral properties of few-particle lattice Hamiltonians and their threshold behaviour.

A fundamental aspect of lattice systems is the analysis of threshold effects, where eigenvalues emerge from or disappear into the edges of the essential spectrum. Such phenomena play a crucial role in the description of bound states and have been extensively studied in the framework of discrete Schrödinger operators [21, 22].

A standard method for analyzing translation-invariant lattice Hamiltonians is the Floquet–Bloch decomposition, which reduces the full operator to a direct integral of fiber operators parametrized by the total quasi-momentum $K \in \mathbb{T}^d$ [2, 3, 26]. The spectral analysis of these fiber operators provides a complete description of bound states and their dependence on the quasi-momentum.

In the present paper, we consider a two-particle Schrödinger operator on the one-dimensional lattice $\mathbb{Z}$ with a symmetric finite-range interaction and focus on the zero quasi-momentum fiber $K = 0$, whose essential spectrum is the interval

[0, 8]. Our main goal is to establish a complete geometric classification of all possible discrete spectrum configurations in terms of the interaction parameters.

1.1. Model and main results

We consider a system of two identical spinless bosons on $\mathbb{Z}$ interacting through a symmetric finite-range potential

$$\hat{v}_\mu(x) = \mu_1 \delta_{|x|,0} + \mu_2 \delta_{|x|,1} + \mu_3 \delta_{|x|,2}, \quad \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3, \quad (1)$$

which generates a rank-three perturbation of the free discrete Laplacian. After Fourier transform and Floquet–Bloch reduction, the corresponding fiber Hamiltonian $H_\mu(0)$ acts in the even-function subspace $L^{2,e}(\mathbb{T})$ and has essential spectrum [0, 8]. Since the perturbation is of rank three, the total number of discrete eigenvalues outside the essential spectrum is finite and does not exceed three.

The spectral problem is reduced to the analysis of an explicit Fredholm determinant $\Delta_\mu(z)$ associated with the Lippmann–Schwinger operator. Its threshold asymptotics at the band edges 0 and 8 are governed by two explicit polynomials $C^\pm(\mu)$, whose zero sets define critical surfaces

$$\Gamma^\pm = \{\mu \in \mathbb{R}^3 : C^\pm(\mu) = 0\}.$$

These surfaces describe precisely the parameter values at which eigenvalues emerge from or disappear into the essential spectrum.

The main result of the paper is a complete geometric classification of all admissible discrete spectrum configurations of $H_\mu(0)$. We prove that the (μ_2, μ_3) -plane is partitioned into three disjoint domains $D_j^\pm$ ($j = 0, 1, 2$), which determine the possible values of the eigenvalue counting functions $n_\pm$. By analyzing the intersections of these domains and combining them with the global rank-three constraint, we obtain an explicit classification of all realizable spectral pairs (n_-, n_+) .

A distinguished role is played by the parameter μ_3 : the critical values $\mu_3 = \pm 2$ separate different global spectral regimes. In particular, for $\mu_3 < -2$ the operator always possesses at least one eigenvalue below the essential spectrum, while for $\mu_3 > 2$ it always possesses at least one eigenvalue above it. This yields a complete partition of the parameter space into spectral phase regions characterized by the exact numbers of bound states below and above the essential spectrum.

By deriving comparison estimates, we extend the results obtained for $K = 0$ to arbitrary quasi-momenta $K \in \mathbb{T}$. These estimates show that the number of eigenvalues $n_\pm(H_\mu(K))$ is controlled by its corresponding value at the origin. In particular, when $|\mu_3| > 2$, they guarantee that the discrete spectrum is non-empty for every $K \in \mathbb{T}$, yielding uniform lower bounds on the number of eigenvalues across the Brillouin zone.

The distribution of the discrete spectrum for two-particle Hamiltonians has been investigated in several recent works, such as [27, 29–33]. While these studies identified various spectral regimes for specific interaction models, a comprehensive global description of the parameter space was not provided. Our approach fills this gap by rigorously partitioning the interaction manifold into disjoint regions where the exact number and location of discrete eigenvalues $n_\pm$ remain invariant. This geometric classification provides a complete characterization of the eigenvalue birth and annihilation mechanisms across the entire range of parameters.

1.2. Outline of the paper

In Section 2, we introduce the model and perform the Floquet–Bloch decomposition. Section 3 establishes the Fredholm determinant representation and derives its threshold asymptotics. Section 4 is devoted to the geometric analysis of the critical surfaces and the partition of the parameter space. In Section 5, we classify all realizable spectral configurations and describe their global dependence on the interaction parameters. Finally, Section 6 summarizes the results and outlines possible generalizations.

2. The Two-Particle Hamiltonian

2.1. Position-space representation

Let $\mathbb{Z}$ be the one-dimensional integer lattice, and denote by

$$\ell^2(\mathbb{Z} \times \mathbb{Z}), \quad \ell^{2,s}(\mathbb{Z} \times \mathbb{Z}) := \{\hat{f} \in \ell^2(\mathbb{Z} \times \mathbb{Z}) : \hat{f}(x_1, x_2) = \hat{f}(x_2, x_1)\}$$

the Hilbert space of square-summable functions and its symmetric subspace, respectively. The total Hamiltonian

$$\hat{\mathbb{H}}_\mu = \hat{\mathbb{H}}_0 + \hat{\mathbb{V}}_\mu, \quad \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3,$$

is a bounded self-adjoint operator on $\ell^{2,s}(\mathbb{Z} \times \mathbb{Z})$. The free Hamiltonian $\hat{\mathbb{H}}_0$ is the two-particle discrete Laplacian on $\mathbb{Z}^2$:

$$\begin{aligned} (\hat{\mathbb{H}}_0 \hat{f})(x_1, x_2) &= [2\hat{f}(x_1, x_2) - \hat{f}(x_1 + 1, x_2) - \hat{f}(x_1 - 1, x_2)] \\ &\quad + [2\hat{f}(x_1, x_2) - \hat{f}(x_1, x_2 + 1) - \hat{f}(x_1, x_2 - 1)]. \end{aligned}$$

The interaction operator $\hat{V}_\mu$ acts as a multiplication operator:

$$(\hat{V}_\mu \hat{f})(x_1, x_2) = \hat{v}_\mu(x_1 - x_2) \hat{f}(x_1, x_2),$$

where the finite-range interaction potential $\hat{v}_\mu$ is defined by (1).

2.2. Momentum-space representation

Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \equiv (-\pi, \pi]$ denote the one-dimensional torus. Applying the Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{T})$ to both particle coordinates, we obtain the momentum-space representation

$$H_\mu := (\mathcal{F} \otimes \mathcal{F}) \hat{\mathbb{H}}_\mu (\mathcal{F}^* \otimes \mathcal{F}^*) = H_0 + V_\mu, \quad \mu \in \mathbb{R}^3,$$

acting on the Hilbert space $L^{2,s}(\mathbb{T} \times \mathbb{T})$ of symmetric square-integrable functions on $\mathbb{T}^2$. The free operator H_0 acts as multiplication by the two-particle dispersion relation,

$$(H_0 f)(p, q) = [\varepsilon(p) + \varepsilon(q)] f(p, q),$$

with

$$\varepsilon(p) = 2(1 - \cos p), \quad p \in \mathbb{T}.$$

The interaction operator V_μ becomes an integral operator,

$$(V_\mu f)(p, q) = \frac{1}{2\pi} \int_{\mathbb{T}} v_\mu(p - u) f(u, p + q - u) du,$$

where the kernel is $v_\mu(p) = \mu_1 + 2\mu_2 \cos p + 2\mu_3 \cos 2p$.

