

## On eigenvalues of the discrete Schrödinger operator on a ladder of a $d$ -dimensional square lattice

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**ABSTRACT** We consider a discrete Schrödinger operator  $H$  on a ladder of  $d$ -dimensional square lattices with a self-adjoint matrix-valued potential whose entries are the scalars  $\lambda$  and  $\mu$ . We investigate conditions for the existence of eigenvalues and describe their exact locations in terms of the parameters  $\lambda$  and  $\mu$  for the cases  $d = 1$  and  $d = 2$ .

**KEYWORDS** discrete Schrödinger operator; eigenvalue; discrete spectrum.

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### 1. Introduction

In paper [1], the authors studied finite graphene nanostructures—graphene nanoflakes and graphene quantum dots—from a quantum-mechanical perspective. They showed that when graphene is reduced to nanoscale dimensions, its electronic properties differ significantly from those of bulk graphene because of finite-size effects, quantum confinement, edge geometry (zigzag versus armchair), and the shape of the nanostructure. This observation is directly relevant to discrete Schrödinger models, since finite lattice Hamiltonians naturally reproduce the size-dependent discrete energy levels arising in such nanosystems. Paper [2] focuses on recent experimental progress in graphene quantum dots. It emphasizes that graphene quantum dots are model nanosystems for studying quantum confinement of Dirac fermions, quasi-bound states, Berry phase effects, edge-induced confinement, and single-electron phenomena. Moreover, this work shows that graphene quantum dots are experimentally realizable nanosystems whose energy levels and localization properties can be naturally described by discrete lattice Hamiltonians.

The existence conditions for eigenvalues of the discrete Schrödinger operator with a  $\delta$  potential were studied in [3]. The same operator was considered in [4], where the authors investigated the behavior of the spectral edges (embedded eigenvalues and resonances) at the two ends of the continuous spectrum, in particular it is shown that an embedded eigenvalue does appear for  $d \geq 5$  but does not for  $1 \leq d \leq 4$ . An exact expression for the number of discrete eigenvalues of discrete Laplacian with finitely supported potential is described in [7]. K. Ando et al. [10] described the spectral properties of the Schrödinger operators with compactly supported potentials on square, triangular, hexagonal, Kagome, diamond, subdivision lattices. The inverse scattering for discrete Schrödinger operators with compactly supported potentials on  $\mathbb{Z}^d$  and on the hexagonal lattice were studied in [5] and [6] only for special cases, in these papers, the discreteness of embedded eigenvalues of these operators was proved.

The absence of embedded eigenvalues for Schrödinger operators with exponentially decaying potentials was proved in [9] using the unique continuation property. Using basic tools from several complex variables and algebraic geometry, in particular Hilbert's Nullstellensatz, Isozaki and Morioka [8] established a Rellich–Vekua type theorem for Schrödinger operators on the square lattice in the case of finitely supported potentials. The Rellich–Vekua type uniqueness theorem, the unique continuation theorem, and the non-existence of eigenvalues embedded in the continuous spectrum for discrete Schrödinger operators with compactly supported potentials on periodic lattices including the square, triangular, and hexagonal lattices were studied in [10], where Kagome lattice and the subdivision of the square lattice, there are examples of Hamiltonians with compactly supported potentials having embedded eigenvalues. Paper [11] extended the results of [10] for Schrödinger operators with exponentially decreasing potentials on a class of lattices including square, triangular, hexagonal lattices and their ladders. The authors discussed the unique continuation property and provided a

counterexample for Hamiltonians on a ladder of a  $d$ -dimensional square lattice with a diagonal matrix potential whose entries are identical scalars, exhibiting an embedded eigenvalue.

Note that the Schrödinger operators with compactly supported potentials acting on the diagonal entries of ladder graphs were also studied in [10] and [12] as simple examples of a general class of perturbed periodic operators possessing embedded eigenvalues. In particular, in [10], the compactly supported potentials were taken to be  $\delta$ -potentials acting on the diagonal entries. In [12], a broad class of operators obtained by coupling  $m$  (possibly different) operators defined on identical graphs was investigated. The resulting operator decouples into  $m$  independent subsystems with different spectral bands. Consequently, a non-embedded eigenvalue of one subsystem may become an embedded eigenvalue of the full coupled system if it lies inside a spectral band of another subsystem. Ladder graphs belong exactly to this class. In particular, Example 2 of [12] considers a non-compact perturbation acting on the off-diagonal entries.

