

Adiabatic conditions for non-stationary optical phase modulation: a non-degenerate case

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ABSTRACT This work examines a non-stationary phase-modulation process characterized by a time-dependent interaction coefficient, confined to the non-degenerate case. We derive an exact non-adiabatic solution to the time-dependent Schrödinger equation. We then establish the criterion under which adiabatic considerations become valid—specifically, the condition permitting a stationary-state solution to approximate the non-stationary problem. Our results hold for an arbitrary number of interacting modes and an arbitrary photon number.

KEYWORDS Phase modulation, non-stationary dynamics, quantum optics, Lie algebra.

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1. Introduction

Electro-optic phase modulation provides a coherent mechanism for coupling optical frequency modes through the linear electro-optic (Pockels) effect [1,2]. Under harmonic driving, an optical carrier is transformed into a discrete frequency comb consisting of sidebands separated by integer multiples of the modulation frequency [3]. From the perspective of quantum optics, this interaction implements a unitary transformation in the Hilbert space of optical frequency modes, enabling deterministic manipulation of high-dimensional photonic states and coherent engineering of multimode optical fields [4–10]. Accordingly, an electro-optic modulator may be viewed as a frequency-domain multiport interferometer whose applications span optical communications, quantum information processing, and precision metrology.

Electro-optic phase modulation is a well-studied process [1,2], especially when a modulation of an optical carrier by a harmonic voltage applied to a nonlinear crystal generates a frequency comb of sidebands offset by integer multiples of the driving signal [3]. This process, rooted in the Pockels effect, is not merely a spectral broadening tool but a fundamental operation for synthesizing arbitrary optical waveforms and engineering multimode quantum states, for instance [4–10]. In essence, the modulator acts as a controllable, lossless multiport interferometer, enabling deterministic transformations across a large number of frequency modes - a capability directly transferable to optical computing, metrology, and high-speed signal processing.

Beyond its well-known role in subcarrier wave quantum key distribution (SCW QKD) [11–15], the technique finds broad utility across diverse fields. In classical communications, it enables dense wavelength-division multiplexing and ultra-flat frequency comb generation for coherent transmission [16–20]. In quantum photonics, it serves as a building block for generating time-bin entanglement and frequency-bin qudits [17,21–25], while in metrology, it provides phase-stable references for optical frequency standards [1,26–29]. The ongoing push toward non-stationary operation - with modulation rates approaching and exceeding 100 GHz and switching times in the picosecond range - demands novel platform developments. Thin-film lithium niobate (TFLN) modulators [30–32] with traveling-wave electrodes now offer unprecedented bandwidth and reduced voltage-length products, while resonant cavity designs enhance efficiency at the cost of mode-count flexibility. These ultrafast regimes [33], however, invalidate adiabatic approximations, requiring time-dependent models that capture transient sideband dynamics. The central challenge thus shifts from mere generation to precise, real-time control of the spectral phase and amplitude under extreme temporal constraints.

This paper continues the work in [15], where a non-stationary phase-modulation process is considered within the framework of SCW QKD. However, the latter study examines modulation in an adiabatic regime. Here we dive deeply into non-adiabatic considerations and study the boundaries of its applicability. Also, here we do not limit ourselves only by QKD applications; current research may be implemented in a broad range of fields, discussed earlier in the introduction.

The article has the structure as follows. In Section 2 preliminaries and essential parts of the theoretical foundation are discussed. Section 3 is dedicated to derivation and studying the non-adiabatic solution of the non-stationary Schrödinger equation, where the interaction constant is dependent on time. In Section 4 analysis of the obtained solution is performed in order to define bounds of adiabatic approximation applicability. In Section 5 generalizations and extensions of the results are considered. Section 6 concludes the article.