2.3. Floquet–Bloch decomposition and fiber Hamiltonians

Since H_μ is translation invariant, we apply the Floquet–Bloch decomposition with respect to the total quasi-momentum $K \in \mathbb{T}$:

$$L^{2,s}(\mathbb{T} \times \mathbb{T}) \simeq \int_{\mathbb{T}}^{\oplus} L^{2,e}(\mathbb{T}) d\nu(K), \quad H_\mu \simeq \int_{\mathbb{T}}^{\oplus} H_\mu(K) d\nu(K),$$

where $d\nu$ is the normalized Lebesgue measure and $L^{2,e}(\mathbb{T}) \subset L^2(\mathbb{T})$ is the subspace of even functions. For each $K \in \mathbb{T}$, the fiber Hamiltonian

$$H_\mu(K) = H_0(K) + V_\mu$$

acts as

$$(H_0(K)f)(p) = \mathcal{E}_K(p) f(p), \quad \mathcal{E}_K(p) = 4\left(1 - \cos \frac{K}{2} \cos p\right),$$

and

$$(V_\mu f)(p) = \sum_{i=1}^3 \mu_i \langle \phi_i, f \rangle \phi_i(p),$$

where we introduce the orthonormal form factors

$$\phi_1(t) = 1, \quad \phi_2(t) = \sqrt{2} \cos t, \quad \phi_3(t) = \sqrt{2} \cos 2t, \quad t \in \mathbb{T}.$$

Remark 1. The Floquet–Bloch decomposition reduces the translation-invariant Hamiltonian H_μ to a direct integral of fiber operators $H_\mu(K)$, $K \in \mathbb{T}$. The essential spectrum of H_μ is the union over K of the essential spectra of the fibers, while discrete eigenvalues of the fibers correspond to bound states with total quasi-momentum K . Because V_μ is of finite rank (at most three), each fiber $H_\mu(K)$ is a finite-rank perturbation of a multiplication operator, which greatly simplifies the spectral analysis.

2.4. Essential spectrum

The free fiber Hamiltonian $H_0(K)$ has purely absolutely continuous spectrum consisting of a single band:

$$\sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)],$$

where

$$\mathcal{E}_{\min}(K) = 4\left(1 - \cos \frac{K}{2}\right) \geq \mathcal{E}_{\min}(0) = 0,$$

$$\mathcal{E}_{\max}(K) = 4\left(1 + \cos \frac{K}{2}\right) \leq \mathcal{E}_{\max}(0) = 8.$$

Since V_μ is a finite-rank operator, Weyl's theorem on the invariance of the essential spectrum (see [26]) under compact perturbations (and, in particular, under finite-rank perturbations) implies

$$\sigma_{\text{ess}}(H_\mu(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)].$$

Consequently, any discrete eigenvalue of $H_\mu(K)$ must lie in the spectral gaps:

$$\sigma_{\text{disc}}(H_\mu(K)) \subset (-\infty, \mathcal{E}_{\min}(K)) \cup (\mathcal{E}_{\max}(K), \infty).$$

Let $n_-(H_\mu(K))$ and $n_+(H_\mu(K))$ denote the number of discrete eigenvalues (counting multiplicities) below and above the essential spectrum, respectively. Because V_μ has rank at most three, we obtain the universal bound

$$n(H_\mu(K)) := n_-(H_\mu(K)) + n_+(H_\mu(K)) \leq 3, \quad \forall K \in \mathbb{T}. \quad (2)$$

Remark 2. The bound (2) is a direct consequence of the finite-rank nature of the perturbation: a finite-rank perturbation can create at most as many eigenvalues (outside the essential spectrum) as its rank, counting multiplicities. This observation will be crucial later when we classify the discrete spectrum, as it restricts the possible pairs (n_-, n_+) that can occur.

3. Fredholm determinant and threshold asymptotics

In this section, we restrict our attention to the specific case $K = 0$, representing zero total quasi-momentum. In this regime, the essential spectrum of $H_0(0)$ is significantly reduced to the interval $[0, 8]$.

3.1. The transposed Lippmann–Schwinger operator

For $z \in \mathbb{R} \setminus [0, 8]$, the resolvent of the free operator

$$R_0(0, z) := (H_0(0) - zI)^{-1}$$

is well-defined as a bounded operator.

Definition 1. For $z \in \mathbb{R} \setminus [0, 8]$, the transposed Lippmann–Schwinger operator associated with $H_\mu(0)$ is defined by

$$L_\mu(0, z) := -V_\mu R_0(0, z).$$

Remark 3. Since V_μ is of finite rank, $L_\mu(0, z)$ acts non-trivially only on the finite-dimensional space spanned by the form factors ϕ_1, ϕ_2, ϕ_3 . Consequently, the eigenvalue problem reduces to a 3×3 matrix equation, and the Fredholm determinant admits an explicit algebraic representation.

The following lemma establishes the spectral correspondence between $H_\mu(0)$ and its Lippmann–Schwinger operator.

Lemma 1. A point $z \in \mathbb{R} \setminus [0, 8]$ is an eigenvalue of $H_\mu(0)$ with multiplicity m if and only if 1 is an eigenvalue of $L_\mu(0, z)$ with the same multiplicity.

Proof. The eigenvalue equation $H_\mu(0)\psi = z\psi$ is equivalent to

$$(H_0(0) - z)\psi = -V_\mu\psi.$$

Applying the resolvent $R_0(0, z)$ yields

$$\psi = -V_\mu R_0(0, z)\psi,$$

which means that ψ is an eigenvector of $L_\mu(0, z)$ corresponding to the eigenvalue 1. The converse follows analogously. $\square$

Lemma 2. A point $z \in \mathbb{R} \setminus [0, 8]$ is an eigenvalue of $H_\mu(0)$ if and only if

$$\Delta_\mu(z) = 0,$$

where

$$\Delta_\mu(z) := \det(I - L_\mu(0, z)) = \det\left(\delta_{ij} + \mu_j a_{ij}(z)\right)_{i,j=1}^3. \quad (3)$$

The matrix elements are given by

$$a_{ij}(z) := \langle R_0(0, z)\phi_j, \phi_i \rangle = \int_{\mathbb{T}} \frac{\phi_i(t)\phi_j(t)}{\mathcal{E}_0(t) - z} d\nu(t), \quad i, j = 1, 2, 3. \quad (4)$$

Moreover, the function $\Delta_\mu(\cdot)$ has at most three zeros in $\mathbb{R} \setminus [0, 8]$.

Proof. By Lemma 1, $z \in \mathbb{R} \setminus [0, 8]$ is an eigenvalue of $H_\mu(0)$ if and only if 1 is an eigenvalue of $L_\mu(0, z)$. Since $L_\mu(0, z)$ is a finite-rank operator, this is equivalent to $\det(I - L_\mu(0, z)) = 0$. Hence

$$z \in \sigma_{\text{disc}}(H_\mu(0)) \iff \Delta_\mu(z) = 0.$$

Finally, since V_μ has rank at most three, the operator $H_\mu(0)$ has at most three discrete eigenvalues in $\mathbb{R} \setminus [0, 8]$. Therefore, $\Delta_\mu(\cdot)$ has at most three zeros in $\mathbb{R} \setminus [0, 8]$. $\square$

Lemma 3. The functions $a_{ij}(z)$, $i, j = 1, 2, 3$, satisfy the following symmetry and asymptotic properties.