In the present paper, we investigate spectral properties of the discrete Schrödinger operator  $H$  on a ladder of  $d$ -dimensional square lattices with a self-adjoint matrix-valued potential whose entries are the scalars  $\lambda$  and  $\mu$ . The main aim of this paper is to determine conditions for the existence of eigenvalues and to describe their exact locations in terms of the parameters  $\lambda$  and  $\mu$  for the cases  $d = 1$  and  $d = 2$ . In particular, we clarify the existence of eigenvalues embedded in the essential spectrum of  $H$ . Due to the symmetry of the ladder graph, the investigated problem reduces to the study of the Schrödinger type operators  $H_1$  and  $H_2$  on the lattice  $\mathbb{Z}^d$ , with delta potentials whose coupling constants are  $\lambda \pm \mu$ . Moreover, a non-embedded eigenvalue of  $H_1$  may become an embedded eigenvalue of the operator  $H$  if it lies within a spectral band of  $H_2$ , and conversely. Remark that this reduced problem is similar to that considered in [3], where the eigenvalues of a Schrödinger operator with a delta potential depending on a coupling constant parameter were investigated. Although the Schrödinger operator with a  $\delta$ -potential has been studied previously, the reduced operators considered in the present paper depends on the parameters  $\lambda \pm \mu$ . Therefore, it is of particular interest to provide a precise description of the eigenvalues in terms of the parameters  $\lambda$  and  $\mu$ .

## 2. Basic statements and the main results

### 2.1. Ladder of $d$ -dimensional square lattice in $\mathbb{R}^{d+1}$

The ladder of  $d$ -dimensional square lattice is realized in  $\mathbb{R}^{d+1}$ . (See [10]) Let  $\mathcal{L}_0 = \{(n_1, \dots, n_d, 0) : n_j \in \mathbb{Z}\}$ . Letting  $p_1 = (0, \dots, 0, 0)$  and  $p_2 = (0, \dots, 0, 1)$ , we put

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2,$$

where

$$\mathcal{V}_j = p_j + \mathcal{L}_0.$$

For  $v \in \mathcal{V}$ , we put

$$N_v = \{\omega \in \mathcal{V} : |\omega - v| = 1\}, \quad j = 1, 2.$$

The degree of a vertex  $v \in \mathcal{V}$  is the number of vertices adjacent to  $v$ . For all  $v \in \mathcal{V}$ ,  $\deg(v) = \#N_v = 2d + 1$ .

There exists a bijection  $\mathcal{V} \ni a \leftrightarrow (j(a), n(a)) \in \{1, 2\} \times \mathbb{Z}^d$  such that  $a = p_{j(a)} + v(n(a))$ . Then  $\ell_2(\mathcal{V})$  is isomorphic to  $\ell_2^{(2)}(\mathbb{Z}^d)$ , where  $\ell_2^{(2)}(\mathbb{Z}^d)$  is the Cartesian product Hilbert space of two copies of  $\ell_2(\mathbb{Z}^d)$ .

Any function  $\hat{f}$  on  $\mathcal{V}$  is written as  $\hat{f}(n) = (\hat{f}_1(n), \hat{f}_2(n)), n \in \mathbb{Z}^d$ , where  $\hat{f}_j(n)$  is identified with a function on  $p_j + \mathcal{L}_0$ . Hence  $\ell_2(\mathcal{V})$  is the Hilbert space equipped with the inner product

$$(\hat{f}, \hat{g})_{\ell_2(\mathcal{V})} = \sum_{j=1}^2 (\hat{f}_j, \hat{g}_j)_{\deg},$$

where  $(f, g)_{\deg}$  denotes the weighted inner product in  $\ell_2(\mathbb{Z}^d)$  defined as  $(f, g)_{\deg} = \sum_{v \in \mathcal{V}} \deg(v) f(v) \overline{g(v)}$ .

Let  $L_2(\mathbb{T}^d)$  be the Hilbert space of square-integrable functions defined on  $\mathbb{T}^d$ ,  $\mathbb{T} = (-\pi, \pi]$ , and let  $L_2^{(2)}(\mathbb{T}^d)$  denote the Cartesian product Hilbert space of two copies of  $L_2(\mathbb{T}^d)$ , where  $L_2^{(2)}(\mathbb{T}^d)$  is equipped with the inner product

$$(f, g)_{L_2^{(2)}(\mathbb{T}^d)} = \sum_{j=1}^2 \int_{\mathbb{T}^d} f_j(x) \overline{g_j(x)} dx.$$

We then define a unitary operator  $\mathcal{F} : \ell_2(\mathcal{V}) \rightarrow L_2^{(2)}(\mathbb{T}^d)$ ,

$$(\mathcal{F}\hat{f})_j = (2\pi)^{-d/2} \sqrt{2d+1} \sum_{n \in \mathbb{Z}^d} \hat{f}_j(n) e^{i(n,x)},$$

where the inverse  $\mathcal{F}^{-1} : L_2^{(2)}(\mathbb{T}^d) \rightarrow \ell_2(\mathcal{V})$  of  $\mathcal{F}$  is defined as

$$(\mathcal{F}^{-1} f_j)(n) = \frac{(2\pi)^{-d/2}}{\sqrt{2d+1}} \int_{\mathbb{T}^d} e^{-i(n,x)} f_j(x) dx.$$