2. Preliminaries

The foundation of current research is presented in [3], where a fully quantum description of modulation with alternating (harmonic) voltage applied to a nonlinear crystal was proposed. As we stated above, we are interested in a broadening of the theory by considering a non-stationary problem, where a modulation depth is changing, presumably rapidly, in time. At first, we recall the essential parts of the developed theory. Consider the non-periodic part of the Hamiltonian (quasi-energy operator) as follows:

$$Q(t) = m_{opt}\omega N + \omega J_0 + \frac{2\gamma(t)}{2S+1}(J_+e^{-i\varphi} + J_-e^{i\varphi}), \quad (1)$$

where m_{opt} stands for the mode number of this operational optical mode, ω is a frequency mismatch of the optical and the microwave modes, the latter is defined by an applied to a crystal alternating voltage with phase φ that produces sidebands, $2S+1$ is a number of interacting optical modes, and $\gamma(t)$ is an interaction constant that is dependent on time, also

$$N = \sum_{j=-S}^S a_j^\dagger a_j, \quad J_0 = \sum_{j=-S}^S j a_j^\dagger a_j, \quad J_- = \sum_{j=-S}^{S-1} f_j a_{j+1}^\dagger a_j, \quad J_+ = (J_-)^\dagger, \quad f_j = \sqrt{(S+j+1)(S-j)}, \quad (2)$$

where a_j ($a_j^\dagger$) is the annihilation (creation) operator for the j^{th} mode.

The operator $Q(t)$ can be diagonalized, utilizing properties of the $su(2)$ algebra; see App. A:

$$R[\varphi, \beta(t)] = e^{-i\varphi J_0} e^{-i\beta(t) J_y}, \quad R^\dagger[\varphi, \beta(t)] R[\varphi, \beta(t)] = I, \quad (3)$$

$$Q_d(t) = R^\dagger[\varphi, \beta(t)] Q(t) R[\varphi, \beta(t)] = m_{opt}\omega N + \Gamma(t) J_0 = \sum_{j=-S}^S (m_{opt}\omega + \Gamma(t)j) a_j^\dagger a_j, \quad (4)$$

$$\Gamma(t) = \sqrt{\omega^2 + \frac{16[\gamma(t)]^2}{(2S+1)^2}}, \quad \cos[\beta(t)] = \frac{\omega}{\Gamma(t)}. \quad (5)$$

Considering that the diagonalized operator, $Q_d(t)$, consists only of diagonal quadratic operators, it is convenient to assume its eigenvector as a basis vector in a Fock subspace, $|\mathbf{v}_k\rangle = \bigotimes_j |v_j^{(k)}\rangle_j$, where each mode j has $v_j^{(k)}$ photons for a given basis vector with index k . Note that the range of the index k indicates the number of eigenvectors and eigenvalues and depends on the dimensions of the system. The latter, in turn, is defined by the total number of modes $2S+1$ and a considered Fock subspace with a fixed total number of photons $V = \sum_{j=-S}^S v_j^{(k)}$, equal for all basis vectors in a given Fock subspace; the range then is defined as follows:

$$k \in \{1, 2, \dots, \binom{V+2S}{2S}\}, \quad (6)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. Inferring from the above, we proceed with the following identities:

$$Q_d(t)|\mathbf{v}_k\rangle = \sum_{j=-S}^S (m_{opt}\omega + \Gamma(t)j) v_j^{(k)} |\mathbf{v}_k\rangle, \quad (7)$$

$$Q(t)R[\varphi, \beta(t)]|\mathbf{v}_k\rangle = \sum_{j=-S}^S (m_{opt}\omega + \Gamma(t)j) v_j^{(k)} R[\varphi, \beta(t)]|\mathbf{v}_k\rangle, \quad (8)$$

$$R[\varphi, \beta(t)]|\mathbf{v}_k\rangle \equiv |\phi_k(t)\rangle, \quad \sum_{j=-S}^S (m_{opt}\omega + \Gamma(t)j) v_j^{(k)} \equiv q_k(t). \quad (9)$$

Then Eq. (9) describes eigenvectors, $|\phi_k(t)\rangle$, and eigenvalues, $q_k(t)$, of the non-periodic part of the Hamiltonian $Q(t)$. It is essential for further analysis of a non-stationary and, presumably, non-adiabatic modulation process. Note, hereinafter we consider the non-degenerate case, characterized by $q_k(\tau_0) \neq q_n(\tau_0)$ for arbitrary τ_0 .