(i) **Parity relation:**

$$a_{ij}(8 - z) = (-1)^{i+j+1} a_{ij}(z), \quad i, j \in \{1, 2, 3\}.$$

(ii) **Asymptotic expansions at the band edges:**

$$a_{ij}(z) = \begin{cases} \frac{\phi_i(0)\phi_j(0)}{2\sqrt{2}\zeta_0(z)} + A_{ij} + O(\zeta_0(z)), & z \nearrow 0, \\ -\frac{\phi_i(\pi)\phi_j(\pi)}{2\sqrt{2}\zeta_8(z)} - A_{ij} + O(\zeta_8(z)), & z \searrow 8, \end{cases} \quad (5)$$

where $\zeta_0(z) = \sqrt{-z}$ (for $z < 0$), $\zeta_8(z) = \sqrt{z-8}$ (for $z > 8$), and the constant matrix A is given by

$$A = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{4} & -\frac{1}{2} & -1 \\ -\frac{\sqrt{2}}{2} & -1 & -1 \end{pmatrix}.$$

Proof. (i) By definition (4),

$$a_{ij}(8 - z) = \int_{\mathbb{T}} \frac{\phi_i(t)\phi_j(t)}{\mathcal{E}_0(t) - (8 - z)} d\nu(t).$$

Using the identities

$$\mathcal{E}_0(t + \pi) = 8 - \mathcal{E}_0(t), \quad \phi_k(t + \pi) = (-1)^{k+1} \phi_k(t),$$

and performing the change of variables $t \mapsto t + \pi$, we obtain

$$a_{ij}(8 - z) = (-1)^{i+j+1} a_{ij}(z).$$

(ii) To derive the asymptotics as $z \nearrow 0$, we write

$$a_{ij}(z) = \int_{\mathbb{T}} \frac{\phi_i(t)\phi_j(t) - \phi_i(0)\phi_j(0)}{\mathcal{E}_0(t) - z} d\nu(t) + \phi_i(0)\phi_j(0) \int_{\mathbb{T}} \frac{d\nu(t)}{\mathcal{E}_0(t) - z}.$$

The first integral is regular at $z = 0$ and therefore admits the expansion $A_{ij} + O(\zeta_0(z))$. For the second integral, using the local expansion $\mathcal{E}_0(t) = 2t^2 + O(t^4)$, $t \rightarrow 0$, we obtain

$$\int_{\mathbb{T}} \frac{d\nu(t)}{\mathcal{E}_0(t) - z} = \frac{1}{2\sqrt{2}\zeta_0(z)} + O(1), \quad z \nearrow 0.$$

Combining these relations, we obtain the asymptotic expansion (5) as $z \nearrow 0$.

The asymptotics as $z \searrow 8$ follows immediately from part (i) by replacing z with $8 - z$ and using

$$\phi_k(\pi) = (-1)^{k+1} \phi_k(0).$$

□

Lemma 4 (Determinant asymptotics). For any $\mu \in \mathbb{R}^3$,

$$\Delta_{\mu}(z) = \begin{cases} C^{-}(\mu) \frac{1}{2\sqrt{2}\zeta_0(z)} + D^{-}(\mu) + O(\zeta_0(z)), & z \nearrow 0, \\ C^{+}(\mu) \frac{1}{2\sqrt{2}\zeta_8(z)} + D^{+}(\mu) + O(\zeta_8(z)), & z \searrow 8, \end{cases}$$

where $\zeta_0(z) = \sqrt{-z}$ ($z < 0$), $\zeta_8(z) = \sqrt{z-8}$ ($z > 8$), and the coefficients are

$$\begin{aligned} C^{\pm}(\mu) &= \mp(\mu_1 + 2\mu_2 + 2\mu_3 + \tfrac{1}{4}\mu_1\mu_2\mu_3) + \tfrac{1}{2}\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \\ D^{\pm}(\mu) &= 1 \pm (\tfrac{1}{2}\mu_2 + \mu_3 + \tfrac{1}{8}\mu_1\mu_2\mu_3) - \tfrac{1}{2}\mu_2\mu_3 - \tfrac{1}{8}\mu_1\mu_2 - \tfrac{1}{2}\mu_1\mu_3. \end{aligned} \quad (6)$$

Proof sketch. The asymptotic formulas follow directly from the determinant representation (3) and the expansions (5) by collecting the leading singular and constant terms. □

4. Geometric Classification of Critical Parameters

In this section, we investigate the geometric structure of the critical parameters where spectral transitions occur. The zeros of the threshold polynomials $C^\pm(\mu)$, introduced in Lemma 4, correspond to the emergence or disappearance of discrete eigenvalues at the band edges. Therefore, we focus on the characterization of the critical surfaces

$$\Gamma^\pm = \{\mu \in \mathbb{R}^3 : C^\pm(\mu) = 0\}. \quad (7)$$

Remark 4. The polynomials $C^\pm(\mu)$ satisfy the symmetry relation

$$C^+(\mu_1, \mu_2, \mu_3) = C^-(-\mu_1, -\mu_2, -\mu_3),$$

which reflects the invariance of the model under the transformation $\mu \mapsto -\mu$. This symmetry allows one to deduce the geometric structure of Γ^+ from that of Γ^- .

4.1. Threshold polynomials and affine structure

A key property that simplifies the analysis of $\Gamma^\pm$ is the affine (linear) dependence of $C^\pm$ on the parameter μ_1 . This allows us to represent the critical surfaces as graphs of functions over the (μ_2, μ_3) -plane.

Lemma 5 (Affine structure). *For each sign $\pm$, the threshold polynomial $C^\pm(\cdot)$ can be uniquely decomposed as*

$$C^\pm(\mu) = Q_0^\pm(\mu_2, \mu_3) + \mu_1 Q_1^\pm(\mu_2, \mu_3), \quad (8)$$

where the coefficients are given by

$$\begin{aligned} Q_0^\pm(\mu_2, \mu_3) &= \mu_2 \mu_3 \mp 2(\mu_2 + \mu_3), \\ Q_1^\pm(\mu_2, \mu_3) &= \mu_3 + \frac{1}{2}\mu_2 \mp (1 + \frac{1}{4}\mu_2 \mu_3). \end{aligned}$$

Proof. The affine linearity in μ_1 (given in (8)) follows directly from the explicit formula for $C^\pm(\mu)$ given in (6). The coefficients are obtained by evaluating $C^\pm$ and its partial derivative at $\mu_1 = 0$:

$$Q_0^\pm(\mu_2, \mu_3) = C^\pm(0, \mu_2, \mu_3), \quad Q_1^\pm(\mu_2, \mu_3) = \left. \frac{\partial C^\pm}{\partial \mu_1} \right|_{\mu_1=0}.$$

A straightforward computation yields the expressions stated above. $\square$

4.2. Partition of the parameter plane

The geometry of the surface $\Gamma^\pm$ is governed by the zero set of the coefficient $Q_1^\pm$. We now partition the (μ_2, μ_3) -plane according to the sign of this polynomial.

Lemma 6. *The zero set of the polynomial $Q_1^\pm(\cdot, \cdot)$ coincides with the graph of the rational function*

$$\mu_2^\pm(\mu_3) = \frac{4(\mu_3 \mp 1)}{\mu_3 \mp 2}, \quad \mu_3 \in \mathbb{R} \setminus \{\pm 2\}. \quad (9)$$

The domain of the function $\mu_2^\pm(\cdot)$ consists of two disjoint intervals:

$$I_0^- = (-2, \infty), \quad I_1^- = (-\infty, -2); \quad I_0^+ = (-\infty, 2), \quad I_1^+ = (2, \infty).$$

Consequently, the graph splits into two disjoint, smooth, unbounded curves

$$\tau_i^\pm := \{(\mu_2, \mu_3) : \mu_2 = \mu_2^\pm(\mu_3), \mu_3 \in I_i^\pm\}, \quad i = 0, 1.$$

These curves partition the (μ_2, μ_3) -plane into three disjoint, open, connected, unbounded components $D_j^\pm$ ($j = 0, 1, 2$) where $Q_1^\pm$ has a constant sign. Explicitly,

$$\begin{aligned} D_0^\pm &:= \{(\mu_2, \mu_3) : \mu_3 \in I_0^\pm, \pm\mu_2 < \pm\mu_2^\pm(\mu_3)\}, \\ D_1^\pm &:= \{(\mu_2, \mu_3) : \mu_3 \in I_0^\pm, \pm\mu_2 > \pm\mu_2^\pm(\mu_3)\} \cup \{(\mu_2, \mu_3) : \mu_3 = \pm 2\} \\ &\quad \cup \{(\mu_2, \mu_3) : \mu_3 \in I_1^\pm, \pm\mu_2 < \pm\mu_2^\pm(\mu_3)\}, \\ D_2^\pm &:= \{(\mu_2, \mu_3) : \mu_3 \in I_1^\pm, \pm\mu_2 > \pm\mu_2^\pm(\mu_3)\}. \end{aligned} \quad (10)$$