Recall that the shift operator  $S_k$  acts on a sequence  $(a(n))_{n \in \mathbb{Z}^d}$  as follows:

$$(S_k(a))(n) = a(n + e_k),$$

where  $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ ,  $k = \overline{1, d}$ . Then we have

$$\mathcal{F}\hat{S}_k = e^{-ix_k} \mathcal{F}.$$

The Laplacian  $\hat{\Delta}$  on the ladder  $\mathcal{V}$  is defined by the following formula

$$(\hat{\Delta}f)(n) = \frac{1}{2d+1} \begin{pmatrix} f_2(n) + \sum_{k=1}^d (f_1(n + e_k) + f_1(n - e_k)) \\ f_1(n) + \sum_{k=1}^d (f_2(n + e_k) + f_2(n - e_k)) \end{pmatrix},$$

Consider the discrete Schrödinger operator on the  $d$ -dimensional ladder  $\mathcal{V}$  defined in the form

$$\hat{H} = -\hat{\Delta} - \hat{V},$$

where  $\hat{V}$  is a potential defined on  $\ell_2^{(2)}(\mathbb{Z}^d)$  as a multiplication operator by real-valued,  $2 \times 2$  matrices:

$$V(n) = \begin{pmatrix} \lambda \hat{v}(n) & \mu \hat{v}(n) \\ \mu \hat{v}(n) & \lambda \hat{v}(n) \end{pmatrix}, \quad \hat{v}(n) = \begin{cases} (2\pi)^d, & \text{for } n = \theta, \\ 0, & \text{for } n \neq \theta, \end{cases}$$

where  $\lambda$  and  $\mu$  are real numbers with  $(\lambda, \mu) \neq (0, 0)$ ,  $\theta$  is zero element in  $\mathbb{Z}^d$ .

To study spectral properties of  $\hat{H}$ , we consider  $H = \mathcal{F}\hat{H}\mathcal{F}^{-1}$ , where  $H$  acts in the Hilbert space  $L_2^{(2)}(\mathbb{T}^d)$  as

$$H = H_0 - V,$$

$H_0$  is a multiplication operator by matrix

$$H_0(x) = \frac{-1}{2d+1} \begin{pmatrix} 2 \sum_{i=1}^d \cos x_i & 1 \\ 1 & 2 \sum_{i=1}^d \cos x_i \end{pmatrix}$$

and  $V$  is an operator matrix

$$V = \begin{pmatrix} \lambda \mathbf{v} & \mu \mathbf{v} \\ \mu \mathbf{v} & \lambda \mathbf{v} \end{pmatrix}$$

with the integral operator  $\mathbf{v}f = \int_{\mathbb{T}^d} f(s)ds$ . Under these consideration,  $H$  is a self-adjoint operator.

We denote by  $\sigma(A)$ ,  $\sigma_p(A)$  and  $\sigma_{ess}(A)$ , respectively, the spectrum, the point spectrum and the essential spectrum of the operator  $A$ .

The operator  $H_0$  was previously analyzed in [6], where it was shown that

$$\sigma(H_0) = [-1, 1].$$

Because  $V$  is a compact operator, Weyl's theorem for the essential spectrum [14] yields

$$\sigma_{ess}(H) = \sigma(H_0).$$

Let

$$\tilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where

$$H_1 = H_{10} - (\lambda + \mu)\mathbf{v}, \quad H_2 = H_{20} - (\lambda - \mu)\mathbf{v},$$

$H_{10}$  and  $H_{20}$  are multiplication operators on  $L_2(\mathbb{T}^d)$  by functions

$$E_{10}(x) = -\frac{2}{2d+1} \sum_{i=1}^d \cos x_i - \frac{1}{2d+1} \quad \text{and} \quad E_{20}(x) = -\frac{2}{2d+1} \sum_{i=1}^d \cos x_i + \frac{1}{2d+1},$$

respectively.

**Lemma 1.** *The operator  $\tilde{H}$  is unitarily equivalent to the operator  $H$ .*

Since the operators  $H$  and  $\tilde{H}$  are unitarily equivalent, they have the same spectrum:

$$\sigma_p(H) = \sigma_p(\tilde{H}), \quad \sigma_{ess}(H) = \sigma_{ess}(\tilde{H}).$$

Therefore, an investigation eigenvalue problem for  $H$  reduces to study eigenvalue problems for  $H_1$  and  $H_2$ , since

$$\sigma(H) = \sigma(H_1) \cup \sigma(H_2).$$

We note that the Weyl theorem for the essential spectrum [14] implies that the essential spectrum  $\sigma_{ess}(H_\alpha)$ ,  $\alpha = 1, 2$  of the operator  $H_\alpha$  coincides with the spectrum of the unperturbed operator  $H_{\alpha 0}$ , since  $\mathbf{v}$  is the one rank operator. In this case,  $\sigma_{ess}(H_1)$  and  $\sigma_{ess}(H_2)$  coincide with the range of the functions  $E_{10}$  and  $E_{20}$ , respectively, i.e.,

$$\sigma_{ess}(H_1) = [-1, \frac{2d-1}{2d+1}] \quad \text{and} \quad \sigma_{ess}(H_2) = [\frac{-2d+1}{2d+1}, 1].$$