3. Non-adiabatic solution

For a time-dependent non-periodic part of the Hamiltonian, $Q(t)$, the solution of the Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = Q(t) |\psi(t)\rangle \quad (10)$$

has the following form:

$$|\psi(t)\rangle = \sum_k c_k(t) e^{-i \int_0^t q_k(t') dt'} |\phi_k(t)\rangle, \quad (11)$$

where the spectral problem solution for $Q(t)$ is known:

$$Q(t) |\phi_k(t)\rangle = q_k(t) |\phi_k(t)\rangle. \quad (12)$$

Then, one needs to define coefficients $c_k(t)$. Substitution of Eq. (11) into Eq. (10) and multiplying both sides of the latter equation by $\langle \phi_n(t) | e^{i q_n t}$ results in

$$\dot{c}_n(t) = - \sum_k c_k(t) \cdot e^{-i \int_0^t (q_k(t') - q_n(t')) dt'} \langle \phi_n(t) | \dot{\phi}_k(t) \rangle. \quad (13)$$

Also, notice that the latter expression can be rearranged by the following observation:

$$\dot{b}_n(t) = \dot{c}_n(t) e^{-i \int_0^t q_n(t') dt'} - i q_n(t) b_n(t), \quad b_n(t) = c_n(t) e^{-i \int_0^t q_n(t') dt'}, \quad (14)$$

that is

$$\dot{b}_n(t) = -i q_n(t) b_n(t) - \sum_k b_k(t) \langle \phi_n(t) | \dot{\phi}_k(t) \rangle. \quad (15)$$

In other words,

$$\dot{b}_n(t) = \sum_k \left(-i q_n(t) \delta_{n,k} - \langle \phi_n(t) | \dot{\phi}_k(t) \rangle \right) b_k(t) = \sum_k W_{nk}(t) \cdot b_k(t), \quad (16)$$

so the solution can be found by defining the matrix W .

We may use (12) in order to define $\langle \phi_n(t) | \dot{\phi}_k(t) \rangle$ by taking the derivative and multiplying by $\langle \phi_n(t) |$ as follows:

$$\langle \phi_n(t) | \dot{\phi}_k(t) \rangle = \frac{\langle \phi_n(t) | \dot{Q}(t) | \phi_k(t) \rangle}{q_k(t) - q_n(t)}, \quad n \neq k, \quad \langle \dot{\phi}_k(t) | \phi_k(t) \rangle = 0. \quad (17)$$

Then, see also App. A,

$$\dot{Q}(t) = \frac{2\dot{\gamma}(t)}{2S+1} (J_+ e^{-i\varphi} + J_- e^{i\varphi}), \quad (18)$$

$$R^\dagger[\varphi, \beta(t)] \dot{Q}(t) R[\varphi, \beta(t)] = \frac{4\dot{\gamma}(t)}{2S+1} e^{i\beta(t)J_y} J_x e^{-i\beta(t)J_y} = \frac{4\dot{\gamma}(t)}{2S+1} \left(\sin[\beta(t)] J_0 + \cos[\beta(t)] J_x \right). \quad (19)$$

In order to define $\langle \phi_n(t) | \dot{Q}(t) | \phi_k(t) \rangle$, it is essential to examine $\langle \mathbf{v}_n | J_0 | \mathbf{v}_k \rangle$ and $\langle \mathbf{v}_n | J_x | \mathbf{v}_k \rangle$ expressions. The action of the diagonal quadratic operator J_0 on $|\mathbf{v}_k\rangle$ does not change the state, and also $|\mathbf{v}_k\rangle$ and $|\mathbf{v}_n\rangle$ are orthogonal; so $\langle \mathbf{v}_n | J_0 | \mathbf{v}_k \rangle$ only defines diagonal terms of a matrix. In turn, the action of J_x on $|\mathbf{v}_k\rangle$ produces a linear combination of “adjacent” states. In other words, $\langle \mathbf{v}_n | J_x | \mathbf{v}_k \rangle$ contributes to entries of secondary diagonals. Then, multiplying Eq. (19) by $\langle \mathbf{v}_n |$ and $|\mathbf{v}_k \rangle$, we obtain the following expression:

$$\langle \phi_n(t) | \dot{Q}(t) | \phi_k(t) \rangle = \frac{2\dot{\gamma}(t)}{2S+1} \left(\sin[\beta(t)] \langle \mathbf{v}_n | J_0 | \mathbf{v}_k \rangle + \cos[\beta(t)] \langle \mathbf{v}_n | J_x | \mathbf{v}_k \rangle \right). \quad (20)$$