Proof. By solving the linear equation $Q_1^\pm(\mu_2, \mu_3) = 0$ for μ_2 , we directly obtain the expression for $\mu_2^\pm(\mu_3)$ in (9). The singularity at $\mu_3 = \pm 2$ generates a vertical asymptote that naturally splits the curve into two continuous branches $\tau_0^\pm$ and $\tau_1^\pm$. The domains $D_j^\pm$ are constructed such that their boundaries are formed by these branches and the line of singularity. Within each domain, the sign of $Q_1^\pm$ is invariant by continuity, as $Q_1^\pm$ is continuous and non-vanishing on $D_j^\pm$. $\square$

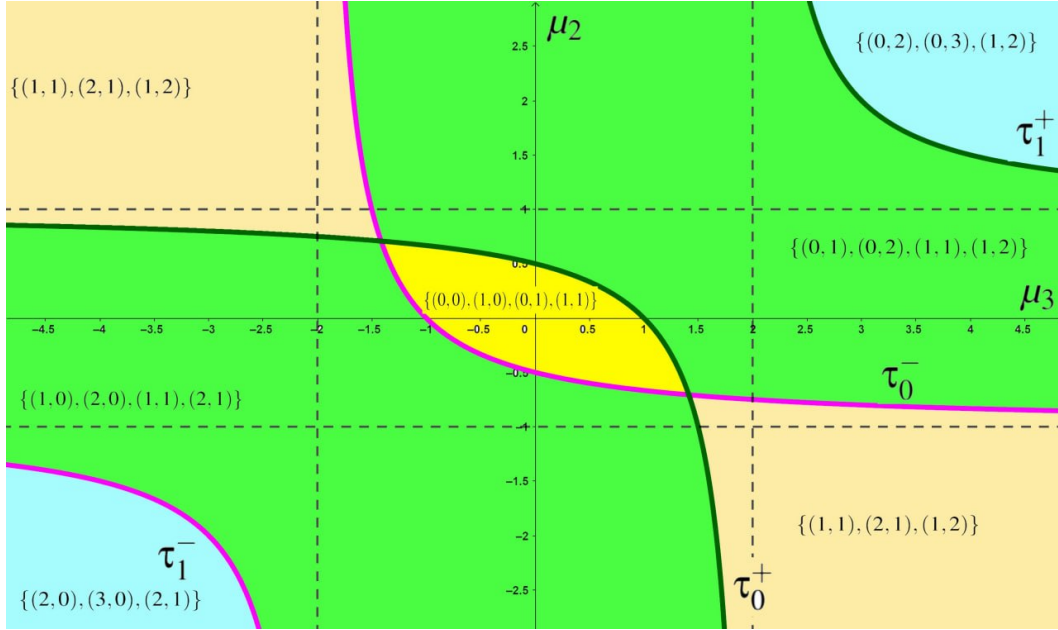


FIG. 1. Geometric structure of the domains D_j^- and D_j^+ in the (μ_2, μ_3) -plane, separated by the critical curves τ_i^- and τ_i^+ , $i = 0, 1$

Remark 5. The components $D_j^\pm$, $j = 0, 1, 2$ are shown schematically in Figure 1. For the minus sign (lower threshold), D_0^- lies to the left of the left branch, D_1^- is the region between the two branches (including the vertical line $\mu_3 = -2$), and D_2^- lies to the right of the right branch. For the plus sign (upper threshold), the roles of left/right are reversed due to the sign conventions in the inequalities.

4.3. Geometry of the critical surfaces

We now characterise the critical surfaces $\Gamma^\pm$ as a union of smooth sheets and describe the resulting partition of the parameter space $\mathbb{R}^3$.

Lemma 7 (Structure of $\Gamma^\pm$). The zero set $\Gamma^\pm$ defined in (7) consists of three disjoint, smooth, unbounded sheets:

$$\Gamma^\pm = \Gamma_0^\pm \cup \Gamma_1^\pm \cup \Gamma_2^\pm, \quad (11)$$

where each sheet $\Gamma_j^\pm$ is the graph of the rational function

$$\mu_1^\pm(\mu_2, \mu_3) = -\frac{Q_0^\pm(\mu_2, \mu_3)}{Q_1^\pm(\mu_2, \mu_3)} \quad (12)$$

over the domain $D_j^\pm$:

$$\Gamma_j^\pm := \{(\mu_1, \mu_2, \mu_3) : \mu_1 = \mu_1^\pm(\mu_2, \mu_3), (\mu_2, \mu_3) \in D_j^\pm\}, \quad j = 0, 1, 2.$$

These sheets partition $\mathbb{R}^3$ into four disjoint, open, connected, unbounded components $\mathcal{C}_k^\pm$ ($k = 0, 1, 2, 3$) defined as follows:

$$\begin{aligned} \mathcal{C}_0^\pm &:= \{\mu : (\mu_2, \mu_3) \in D_0^\pm, \pm\mu_1 < \pm\mu_1^\pm(\mu_2, \mu_3)\}, \\ \mathcal{C}_1^\pm &:= \{\mu : (\mu_2, \mu_3) \in D_0^\pm, \pm\mu_1 > \pm\mu_1^\pm(\mu_2, \mu_3)\} \cup \{\mu : (\mu_2, \mu_3) \in \tau_0^\pm\} \\ &\quad \cup \{\mu : (\mu_2, \mu_3) \in D_1^\pm, \pm\mu_1 < \pm\mu_1^\pm(\mu_2, \mu_3)\}, \\ \mathcal{C}_2^\pm &:= \{\mu : (\mu_2, \mu_3) \in D_1^\pm, \pm\mu_1 > \pm\mu_1^\pm(\mu_2, \mu_3)\} \cup \{\mu : (\mu_2, \mu_3) \in \tau_1^\pm\} \\ &\quad \cup \{\mu : (\mu_2, \mu_3) \in D_2^\pm, \pm\mu_1 < \pm\mu_1^\pm(\mu_2, \mu_3)\}, \\ \mathcal{C}_3^\pm &:= \{\mu : (\mu_2, \mu_3) \in D_2^\pm, \pm\mu_1 > \pm\mu_1^\pm(\mu_2, \mu_3)\}. \end{aligned} \quad (13)$$

Within each region $\mathcal{C}_k^\pm$, the threshold polynomial $C^\pm$ has a constant sign (alternating as k increases).

Proof. The threshold equation $C^\pm(\mu) = 0$ is affine in μ_1 and can be expressed as $\mu_1 Q_1^\pm(\mu_2, \mu_3) = -Q_0^\pm(\mu_2, \mu_3)$. Since the system of equations $Q_0^\pm(\mu_2, \mu_3) = Q_1^\pm(\mu_2, \mu_3) = 0$ has no real solutions, it follows that $Q_1^\pm$ is non-vanishing on the critical surface $\Gamma^\pm$. Thus, $\Gamma^\pm$ is uniquely represented as the graph of the rational function $\mu_1^\pm(\mu_2, \mu_3) = -Q_0^\pm/Q_1^\pm$ over the domain $\mathbb{R}^2 \setminus (\tau_0^\pm \cup \tau_1^\pm)$.

By Lemma 6, the domain of $\mu_1^\pm$ is the union of three disjoint, open components $D_0^\pm$, $D_1^\pm$, and $D_2^\pm$ (see Figure 1). Since $\mu_1^\pm$ is continuous on each $D_i^\pm$, its graph $\Gamma^\pm$ naturally inherits this topological structure, consisting of three disjoint, smooth sheets $\Gamma_i^\pm$.

The sign of $C^\pm(\mu)$ is governed by the factorization $C^\pm(\mu) = Q_1^\pm \cdot [\mu_1 - \mu_1^\pm(\mu_2, \mu_3)]$. Within each region $\mathcal{C}_k^\pm$, the continuity of $Q_1^\pm$ and $\mu_1^\pm$ ensures that $C^\pm$ remains non-vanishing. A sign change occurs only when crossing a sheet $\Gamma_i^\pm$ or passing through the singular boundaries $\tau_k^\pm$, which precisely defines the transitions between the four spectral regions. $\square$

4.4. Discrete spectrum classification

We now characterise the dependence of the number of discrete eigenvalues on the interaction parameter vector $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$. The discrete spectrum phases are determined by the sign of the threshold polynomials $C^\pm(\mu)$, which in turn depends on the position of μ relative to the critical surfaces $\Gamma^\pm$ (see (11)).