In particular cases for  $d$  we have

$$\begin{aligned} \sigma_{ess}(H_1) &= [-1, \frac{1}{3}] \quad \text{and} \quad \sigma_{ess}(H_2) = [-\frac{1}{3}, 1] \quad \text{for} \quad d = 1, \\ \sigma_{ess}(H_1) &= [-1, \frac{3}{5}] \quad \text{and} \quad \sigma_{ess}(H_2) = [-\frac{3}{5}, 1] \quad \text{for} \quad d = 2. \end{aligned}$$

Let

$$\begin{aligned} \Delta_1(z) &= 1 - (\lambda + \mu) \int_{\mathbb{T}^d} \frac{dx}{E_{10}(x) - z}, \quad z \notin \sigma_{ess}(H_1), \\ \Delta_2(z) &= 1 - (\lambda - \mu) \int_{\mathbb{T}^d} \frac{dx}{E_{20}(x) - z}, \quad z \notin \sigma_{ess}(H_2). \end{aligned}$$

**Lemma 2.** *The number  $z \in \mathbb{R} \setminus \sigma_{ess}(H_\alpha)$ ,  $\alpha = 1, 2$ , is an eigenvalue of the operator  $H_\alpha$  if and only if  $\Delta_\alpha(z) = 0$ .*

## 2.2. The main results for the case $d = 1$

The following lemmas describe the location of the eigenvalue  $z_\alpha$  of the operator  $H_\alpha$  with respect to the constants  $\lambda$  and  $\mu$ .

**Lemma 3.** *For any  $\lambda + \mu \neq 0$ , the operator  $H_1$  has a unique eigenvalue  $z_1$  lying on  $\mathbb{R} \setminus [-1, \frac{1}{3}]$  and  $H_1$  has no any eigenvalue for  $\lambda = -\mu$ . More precisely,*

- (1)  $z_1 \in (-\infty, -1)$ ,  $z_1 = \frac{-1 - 2\sqrt{1 + 9\pi^2(\lambda + \mu)^2}}{3}$  for  $\lambda + \mu > 0$ ;
- (2)  $z_1 \in (\frac{1}{3}, 1)$ ,  $z_1 = \frac{-1 + 2\sqrt{1 + 9\pi^2(\lambda + \mu)^2}}{3}$  for  $-\frac{1}{\sqrt{3}\pi} < \lambda + \mu < 0$ . Moreover, in this case  $z_1$  lies in the essential spectrum  $\sigma_{ess}(H_2)$ ;
- (3)  $z_1 = 1$  for  $\lambda + \mu = -\frac{1}{\sqrt{3}\pi}$ , i.e.  $z_1 = 1$  is the threshold of  $\sigma_{ess}(H_2)$ ;
- (4)  $z_1 \in (1, \infty)$ ,  $z_1 = \frac{-1 + 2\sqrt{1 + 9\pi^2(\lambda + \mu)^2}}{3}$  for  $\lambda + \mu < -\frac{1}{\sqrt{3}\pi}$ .

**Lemma 4.** *For any  $\lambda - \mu \neq 0$ , the operator  $H_2$  has a unique eigenvalue  $z_2$  lying on  $\mathbb{R} \setminus [-\frac{1}{3}, 1]$  and  $H_2$  has no any eigenvalue for  $\lambda = \mu$ . More precisely,*

- (1)  $z_2 \in (1, \infty)$ ,  $z_2 = \frac{1 + 2\sqrt{1 + 9\pi^2(\lambda - \mu)^2}}{3}$  for  $\lambda - \mu < 0$ ;
- (2)  $z_2 \in (-1, -\frac{1}{3})$ ,  $z_2 = \frac{1 - 2\sqrt{1 + 9\pi^2(\lambda - \mu)^2}}{3}$  for  $0 < \lambda - \mu < \frac{1}{\sqrt{3}\pi}$ . Moreover, in this case,  $z_2$  lies in the essential spectrum  $\sigma_{ess}(H_1)$ ;
- (3)  $z_2 = -1$  for  $\lambda - \mu = \frac{1}{\sqrt{3}\pi}$ , i.e.  $z_2 = -1$  is the threshold of  $\sigma_{ess}(H_1)$ ;
- (4)  $z_2 \in (-\infty, -1)$ ,  $z_2 = \frac{1 - 2\sqrt{1 + 9\pi^2(\lambda - \mu)^2}}{3}$  for  $\lambda - \mu > \frac{1}{\sqrt{3}\pi}$ .

We introduce the following sets:

$$l_1 = \{(\lambda, \mu) : \lambda + \mu = 0\}, \quad l_2 = \{(\lambda, \mu) : \lambda - \mu = 0\},$$

$$D_1 = \{(\lambda, \mu) : -\frac{1}{\sqrt{3}\pi} \leq \lambda + \mu < 0\}, \quad D_2 = \{(\lambda, \mu) : 0 < \lambda - \mu \leq \frac{1}{\sqrt{3}\pi}\}.$$

Combining Lemma 3 and Lemma 4, we immediately obtain the following theorem.