Note, the latter term enters the matrix W only if $k \neq n$, keeping in mind the discussion above, the latter expression results in

$$\langle \phi_n(t) | \dot{Q}(t) | \phi_k(t) \rangle = \frac{4\dot{\gamma}(t) \cos[\beta(t)] \langle \mathbf{v}_n | J_x | \mathbf{v}_k \rangle}{2S+1}. \quad (21)$$

Finally, the solution of time-dependent equation (10) can be expressed as follows in terms of time-ordered exponentiated matrix W :

$$|\psi(t)\rangle = \sum_k \left(e^{\int_0^t W(\tau) d\tau} \cdot \vec{b}(0) \right)_k |\phi_k(t)\rangle, \quad (22)$$

where

$$W_{nk}(t) = -i q_n(t) \delta_{n,k} - \frac{4\dot{\gamma}(t) \cos[\beta(t)] \langle \mathbf{v}_n | J_x | \mathbf{v}_k \rangle}{(2S+1)[q_k(t) - q_n(t)]} \delta_{n \neq k}, \quad (23)$$

where $\delta_{n,k}$ and $\delta_{n \neq k}$ indicate that a term is considered only if $k = n$ or $k \neq n$, correspondingly.

4. Analysis

Here we consider a rather simple case, where $S = 1$, $V = 1$, $j \in \{-1, 0, 1\}$, and $|\mathbf{v}_1\rangle = |1, 0, 0\rangle$, $|\mathbf{v}_2\rangle = |0, 1, 0\rangle$, $|\mathbf{v}_3\rangle = |0, 0, 1\rangle$. Despite being simple, its analysis can be utilized as a tool to define bounds of (non-) adiabatic considerations' applicability as well as to study the potential for generalizations. It is considerably easier to analyze a close-to-adiabatic regime and explore whether it can not be considered so. In this case, the matrix W can be expressed as follows:

$$W(t) = \begin{pmatrix} -iq_{-1}(t) & \frac{w(t)}{\sqrt{2}} & 0 \\ -\frac{w(t)}{\sqrt{2}} & -iq_0(t) & \frac{w(t)}{\sqrt{2}} \\ 0 & -\frac{w(t)}{\sqrt{2}} & -iq_1(t) \end{pmatrix}, \quad (24)$$

$$-iq_j = -i[m_{opt}\omega + \Gamma(t)j], \quad w(t) = \frac{z\omega\dot{\gamma}(t)}{\Gamma(t)^2}, \quad z \equiv \frac{4}{2S+1} \quad (25)$$

Further, we investigate whether the absolute values of exponentiated matrix $E = e^{\int_0^t W(\tau)d\tau}$ entries can be close to those of the identity matrix. If the latter is true, it indicates that the system is in an adiabatic regime, so we may consider the time-ordered exponential matrix E as the first-order approximation of the Magnus expansion [34], implying direct exponentiation of the following matrix:

$$\begin{aligned} \int_0^t W(\tau)d\tau &= \begin{pmatrix} -i[m_{opt}\omega t - g(t)] & \frac{a(t)}{\sqrt{2}} & 0 \\ -\frac{a(t)}{\sqrt{2}} & -i[m_{opt}\omega t] & \frac{a(t)}{\sqrt{2}} \\ 0 & -\frac{a(t)}{\sqrt{2}} & -i[m_{opt}\omega t + g(t)] \end{pmatrix} = \\ &= D \begin{pmatrix} -i[m_{opt}\omega t - b(t)] & 0 & 0 \\ 0 & -i[m_{opt}\omega t] & 0 \\ 0 & 0 & -i[m_{opt}\omega t + b(t)] \end{pmatrix} D^{-1}, \end{aligned} \quad (26)$$