Theorem 1. *The numbers of discrete eigenvalues $n_\pm(H_\mu(0))$ are determined by the interaction parameters as follows:*

- (i) *Within each open connected component $\mathcal{C}_k^\pm$, where $C^\pm(\mu) \neq 0$, the count is constant:*

$$n_\pm(H_\mu(0)) = k \quad \text{if } \mu \in \mathcal{C}_k^\pm, \quad k \in \{0, 1, 2, 3\}.$$

- (ii) *On the critical surfaces $\Gamma_j^\pm$, where $C^\pm(\mu) = 0$, the count is given by*

$$n_\pm(H_\mu(0)) = j \quad \text{if } \mu \in \Gamma_j^\pm, \quad j \in \{0, 1, 2\}.$$

Proof. The proof is based on the analytic properties of the Fredholm determinant $\Delta_\mu(z)$ and the continuity of discrete eigenvalues.

(i) For any $\mu \in \mathcal{C}_k^\pm$, the condition $C^\pm(\mu) \neq 0$ ensures that the threshold $z = 0$ (resp. $z = 8$) is not a zero of the determinant. Since $\Delta_\mu(z)$ is analytic outside the essential spectrum and depends continuously on μ , the number of its zeros in $(-\infty, 0)$ (resp. $(8, \infty)$) is locally constant by the argument principle. The connectivity of each component $\mathcal{C}_k^\pm$ then implies that this count remains globally constant within the region. The specific values of k are determined by evaluating $\Delta_\mu(z)$ at representative points in each domain or by considering the asymptotic behavior as $|\mu_1| \rightarrow \infty$. A detailed proof can be obtained by referring to the proofs of a similar theorem presented in [27].

(ii) On the critical surface $\Gamma_j^\pm$, the leading singular coefficient $C^\pm$ vanishes, so the corresponding eigenvalue lies exactly at the threshold and is not counted as a discrete eigenvalue. The remaining eigenvalues are those that have already crossed the threshold, giving the count j . $\square$

Remark 6. *The components $\mathcal{C}_k^\pm$, $k = 0, 1, 2, 3$ are ordered by the number of discrete eigenvalues they host: as k increases, the sign of $C^\pm$ changes, and one additional eigenvalue appears at the corresponding threshold. The boundaries $\Gamma_j^\pm$ correspond to the moment when the j -th eigenvalue (counting from the threshold) is exactly at the band edge. The singular curves $\tau_i^\pm$ (where $Q_1^\pm = 0$) correspond to vertical asymptotes of $\mu_1^\pm$; they separate regions where the graph $\Gamma^\pm$ has different asymptotic behaviour.*

Since the collection of regions $\{\mathcal{C}_k^\pm\}_{k=0}^3$ together with their boundaries $\{\Gamma_j^\pm\}_{j=0}^2$ covers $\mathbb{R}^3$, we define the combined spectral sets

$$\mathcal{R}_\alpha^\pm := \mathcal{C}_\alpha^\pm \cup \Gamma_\alpha^\pm, \quad \alpha \in \{0, 1, 2, 3\},$$

where we set $\Gamma_3^\pm = \emptyset$ for consistency.

Corollary 1. *For any $\mu \in \mathbb{R}^3$, the operator $H_\mu(0)$ has exactly α discrete eigenvalues below (resp. above) the essential spectrum if and only if $\mu \in \mathcal{R}_\alpha^-$ (resp. $\mu \in \mathcal{R}_\alpha^+$):*

$$n_\pm(H_\mu(0)) = \alpha \iff \mu \in \mathcal{R}_\alpha^\pm, \quad \alpha \in \{0, 1, 2, 3\}.$$

Proof. The equivalence follows directly from Theorem 1: on $\mathcal{C}_\alpha^\pm$ we have $n_\pm = \alpha$; on $\Gamma_\alpha^\pm$ (for $\alpha = 0, 1, 2$) we also have $n_\pm = \alpha$; and for $\alpha = 3$, $\Gamma_3^\pm = \emptyset$ and $\mathcal{C}_3^\pm$ gives $n_\pm = 3$. The sets $\mathcal{R}_\alpha^\pm$ are disjoint for different α and cover $\mathbb{R}^3$. $\square$

The intersection regions $\mathcal{G}_{\alpha\beta}$, representing the sets of parameters where the spectral indices α and β are simultaneously realised, are defined by

$$\mathcal{G}_{\alpha\beta} := \mathcal{R}_\alpha^- \cap \mathcal{R}_\beta^+, \quad \alpha, \beta \in \{0, 1, 2, 3\}.$$

The collection of these intersections forms a partition of the parameter space:

$$\mathbb{R}^3 = \bigcup_{\alpha, \beta=0}^3 \mathcal{G}_{\alpha\beta}, \quad \mathcal{G}_{\alpha\beta} \cap \mathcal{G}_{\alpha'\beta'} = \emptyset \text{ for } (\alpha, \beta) \neq (\alpha', \beta'). \quad (14)$$

Theorem 2. *The operator $H_\mu(0)$ has exactly α discrete eigenvalues below the essential spectrum and β discrete eigenvalues above it if and only if $\mu \in \mathcal{G}_{\alpha\beta}$. Moreover, $\mathcal{G}_{\alpha\beta}$ is non-empty precisely when $\alpha + \beta \leq 3$.*

Proof. The first statement follows from Corollary 1 and the definition of $\mathcal{G}_{\alpha\beta}$. The partition property (14) ensures that each μ gives a unique pair (α, β) . The condition $\alpha + \beta \leq 3$ is a consequence of the rank-three nature of the perturbation (see (2)), which limits the total number of discrete eigenvalues. Conversely, for any pair (α, β) with $\alpha + \beta \leq 3$, one can construct parameters achieving this configuration (e.g., by choosing μ_1 appropriately in the corresponding region), so the intersection is non-empty. $\square$

Remark 7. The geometric classification provided in this section is exhaustive. The sets $\mathcal{G}_{\alpha\beta}$ are either open (when both α and β come from open regions) or include boundary parts (when one or both indices correspond to critical surfaces). Their explicit description can be obtained by intersecting the corresponding $\mathcal{R}_{\alpha}^{\pm}$ and $\mathcal{R}_{\beta}^{\pm}$; the resulting regions are semi-algebraic sets.

5. Spectral configurations in the (μ_2, μ_3) plane and their classification

In this section, we provide a geometric classification of the spectral regimes for the operator $H_{\mu}(0)$. The partition of the (μ_2, μ_3) -plane into the domains $D_k^{\pm}$, $k = 0, 1, 2$ (see (10)) plays a crucial role in determining the range of the spectral configuration mapping. Before establishing the main spectral counts, we characterize the fundamental geometric properties and symmetries of these regions in the following lemma.

Lemma 8 (Geometric structure of $D_k^{\pm}$). *The domains $D_k^{\pm}$, $k = 0, 1, 2$, satisfy the following properties:*

- (i) *Origin symmetry:* $D_k^{\pm} = \{(\mu_2, \mu_3) \in \mathbb{R}^2 : (-\mu_2, -\mu_3) \in D_k^{\mp}\}$;
- (ii) *Sub-domain inclusion:* $D_2^{+} \subset D_0^{-}$ and $D_2^{-} \subset D_0^{+}$;
- (iii) *Maximal index disjointness:* $D_2^{-} \cap D_2^{+} = \emptyset$.

Figure 2 illustrates the intersections and relative positions of these domains.