TABLE 1. Locations of the eigenvalues of the operator  $H$  for all  $(\lambda, \mu) \in \mathbb{R}$ 

| $\lambda - \mu$                   | $\lambda + \mu$                           |                                           |                                                            |                                                |                                            |
|-----------------------------------|-------------------------------------------|-------------------------------------------|------------------------------------------------------------|------------------------------------------------|--------------------------------------------|
|                                   | $(-\infty, -\frac{1}{\sqrt{3\pi}})$       | $(-\frac{1}{\sqrt{3\pi}})$                | $(-\frac{1}{\sqrt{3\pi}}, 0)$                              | 0                                              | $(0, \infty)$                              |
| $(-\infty, 0)$                    | $z_1 > 1$<br>$z_2 > 1$                    | $z_1 = 1$<br>$z_2 > 1$                    | $z_1 \in (\frac{1}{3}, 1)$<br>$z_2 > 1$                    | $\nexists z_1$<br>$z_2 > 1$                    | $z_1 < -1$<br>$z_2 > 1$                    |
| 0                                 | $z_1 > 1$ ,<br>$\nexists z_2$             | $z_1 = 1$ ,<br>$\nexists z_2$             | $z_1 \in (\frac{1}{3}, 1)$ ,<br>$\nexists z_2$             | —                                              | $z_1 < -1$<br>$\nexists z_2$               |
| $(0, \frac{1}{\sqrt{3\pi}})$      | $z_1 > 1$<br>$z_2 \in (-1, -\frac{1}{3})$ | $z_1 = 1$<br>$z_2 \in (-1, -\frac{1}{3})$ | $z_1 \in (\frac{1}{3}, 1)$<br>$z_2 \in (-1, -\frac{1}{3})$ | $\nexists z_1$<br>$z_2 \in (-1, -\frac{1}{3})$ | $z_1 < -1$<br>$z_2 \in (-1, -\frac{1}{3})$ |
| $\frac{1}{\sqrt{3\pi}}$           | $z_1 > 1$<br>$z_2 = -1$                   | $z_1 = 1$<br>$z_2 = -1$                   | $z \in (\frac{1}{3}, 1)$<br>$z_2 = -1$                     | $\nexists z_1$<br>$z_2 = -1$                   | $z_1 < -1$<br>$z_2 = -1$                   |
| $(\frac{1}{\sqrt{3\pi}}, \infty)$ | $z_1 > 1$<br>$z_2 < -1$                   | $z_1 = 1$<br>$z_2 < -1$                   | $z_1 \in (\frac{1}{3}, 1)$ ,<br>$z_2 = -1$                 | $\nexists z_1$<br>$z_2 < -1$                   | $z_1 < -1$<br>$z_2 < -1$                   |

**Theorem 2.1.** (1) For  $(\lambda, \mu) \in \mathbb{R}^2 \setminus (l_1 \cup l_2)$  the operator  $H$  has two eigenvalues  $z_1$  and  $z_2$ . Moreover, if  $(\lambda, \mu) \in D_1 \cap D_2$ , then  $z_1, z_2 \in \sigma_{ess}(H)$  and if  $(\lambda, \mu) \in D_\alpha \setminus (D_1 \cap D_2)$ , then  $z_\alpha \in \sigma_{ess}(H)$ ,  $\alpha = 1, 2$ ;

(2) For  $(\lambda, \mu) \in (l_1 \cup l_2) \setminus (0, 0)$  the operator  $H$  has a unique eigenvalue. Moreover, if  $(\lambda, \mu) \in D_1 \cap l_2$  then  $z_1 \in \sigma_{ess}(H)$  and if  $(\lambda, \mu) \in D_2 \cap l_1$  then  $z_2 \in \sigma_{ess}(H)$ .

Table 1 presents the precise locations of the eigenvalues of the operator  $H$  for all  $(\lambda, \mu) \in \mathbb{R}$ , where in some cases the eigenvalues lie in  $\sigma_{ess}(H)$ .

### 2.3. The main results for the case $d = 2$

The following lemmas describe the location of the eigenvalue  $z_\alpha$  of  $H_\alpha$  with respect to constants  $\lambda$  and  $\mu$ .

Let  $K(\cdot)$  be the first-kind Elliptic function then  $K(\frac{2}{3}) \approx 1, 8096675$ .

**Lemma 5.** For any  $\lambda + \mu \neq 0$ , the operator  $H_1$  has a unique eigenvalue  $z_1$  lying on  $\mathbb{R} \setminus [-1, \frac{3}{5}]$  and  $H_1$  has no any eigenvalue for  $\lambda = -\mu$ . More precisely,

- (1)  $z_1 \in (-\infty, -1)$  for  $\lambda + \mu > 0$ ;
- (2)  $z_1 \in (\frac{3}{5}, 1)$  for  $-\frac{3}{20\pi K(\frac{2}{3})} < \lambda + \mu < 0$ . Moreover, in this case,  $z_1$  lies in the essential spectrum  $\sigma_{ess}(H_2)$ ;
- (3)  $z_1 = 1$  for  $\lambda + \mu = -\frac{3}{20\pi K(\frac{2}{3})}$ , i.e.  $z_1 = 1$  is the threshold of  $\sigma_{ess}(H_2)$ ;
- (4)  $z_1 \in (1, \infty)$  for  $\lambda + \mu < -\frac{3}{20\pi K(\frac{2}{3})}$ .