$$D = \frac{1}{[a(t)]^2} \begin{pmatrix} -[a(t)]^2 - 2[g(t)]^2 - 2g(t)b(t) & [a(t)]^2 & -[a(t)]^2 - 2[g(t)]^2 + 2g(t)b(t) \\ -i\sqrt{2}a(t)[g(t) + b(t)] & -i\sqrt{2}a(t)g(t) & -i\sqrt{2}a(t)[g(t) - b(t)] \\ [a(t)]^2 & [a(t)]^2 & [a(t)]^2 \end{pmatrix}, \quad (27)$$

where

$$a(t) = \int_0^t w(\tau)d\tau, \quad g(t) = \int_0^t \Gamma(\tau)d\tau, \quad b(t) = \sqrt{[a(t)]^2 + [g(t)]^2}, \quad (28)$$

resulting in

$$E = e^{-im_{opt}\omega t} D \begin{pmatrix} e^{ib(t)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-ib(t)} \end{pmatrix} D^{-1}. \quad (29)$$

Note that a closed-form solution for $a(t)$ can be found as it is shown further. However, the same cannot be done for $g(t)$: it involves a general elliptic integral that does not have a closed-form solution (except for a few special cases). The latter fact significantly limits the direct analytical approach.

The absolute values of E entries (hereinafter we utilize notation $|E|_{ij} \equiv |E_{ij}|$) are as follows:

$$|E| = I + \begin{pmatrix} -y & \sqrt{2}x\sqrt{1-y} & x^2 \\ \sqrt{2}x\sqrt{1-y} & -2x^2 & \sqrt{2}x\sqrt{1-y} \\ x^2 & \sqrt{2}x\sqrt{1-y} & -y \end{pmatrix}, \quad (30)$$

where

$$x = \frac{|a(t) \sin\left(\frac{b(t)}{2}\right)|}{b(t)}, \quad y = \frac{[a(t)]^2(1 - \cos[b(t)])}{2[b(t)]^2}. \quad (31)$$

In order to show closeness of $|E|$ to the identity matrix, a case where non-adiabatic effects are negligible, one needs to demonstrate, that $|a(t)| \ll b(t)$. If $|a(t)| \ll |g(t)|$, then $b(t) \approx |g(t)|$ and

$$\frac{|a(t)|}{b(t)} \approx \frac{|a(t)|}{|g(t)|} = r(t) \ll 1. \quad (32)$$

We consider a change of $\gamma(t)$ between two times. Then $\dot{\gamma}(t_1) = \dot{\gamma}(t_2) = 0$, $\gamma_{\max} = \max_t[\gamma(t)] = \gamma(t_1)$ and $\gamma_{\min} = \min_t[\gamma(t)] = \gamma(t_2)$ or vice versa. Note, we do not consider a degenerate case, where $\gamma_{\min} = 0$ and $\omega = 0$; this requires special methods; see a remark after Eq. (38). It is challenging to find in the general case the exact value of t_0 analytically, where the latter is defined by $\max_t[r(t)] = r(t_0)$ as well as precisely bounds the expression of interest as $r(t) \leq r(t_0)$. In order to do so, one should solve the following equation:

$$w(t) \int_0^t \Gamma(\tau) d\tau - \Gamma(t) \int_0^t w(\tau) d\tau = 0, \quad t \neq 0. \quad (33)$$

Note, however, this equation can be easily solved numerically. Then, we should find another approach to study the bounds of the (non-) adiabatic regime.

Observe that the maximal value of $|a(t)|$ at some point t_0 can be bounded from above as follows:

$$|a(t_0)| = \left| \int_0^{t_0} w(\tau) d\tau \right| = \left| \int_0^{t_0} \frac{z\omega\dot{\gamma}(\tau)}{\omega^2 + z^2[\gamma(\tau)]^2} d\tau \right| = \left| \frac{\omega}{z} \int_{\gamma(0)}^{\gamma(t_0)} \frac{d[\gamma(t)]}{\frac{\omega^2}{z^2} + [\gamma(t)]^2} \right| = \quad (34)$$

$$= \left| \arctan\left(\frac{z\gamma(t_0)}{\omega}\right) - \arctan\left(\frac{z\gamma(0)}{\omega}\right) \right| \leq \frac{\pi}{2}. \quad (35)$$