Sketch of Proof. The symmetry (i) follows directly from the parity of the threshold polynomials and the definitions of $\mu_2^{\pm}(\mu_3)$. The inclusion (ii) and disjointness (iii) are established by analyzing the boundaries $\mu_2^{\pm}(\mu_3)$ and the intervals $I_k^{\pm}$, which ensure that the growth conditions for $D_2^{\pm}$ do not overlap and are strictly bounded by the conditions defining $D_0^{\mp}$. $\square$

Following the geometric characterization of the domains $D_k^{\pm}$, we now determine the admissible spectral configurations (n_-, n_+) within their various intersections. While the parameter μ_1 shifts the eigenvalues within the gaps, the set of all possible values for the pair (n_-, n_+) is strictly governed by the location of (μ_2, μ_3) in the parameter plane.

For any fixed $j, k \in \{0, 1, 2\}$, let us consider a fixed pair of interaction parameters $(\mu_2, \mu_3) \in D_j^{-} \cap D_k^{+}$. For a given $\mu_1 \in \mathbb{R}$, we denote by $n_-(\mu_1)$ and $n_+(\mu_1)$ the number of discrete eigenvalues of the operator $H_{\mu}(0)$ lying below and above the essential spectrum, respectively:

$$n_-(\mu_1) := n_-(H_{\mu}(0)), \quad n_+(\mu_1) := n_+(H_{\mu}(0)).$$

The pair $(n_-(\mu_1), n_+(\mu_1))$ characterizes the spectral configuration of the operator for a specific intensity μ_1 . To analyze the evolution of the spectrum, we define the spectral configuration mapping $n : \mathbb{R} \rightarrow \mathbb{Z}_{\geq 0}^2$ as follows:

$$n(\mu_1) = (n_-(\mu_1), n_+(\mu_1)), \quad \mu_1 \in \mathbb{R}.$$

The central object of our analysis is the range of this mapping, which represents the set of all spectral configurations attainable for the given geometry (μ_2, μ_3) . We denote this set by $\mathcal{S}_{jk}$ and define it as the image of the function n over the entire real line:

$$\mathcal{S}_{jk} := \text{Im}(n) = \{n(\mu_1) : \mu_1 \in \mathbb{R}\}. \quad (15)$$

By identifying the elements of $\mathcal{S}_{jk}$, we effectively determine all possible spectral phases that the operator $H_{\mu}(0)$ can exhibit as the interaction μ_1 varies over $\mathbb{R}$.

The following theorem provides a complete classification of the spectral configuration sets $\mathcal{S}_{jk}$. This result is based on two fundamental constraints: the independent component-wise bounds established in Lemma 9 (and further refined in Remark 8), and the global rank-three condition, which limits the total number of discrete eigenvalues.

Lemma 9 (Independent bounds). *Let $j, k \in \{0, 1, 2\}$. For the fixed $(\mu_2, \mu_3) \in D_j^{-} \cap D_k^{+}$ and for every $\mu_1 \in \mathbb{R}$, the followings are true:*

- (i) $n_-(\mu_1) \in \{j, j+1\}$;
- (ii) $n_+(\mu_1) \in \{k, k+1\}$;

Proof. We prove the assertion for n_- ; the case for n_+ follows by symmetry. Fix $(\mu_2, \mu_3) \in D_j^{-}$. According to the definition of the disjoint components $\mathcal{C}_k^{-}$ in (13), the parameter vector $\mu = (\mu_1, \mu_2, \mu_3)$ must belong to one of the following sets as μ_1 varies over $\mathbb{R}$:

- (1) The region where $\mu_1 > \mu_1^{-}(\mu_2, \mu_3)$, which corresponds to $\mu \in \mathcal{C}_j^{-}$;
- (2) The threshold hypersurface $\Gamma_j^{-} := \{\mu : \mu_1 = \mu_1^{-}(\mu_2, \mu_3)\}$;

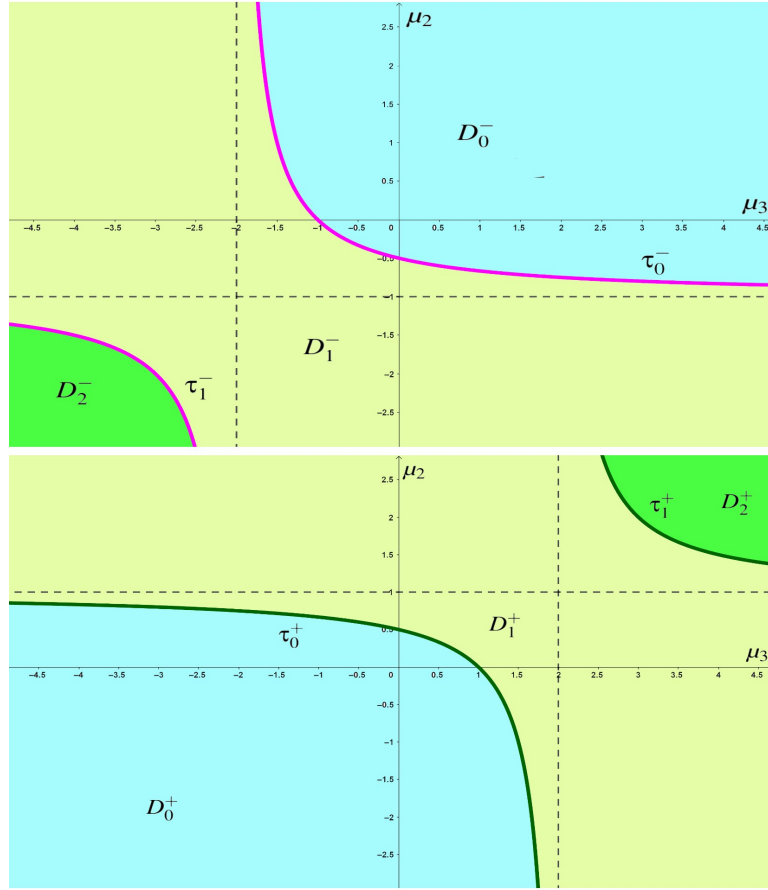


FIG. 2. Geometric structure of the intersections $D_j^- \cap D_k^+$ in the (μ_2, μ_3) -plane. Each overlapping region $D_j^- \cap D_k^+$ corresponds to a specific set of attainable spectral configurations $\mathcal{S}_{jk}$, as characterized in Theorem 3

(3) The region where $\mu_1 < \mu_1^-(\mu_2, \mu_3)$, which corresponds to $\mu \in \mathcal{C}_{j+1}^-$.

Thus, for any $\mu_1 \in \mathbb{R}$, we have the inclusion $\mu \in \mathcal{C}_j^- \cup \Gamma_j^- \cup \mathcal{C}_{j+1}^-$. By Theorem 1, the eigenvalue count n_- is exactly j in $\mathcal{C}_j^- \cup \Gamma_j^-$ and $j+1$ in $\mathcal{C}_{j+1}^-$. This implies that $n_-(\mu_1) \in \{j, j+1\}$. $\square$

Remark 8. For fixed $(\mu_2, \mu_3) \in D_k^-$, let $\mu_1^-(\mu_2, \mu_3)$ be the threshold function defined in (12). By Lemma 9, the count n_- depends on the position of μ_1 relative to the threshold graph $\mu_1^-(\mu_2, \mu_3)$:

$$n_- = \begin{cases} k, & \mu_1 \geq \mu_1^-(\mu_2, \mu_3), \\ k+1, & \mu_1 < \mu_1^-(\mu_2, \mu_3). \end{cases}$$

Similarly, for fixed $(\mu_2, \mu_3) \in D_k^+$, let $\mu_1^+(\mu_2, \mu_3)$ be defined in (12). Then

$$n_+ = \begin{cases} k, & \mu_1 \leq \mu_1^+(\mu_2, \mu_3), \\ k+1, & \mu_1 > \mu_1^+(\mu_2, \mu_3). \end{cases}$$

Theorem 3 (Classification of configurations on domain intersections). Let $(\mu_2, \mu_3) \in D_j^- \cap D_k^+$ with $j, k \in \{0, 1, 2\}$, and let $\mathcal{S}_{jk}$ be the spectral configuration set defined by (15). Then

$$\mathcal{S}_{jk} = (\{j, j+1\} \times \{k, k+1\}) \cap \{(a, b) \in \mathbb{N}_0^2 : a + b \leq 3\}. \quad (16)$$