**Lemma 6.** For any  $\lambda - \mu \neq 0$ , the operator  $H_2$  has a unique eigenvalue  $z_2$  lying on  $\mathbb{R} \setminus [-\frac{3}{5}, 1]$  and  $H_2$  has no any eigenvalue for  $\lambda = \mu$ . More precisely,

- (1)  $z_2 \in (1, \infty)$  for  $\lambda - \mu < 0$ ;
- (2)  $z_2 \in (-1, -\frac{3}{5})$  for  $0 < \lambda - \mu < \frac{3}{20\pi K(\frac{2}{3})}$ . Moreover, in this case  $z_2$  lies in the essential spectrum  $\sigma_{ess}(H_1)$ ;
- (3)  $z_2 = -1$  for  $\lambda - \mu = \frac{3}{20\pi K(\frac{2}{3})}$ , i.e.  $z_2 = -1$  is the threshold of  $\sigma_{ess}(H_1)$ ;
- (4)  $z_2 \in (-\infty, -1)$  for  $\lambda - \mu > \frac{3}{20\pi K(\frac{2}{3})}$ .

To describe locations of eigenvalues of the operator  $H$ , we introduce the following sets:

$$l_1 = \{(\lambda, \mu) : \lambda + \mu = 0\} \quad \text{and} \quad l_2 = \{(\lambda, \mu) : \lambda - \mu = 0\},$$

$$D_1 = \{(\lambda, \mu) : -\frac{3}{20\pi K(\frac{2}{3})} \leq \lambda + \mu < 0\} \quad \text{and} \quad D_2 = \{(\lambda, \mu) : 0 < \lambda - \mu \leq \frac{3}{20\pi K(\frac{2}{3})}\}.$$

The following Theorem 2.2 is established by combining the results obtained in Lemma 5 and Lemma 6.

**Theorem 2.2.** (1) For  $(\lambda, \mu) \in \mathbb{R}^2 \setminus (l_1 \cup l_2)$  the operator  $H$  has two eigenvalues. Moreover, if  $(\lambda, \mu) \in D_1 \cap D_2$  then  $z_1, z_2 \in \sigma_{ess}(H)$  and if  $(\lambda, \mu) \in D_\alpha \setminus (D_1 \cap D_2)$  then  $z_\alpha \in \sigma_{ess}(H)$ ,  $\alpha = 1, 2$ ;

(2) For  $(\lambda, \mu) \in (l_1 \cup l_2) \setminus (0, 0)$ , the operator  $H$  has exactly one eigenvalue. Moreover, if  $(\lambda, \mu) \in D_1 \cap l_2$  then  $z_1 \in \sigma_{ess}(H)$  and if  $(\lambda, \mu) \in D_2 \cap l_1$  then  $z_2 \in \sigma_{ess}(H)$ .

### 3. Proof of the main theorems and lemmas

*Proof of the Lemma 1.* Consider a unitary operator  $U$  acting in  $L_2^{(2)}(\mathbb{T}^d)$  as

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix},$$

where  $I$  is a unit operator on  $L_2(\mathbb{T}^d)$ . Then, we verify the following equality

$$HU^* = (H_0 - V)U^* = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{10} - \lambda \mathbf{v} - \mu \mathbf{v} & H_{20} - \lambda \mathbf{v} + \mu \mathbf{v} \\ H_{10} - \lambda \mathbf{v} - \mu \mathbf{v} & -H_{20} + \lambda \mathbf{v} - \mu \mathbf{v} \end{pmatrix}.$$

Therefore,

$$UHU^* = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \tilde{H}.$$

□

*Proof of Lemma 2. Necessity.* Let a number  $z$  be an eigenvalue of the operator  $H_1$  corresponding to the eigenfunction  $\varphi \in L_2(\mathbb{T}^d)$ , i.e

$$H_1 \varphi = z \varphi, \quad \varphi \neq 0.$$

Then

$$(H_{10} - zI)\varphi = (\lambda + \mu)V\varphi$$

or

$$\varphi(x) = (\lambda + \mu) \frac{C_\varphi}{E_{10}(x) - z}, \quad (1)$$

since  $E_{10}(x) \neq z$  for  $z \in \mathbb{R} \setminus \sigma_{ess}(H_1)$ , where

$$C_\varphi = \int_{\mathbb{T}^d} \varphi(x) dx. \quad (2)$$

Substituting (1) to (2), we obtain

$$C_\varphi \Delta_1(z) = 0, \quad \Delta_1(z) = 1 - (\lambda + \mu) \int_{\mathbb{T}^d} \frac{dx}{E_{10}(x) - z}.$$

Since  $\varphi \neq 0$ , so  $C_\varphi \neq 0$ . Hence  $\Delta_1(z) = 0$ .

**Sufficiency.** Let  $\Delta_1(z) = 0$  for  $z \in \mathbb{R} \setminus \sigma_{ess}(H_1)$ . Then, the function  $\varphi(x) = \frac{\lambda + \mu}{E_{10}(x) - z}$  belongs to  $L_2(\mathbb{T}^d)$  and satisfies the equation  $H_1 \varphi = x \varphi$ , i.e.  $\varphi$  is an eigenfunction of the operator  $H_1$  corresponding to the eigenvalue  $z$ .

The theorem for the case  $\alpha = 2$  can be proved similarly. □

*Proof of Lemma 3.* To prove lemma, according to the Lemma 2, we investigate zeros of the function

$$\Delta_1(z) = 1 - (\lambda + \mu) \int_{-\pi}^{\pi} \frac{dx}{-\frac{2}{3} \cos x - \frac{1}{3} - z}.$$

We need the classical identity

$$\int_{-\pi}^{\pi} \frac{dx}{A + B \cos x} = \frac{2\pi \operatorname{sgn}(A)}{\sqrt{A^2 - B^2}}, \quad |A| > |B| > 0. \quad (3)$$

Choosing  $A = -\frac{1}{3} - z$  and  $B = -\frac{2}{3}$  for  $z \in \mathbb{R} \setminus (-1, \frac{1}{3})$ , from (3) we have

$$\int_{-\pi}^{\pi} \frac{dx}{-\frac{2}{3} \cos x - \frac{1}{3} - z} = \frac{2\pi \operatorname{sgn}(-\frac{1}{3} - z)}{\sqrt{(-\frac{1}{3} - z)^2 - (-\frac{2}{3})^2}} = \frac{2\pi \operatorname{sgn}(-\frac{1}{3} - z)}{\sqrt{z^2 + \frac{2}{3}z - \frac{1}{3}}}.$$

Hence, the function  $\Delta_1(z)$  represents as

$$\Delta_1(z) = 1 - (\lambda + \mu) \frac{2\pi \operatorname{sgn}(-\frac{1}{3} - z)}{\sqrt{z^2 + \frac{2}{3}z - \frac{1}{3}}}. \quad (4)$$

(1) Let  $\lambda + \mu > 0$ . Then according to (4)  $\Delta_1(z) > 0$  for  $z > \frac{1}{3}$ . For  $z < -1$ , the equation  $\Delta_1(z) = 0$  is

$$\sqrt{z^2 + \frac{2}{3}z - \frac{1}{3}} = 2\pi(\lambda + \mu).$$

Squaring both sides yields

$$z^2 + \frac{2}{3}z - \frac{1}{3} = 4\pi^2(\lambda + \mu)^2.$$

The roots of this quadratic equation are

$$z_{\pm} = \frac{-1 \pm 2\sqrt{1 + 9\pi^2(\lambda + \mu)^2}}{3}.$$

Due to  $z_+ > -1$ , we identify that  $z_- = \frac{-1 - 2\sqrt{1 + 9\pi^2((\lambda + \mu)^2)}}{3}$  as a unique root of  $\Delta_1(z) = 0$ . Hence by the Lemma 2, the number  $z_-$  is the unique eigenvalue lying in  $(-\infty, -1)$ .

Let  $\lambda + \mu < 0$ . Then  $\Delta_1(z) > 0$  for  $z < -1$ . For  $z > \frac{1}{3}$  the equation  $\Delta_1(z) = 0$  is equivalent to

$$\sqrt{z^2 + \frac{2}{3}z - \frac{1}{3}} = -2\pi(\lambda + \mu).$$

Then, the number  $z_+ = \frac{-1 + 2\sqrt{1 + 9\pi^2(\lambda + \mu)^2}}{3}$  is a unique root of the equation  $\Delta_1(z) = 0$  for  $z > \frac{1}{3}$ . So, by Lemma 2, the number  $z_+$  is the unique eigenvalue lying in  $(\frac{1}{3}, \infty)$ .

(2) Let  $-\frac{1}{\sqrt{3}\pi} < \lambda + \mu < 0$ . Then  $z_+ = \frac{-1 + 2\sqrt{1 + 9\pi^2(\lambda + \mu)^2}}{3} < 1$ , i.e.  $z_+ \in (\frac{1}{3}, 1)$ .

One can see that

(3)  $z_+ = 1$  for  $\lambda + \mu = -\frac{1}{\sqrt{3}\pi}$

and

(4)  $z_+ = \frac{-1 + 2\sqrt{1 + 9\pi^2((\lambda + \mu)^2)}}{3} > 1$  for  $\lambda + \mu < -\frac{1}{\sqrt{3}\pi}$ . □

We omit the proof of Lemma 4, since it proceeds analogously to the proof of Lemma 3.

*Proof of Lemma 5.* We investigate zeros of the function

$$\Delta_1(z) = 1 - (\lambda + \mu)I(z),$$

where

$$I(z) = \int_{\mathbb{T}^2} \frac{dx}{E_{10}(x) - z}.$$

(1) Let  $\lambda + \mu > 0$ . Then the function  $I(z)$  is positive on  $(-\infty, -1)$  and strictly increasing, with range  $(0, \infty)$ . Hence, from the continuity  $I(z)$ , the equation

$$1 - (\lambda + \mu)I(z) = 0$$

has unique solution in  $(-\infty, -1)$ . Since  $I(z) < 0$  for  $z \in (\frac{3}{5}, \infty)$ ,  $\Delta_1(z) > 1$ . Therefore, according to the Lemma 2 the operator  $H_1$  has unique eigenvalue  $z$  and it lies on  $(-\infty, -1)$ .