More explicitly,

$$\begin{aligned}\mathcal{S}_{00} &= \{(0, 0), (1, 0), (0, 1), (1, 1)\}, \\ \mathcal{S}_{10} &= \{(1, 0), (2, 0), (1, 1), (2, 1)\}, \\ \mathcal{S}_{11} &= \{(1, 1), (2, 1), (1, 2)\}, \\ \mathcal{S}_{20} &= \{(2, 0), (3, 0), (2, 1)\}, \\ \mathcal{S}_{01} &= \{(0, 1), (0, 2), (1, 1), (1, 2)\}, \\ \mathcal{S}_{02} &= \{(0, 2), (0, 3), (1, 2)\}.\end{aligned}$$

Proof. Let $(\mu_2, \mu_3) \in D_j^- \cap D_k^+$. From Lemma 9, any spectral configuration $(n_-, n_+) \in \mathcal{S}_{jk}$ must satisfy $n_- \in \{j, j+1\}$ and $n_+ \in \{k, k+1\}$. Additionally, the global rank-three constraint $n_- + n_+ \leq 3$ must hold (see (2)). This implies the inclusion:

$$\mathcal{S}_{jk} \subseteq (\{j, j+1\} \times \{k, k+1\}) \cap \{(a, b) : a + b \leq 3\}.$$

To prove that $\mathcal{S}_{jk}$ coincides with the intersection of these two conditions, we analyze the configuration as a function of μ_1 by comparing it to the threshold values of the functions $\mu_1^- := \mu_1^-(\mu_2, \mu_3)$ and $\mu_1^+ := \mu_1^+(\mu_2, \mu_3)$. According to the Remark 8, for a fixed $(\mu_2, \mu_3) \in \mathbb{R}^2$, the spectral counts are step functions of μ_1 :

$$n_-(\mu_1) = \begin{cases} j, & \mu_1 \geq \mu_1^- \\ j+1, & \mu_1 < \mu_1^- \end{cases} \quad \text{and} \quad n_+(\mu_1) = \begin{cases} k, & \mu_1 \leq \mu_1^+ \\ k+1, & \mu_1 > \mu_1^+ \end{cases}$$

The set of realized configurations $\mathcal{S}_{jk}$ is thus determined by the relative order of μ_1^- and μ_1^+ . Specifically:

- If $\mu_1 \in [\mu_1^-, \mu_1^+]$ (assuming $\mu_1^- \leq \mu_1^+$), the configuration is (j, k) .
- If $\mu_1 > \max\{\mu_1^-, \mu_1^+\}$, the configuration is $(j, k+1)$.
- If $\mu_1 < \min\{\mu_1^-, \mu_1^+\}$, the configuration is $(j+1, k)$.
- If μ_1 lies between μ_1^- and μ_1^+ in the case where $\mu_1^+ < \mu_1 < \mu_1^-$, the configuration is $(j+1, k+1)$, provided that $(j+1) + (k+1) \leq 3$.

Since μ_1 can take any real value, all four combinations $\{(j, k), (j, k+1), (j+1, k), (j+1, k+1)\}$ are attainable if and only if they satisfy the total rank constraint $n_- + n_+ \leq 3$. If a pair violates this constraint, the corresponding interval for μ_1 becomes empty. Therefore the inclusion above is actually an equality.

It remains to compute the finite intersection for each pair (j, k) . For example, when $j = 1, k = 1$:

$$\{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

The condition $a + b \leq 3$ excludes $(2, 2)$, giving $\mathcal{S}_{11} = \{(1, 1), (2, 1), (1, 2)\}$. Similarly,

$$\mathcal{S}_{20} = \{2, 3\} \times \{0, 1\} \cap \{a + b \leq 3\} = \{(2, 0), (3, 0), (2, 1)\}.$$

The other cases are computed in the same way, yielding the explicit lists stated in the theorem. $\square$

5.1. An algebraic criterion for domain intersections

For the rank-three case the intersections $D_j^- \cap D_k^+$, $j, k \in \{0, 1, 2\}$ can be visualized directly (see Figure 2), but such a geometric approach becomes impractical as the rank of the perturbation increases. To provide a robust analytical framework that transcends low-dimensional geometry, we establish an algebraic criterion for the non-emptiness of domain intersections. This criterion is a first step toward extending the analysis to higher-rank perturbations (rank- N), where spectral regimes must be identified through the consistency of reachable state sets rather than graphical overlaps.

Theorem 4. Let $j, k \in \{0, 1, 2\}$ be arbitrary integers, and let D_j^- and D_k^+ be the domains in the (μ_2, μ_3) plane defined in (10). The intersection $D_j^- \cap D_k^+$ is non-empty if and only if the set of reachable spectral configurations $\mathcal{S}_{jk}$, defined in (16), is non-empty:

$$D_j^- \cap D_k^+ \neq \emptyset \iff \mathcal{S}_{jk} \neq \emptyset.$$

Proof. ($\Rightarrow$) Suppose that $D_j^- \cap D_k^+ \neq \emptyset$ and fix a point $(\mu_2, \mu_3) \in D_j^- \cap D_k^+$. Then, for any $\mu_1 \in \mathbb{R}$, the parameter vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ corresponds to a spectral configuration $(n_-(\mu_1), n_+(\mu_1))$ which, by definition, belongs to $\mathcal{S}_{jk}$. Thus, $\mathcal{S}_{jk} \neq \emptyset$.

($\Leftarrow$) Conversely, suppose that $\mathcal{S}_{jk} \neq \emptyset$. This implies there exists at least one configuration $(n_-, n_+) \in \mathcal{S}_{jk}$ that is reachable for some parameter vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$. According to the definition of $\mathcal{S}_{jk}$ in (16), the existence of such a configuration requires $(\mu_2, \mu_3) \in D_j^- \cap D_k^+$. Consequently,

$$D_j^- \cap D_k^+ \neq \emptyset.$$

$\square$

Example 1 (Non-existence of specific regimes). *The power of the algebraic criterion is best illustrated by considering the case $j = 2$ and $k = 2$. Geometrically, it is established in Lemma 1 that $D_2^- \cap D_2^+ = \emptyset$, meaning there is no point in the parameter plane that simultaneously supports the indices $j = 2$ and $k = 2$.*

Algebraically, this is reflected in the emptiness of the set $\mathcal{S}_{22}$. Indeed, according to (16):

$$\mathcal{S}_{22} = (\{2, 3\} \times \{2, 3\}) \cap \{(a, b) \in \mathbb{N}_0^2 : a + b \leq 3\}.$$

The Cartesian product $\{2, 3\} \times \{2, 3\}$ contains the configurations

$$\{(2, 2), (2, 3), (3, 2), (3, 3)\}.$$

For all these pairs, the sum $a + b \geq 4$, which violates the rank-three constraint $a + b \leq 3$. Consequently, $\mathcal{S}_{22} = \emptyset$, which is in perfect agreement with the geometric fact that $D_2^- \cap D_2^+ = \emptyset$.

Remark 9. *The algebraic criterion is particularly useful when the domains $D_j^\pm, j = 0, 1, 2$ are defined by polynomial inequalities; one can then check feasibility of the combined system without explicit geometric construction. This idea generalises naturally to higher-rank perturbations.*

5.2. Global spectral classifications

After characterizing the local configurations in the (μ_2, μ_3) -plane, we now establish a global classification of the discrete spectrum. In this context, we treat μ_3 as a fixed structural parameter and analyze the attainable spectral states as the remaining coupling intensities (μ_1, μ_2) vary over the entire real plane.