(3) Let  $\lambda + \mu = \frac{-3}{20\pi K(\frac{2}{3})}$ . We show that  $\Delta_1(1) = 0$ , i.e.  $I(1) = \frac{-20\pi}{3}K(\frac{2}{3})$ . In the calculation of the integral

$$I(1) = \int_{\mathbb{T}^2} \frac{dx}{\frac{-2}{5}(\cos x_1 + \cos x_2) - \frac{6}{5}},$$

using (3), we obtain

$$I(1) = \int_{-\pi}^{\pi} \frac{-10\pi dx_1}{\sqrt{(2\cos x_1 + 6)^2 - 4}}.$$

By applying the substitution  $\tan x_1 = t$ , we arrive at the first-kind elliptic integral in the form

$$I(1) = \int_{-\infty}^{\infty} \frac{-20\pi dt}{\sqrt{(6t^2 + 10)(2t^2 + 6)}}.$$

To compute  $I(1)$ , we use the following identity (See formula 3.147.5 in [13])

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(At^2 + C)(Bt^2 + D)}} = \frac{1}{\sqrt{CD}} \frac{2}{\sqrt{a}} K(k)$$

with

$$k = \sqrt{1 - \frac{b}{a}}, \quad a > b > 0, \quad a = \frac{A}{C}, \quad b = \frac{B}{D},$$

where  $K(k) = \frac{\pi}{2} \left[ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots \right]$ . Then we have  $I(1) = \frac{-20\pi}{3} K(\frac{2}{3})$ ,  $K(\frac{2}{3}) \approx 1,8096675$ . Thus,  $\Delta_1(1) = 0$ , hence the number  $z = 1$  is the unique eigenvalue of  $H_1$ .

(2) Let  $\frac{-3}{20\pi K(\frac{2}{3})} < \lambda + \mu < 0$ . Then for  $z \in (\frac{3}{5}, 1)$ , we have

$$1 - (\lambda + \mu)I(z) < 1 + \frac{3}{20\pi K(\frac{2}{3})} I(z) < 1 + \frac{3}{20\pi K(\frac{2}{3})} I(1) = 0.$$

The function  $I(z)$  is continuous and strictly increasing on  $(\frac{3}{5}, 1)$ . Then from the equality

$$\lim_{z \rightarrow \frac{3}{5} + 0} I(z) = -\infty,$$

the equation

$$\Delta_1(z) = 1 - (\lambda + \mu)I(z) = 0$$

has exactly one solution in  $(\frac{3}{5}, 1)$ .

For  $(1, \infty)$  the relation  $I(z) \in (\frac{-20\pi}{3} K(\frac{2}{3}), 0)$  holds. Then  $\Delta_1(z) = 1 - (\lambda + \mu)I(z) < 0$  for  $z \in (1, \infty)$ . So, according to Lemma 2, the operator  $H_1$  has a unique eigenvalue  $z_1 \in (\frac{3}{5}, 1)$  for  $\lambda + \mu \in (\frac{-3}{20\pi K(\frac{2}{3})}, 0)$ .

(4) Let  $\lambda + \mu < \frac{-3}{20\pi K(\frac{2}{3})}$ . Since  $\Delta_1(z) = 1 - (\lambda + \mu)I(z) < 0$ ,

$$\lim_{z \rightarrow \infty} I(z) = 1$$

and  $I(z)$  increases on  $(1, \infty)$ , the function  $\Delta_1(z)$  has a unique zero in  $(1, \infty)$ . □

The proof of Lemma 6 is similar to the proof of Lemma 5.

#### 4. Conclusion

In the present paper, we investigate the spectral properties of the discrete Schrödinger operator  $H$  on a ladder graph of  $d$ -dimensional square lattices with a self-adjoint matrix-valued potential whose entries are real parameters  $\lambda$  and  $\mu$ . The considered operator  $H$  can be regarded as a perturbation of the operator studied in [10] by a matrix operator with off-diagonal  $\delta$ -potentials having coupling constant  $\lambda$ . Due to the symmetry of the ladder graph, the investigated problem is reduced to the study of the Schrödinger-type operators  $H_1$  and  $H_2$  on the lattice  $\mathbb{Z}^d$  with  $\delta$ -potentials whose coupling constants are  $\lambda \pm \mu$ . Moreover, a non-embedded eigenvalue of  $H_1$  may become an embedded eigenvalue of the operator  $H$  if it lies within a spectral band of  $H_2$ , and conversely. This reduced problem is similar to that considered in [3], where the eigenvalues of the Schrödinger operator with  $\delta$ -potential depending on coupling constant parameter were investigated. Furthermore, by applying methods similar to those used in [3], we obtained a complete description of the eigenvalues in terms of the parameters  $\lambda$  and  $\mu$ .

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