To formalize this, for a fixed $\mu_3 \in \mathbb{R}$, we define the global spectral configuration mapping $N_{\mu_3} : \mathbb{R}^2 \rightarrow \mathbb{Z}_{\geq 0}^2$ as follows:

$$N_{\mu_3} : (\mu_1, \mu_2) \rightarrow (n_-(H_{\mu}(0)), n_+(H_{\mu}(0))), \mu = (\mu_1, \mu_2, \mu_3), (\mu_1, \mu_2) \in \mathbb{R}^2.$$

The main object of interest is the set of all reachable configurations for a given μ_3 , which corresponds to the image (range) of this mapping:

$$\mathcal{S}_{\mu_3} := \text{Im}(N_{\mu_3}) = \{N_{\mu_3}(\mu_1, \mu_2) : (\mu_1, \mu_2) \in \mathbb{R}^2\}.$$

The following theorem demonstrates that the topology of the parameter space and the resulting range $\mathcal{S}_{\mu_3}$ undergo qualitative shifts at the critical thresholds $\mu_3 = \pm 2$.

Theorem 5 (Global Classification of Spectral Configurations). *The set of reachable spectral configurations $\mathcal{S}_{\mu_3}$ is uniquely determined by the value of μ_3 as follows:*

- (i) *If $\mu_3 < -2$, then $\mathcal{S}_{\mu_3} = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0)\}$;*
- (ii) *If $|\mu_3| \leq 2$, then $\mathcal{S}_{\mu_3} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1)\}$;*
- (iii) *If $\mu_3 > 2$, then $\mathcal{S}_{\mu_3} = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 1)\}$.*

Proof. By construction, the global set $\mathcal{S}_{\mu_3}$ can be expressed as the union of the local configuration sets $\mathcal{S}_{jk}$ over all indices j, k for which the parameter pair (μ_2, μ_3) can actually reside in the intersection $D_j^- \cap D_k^+$. Thus, for a fixed μ_3 , we have:

$$\mathcal{S}_{\mu_3} = \bigcup \{ \mathcal{S}_{jk} : \exists \mu_2 \in \mathbb{R} \text{ such that } (\mu_2, \mu_3) \in D_j^- \cap D_k^+ \}.$$

The proof then proceeds by analyzing the non-emptiness of the domains $D_j^\pm$ relative to the value of μ_3 :

Case (i): $\mu_3 < -2$. In this regime, the domain D_0^- is empty for all $\mu_2 \in \mathbb{R}$. This implies that $j \geq 1$ must hold, ensuring at least one eigenvalue exists below the spectrum ($n_- \geq 1$). Combining the local sets $\mathcal{S}_{jk}$ for $j \in \{1, 2\}$ and $k \in \{0, 1\}$ from Theorem 3, and filtering by the global rank-three condition $n_- + n_+ \leq 3$, we obtain the set specified in (i).

Case (ii): $|\mu_3| \leq 2$. In this central strip, the intersection $D_0^- \cap D_0^+$ is non-empty, which allows for the vacuum configuration $(0, 0)$. All local sets $\mathcal{S}_{jk}$ contribute to the union, and the exhaustive search over $j, k \in \{0, 1, 2\}$ under the constraint $a + b \leq 3$ yields the collection in (ii).

Case (iii): $\mu_3 > 2$. Due to the origin symmetry established in Lemma 1, the domain D_0^+ is empty in this region. This forces $n_+ \geq 1$ for any choice of (μ_1, μ_2) . The union of $\mathcal{S}_{jk}$ for $k \in \{1, 2\}$ and $j \in \{0, 1\}$ leads directly to the set listed in (iii). $\square$

Remark 10. *The critical values $\mu_3 = \pm 2$ act as bifurcation points for the spectral template. At $\mu_3 = -2$, the domain D_0^- collapses, leading to a "spectral jump" where the zero-eigenvalue state $(0, n_+)$ is no longer attainable. A symmetric transition occurs at $\mu_3 = 2$ for the upper eigenvalues.*

5.3. Comparison with nonzero quasimomentum

Theorem 6. For any $\mu \in \mathbb{R}^3$ and $K \in \mathbb{T} \setminus \{0\}$,

$$n_{\pm}(H_{\mu}(K)) \geq n_{\pm}(H_{\mu}(0)).$$

Moreover, if $H_{\mu}(0)$ possesses a threshold resonance at $z = 0$ (resp. $z = 8$), then for any $K \neq 0$,

$$n_{-}(H_{\mu}(K)) > n_{-}(H_{\mu}(0)) \quad (\text{resp. } n_{+}(H_{\mu}(K)) > n_{+}(H_{\mu}(0))).$$

Sketch of proof. The proof of Theorem 6 is established by analyzing the displacement of the essential spectrum and the monotonicity of the discrete eigenvalues with respect to the total momentum K . The core argument relies on the comparison of the Lippman-Schwinger operators associated with $H_{\mu}(K)$ and $H_{\mu}(0)$. Specifically, the inequality follows from the fact that the minimum of the two-particle energy band is attained at $K = 0$. A detailed rigorous derivation is provided in [27, Theorem 3.1]. $\square$

Corollary 2. If $\mu \in \mathcal{G}_{\alpha\beta}$ for $\alpha, \beta \in \{0, 1, 2\}$, then the following lower bounds hold:

$$n_{-}(H_{\mu}(K)) \geq \alpha \quad \text{and} \quad n_{+}(H_{\mu}(K)) \geq \beta$$

for all $K \in \mathbb{T}$. Moreover, if $\alpha + \beta = 3$, then these inequalities become equalities for all $K \in \mathbb{T}$.

Proof. The result follows directly from Theorem 6 and Theorem 2. $\square$

A key observation from the above classification is the persistence of the discrete spectrum in the extreme regimes. Specifically, for any quasi-momentum K , the existence of at least one eigenvalue is guaranteed when $|\mu_3|$ is sufficiently large. This is summarized in the following corollary.

Corollary 3. For any $K \in \mathbb{T}$, the following lower bounds on the number of eigenvalues hold:

- If $\mu_3 < -2$, then $n_{-}(H_{\mu}(K)) \geq 1$.
- If $\mu_3 > 2$, then $n_{+}(H_{\mu}(K)) \geq 1$.

In these regimes, the discrete spectrum is non-empty for all values of the total momentum K .

Proof. The proof follows directly from the global classification established in Theorem 5 and the monotonicity properties with respect to the total momentum K provided in Theorem 6. $\square$

6. Conclusion

In this study we have provided a complete global classification of the discrete spectrum configurations (n_{-}, n_{+}) for the fiber Hamiltonian $H_{\mu}(0)$ with a symmetric rank-three potential. By exploiting the affine dependence of the Fredholm determinant on μ_1 , we have systematically mapped the transitions between different spectral phases.

The main conclusions are as follows:

- **Necessary and sufficient conditions:** We have established a partition of $\mathbb{R}^3$ into disjoint sets $\mathcal{G}_{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2, 3, \alpha + \beta \leq 3$) that provide necessary and sufficient conditions for the realisation of any spectral state (α, β) . These conditions are consistent with the rank-three nature of the perturbation, which imposes the fundamental bound $n_{-} + n_{+} \leq 3$.
- **Spectral phase determination:** The (μ_2, μ_3) plane is divided into domains $D_j^{\pm}$ ($j = 0, 1, 2$) that act as “spectral templates”. For any μ_1 , the possible numbers of eigenvalues $n_{\pm}$ are restricted to a specific set determined solely by the (μ_2, μ_3) region. Thus the interaction range (governed by μ_2, μ_3) defines the potential’s capacity to support bound states, while the central intensity μ_1 acts as a secondary trigger for eigenvalue emergence within those bounds.
- **Global role of μ_3 :** The parameter μ_3 controls the global topological structure of the spectrum. Transitions at the critical values $\mu_3 = \pm 2$ fundamentally alter the admissible configurations, e.g., forcing at least one eigenvalue below the lower threshold when $\mu_3 < -2$ or above the upper threshold when $\mu_3 > 2$.

These findings offer a robust framework for predicting the multiplicity and configuration of the discrete spectrum in lattice systems with multi-site interactions. The methods developed here can be extended to analyse spectral properties in higher dimensions or in systems with more complex interaction geometries.

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