Since $|g(t)|$ in $r(t)$ is in the denominator, its value can be bounded from below as follows:

$$|g(t)| = \left| \int_0^t \sqrt{\omega^2 + z^2[\gamma(\tau)]^2} d\tau \right| \geq |t| \sqrt{\omega^2 + z^2\gamma_{\min}^2}. \quad (36)$$

The latter expression grows in time, so conservatively we may estimate the relation $r(t)$ at $t = 0$ as the following limit:

$$\lim_{t \rightarrow 0} r(t) = \left| \frac{\omega z \dot{\gamma}(0)}{[\Gamma(0)]^3} \right| = \left| \frac{z \dot{\gamma}(0) \cos[\beta(t)]}{[\Gamma(0)]^2} \right| \leq \left| \frac{z \dot{\gamma}(0)}{[\Gamma(0)]^2} \right|, \quad (37)$$

where we use identity (5). Due to an ambiguity of the initial time choice, we may bound the latter expression as follows:

$$r(0) \leq \left| \frac{z \dot{\gamma}(0)}{[\Gamma(0)]^2} \right| \leq \frac{z |\dot{\gamma}|_{\max}}{\Gamma_{\min}^2}, \quad (38)$$

where $|\dot{\gamma}|_{\max} = \max_t(|\dot{\gamma}(t)|)$ and $\Gamma_{\min} = \min_t[\Gamma(t)]$. Here we assume that $\Gamma_{\min} \neq 0$, otherwise, it is a highly degenerate case (both $\omega = 0$ and $\gamma_{\min} = 0$), and another approach should be implemented right from the beginning (e.g. local algebraic geometry and singularity theory [35], generalized Wigner-Smith theory [36] or similar methods).

As a final remark, we consider a special case of resonant modulation. Note that if $\omega \rightarrow 0$, Eq. (33) has no solutions other than $t_0 = 0$; it can be explicitly seen as follows by alternating the latter equation:

$$w(t) \int_0^t \Gamma(\tau) d\tau - \Gamma(t) \int_0^t w(\tau) d\tau = 0 \xrightarrow{\omega \rightarrow 0} 1 - \frac{\gamma(t)}{\gamma(0)} + \frac{\dot{\gamma}(t)}{\gamma(t)^2} \int_0^t \gamma(\tau) d\tau = 0. \quad (39)$$

Also $\gamma(0) \neq 0$ and $\dot{\gamma}(0) \neq 0$, based on initial restrictions of $\gamma(t)$. Then, an alternative boundary is as follows:

$$\left| \frac{\omega z \dot{\gamma}(0)}{[\Gamma(0)]^3} \right| \leq \frac{\omega |\dot{\gamma}(0)|}{z^2 [\gamma(0)]^3} \xrightarrow{\omega \rightarrow 0} 0. \quad (40)$$

Then, if one considers resonant modulation (assuming $\gamma_{\min} \neq 0$ and finite amount of interacting mode $2S + 1$, i.e., $z \neq 0$, in order to avoid degeneracy), then the dynamics of a system can be treated as an adiabatic process regardless.

And for a general (non-degenerate) case, the restriction condition for adiabatic considerations is as follows based on Eq. (38):

$$|\dot{\gamma}|_{\max} \ll \left(\frac{\Gamma_{\min}}{2} \right)^2 (2S + 1). \quad (41)$$

The latter implies the utilization of a stationary solution for a non-stationary problem of interest; otherwise, a full non-adiabatic solution should be introduced, including consideration of higher orders of the Magnus expansion. Figure 1 shows time-dependent probabilities, p_j and p'_j , of observing a photon in j^{th} mode with $E \equiv I$ and where E is expressed as in Eq. (29) correspondingly, where inequality (41) does not hold. The frequency of observed oscillatory deviation between p_j and p'_j is $b(t)$. So inequality (41) can be reformulated as a rule of thumb: oscillation frequency $b(t)$ should be

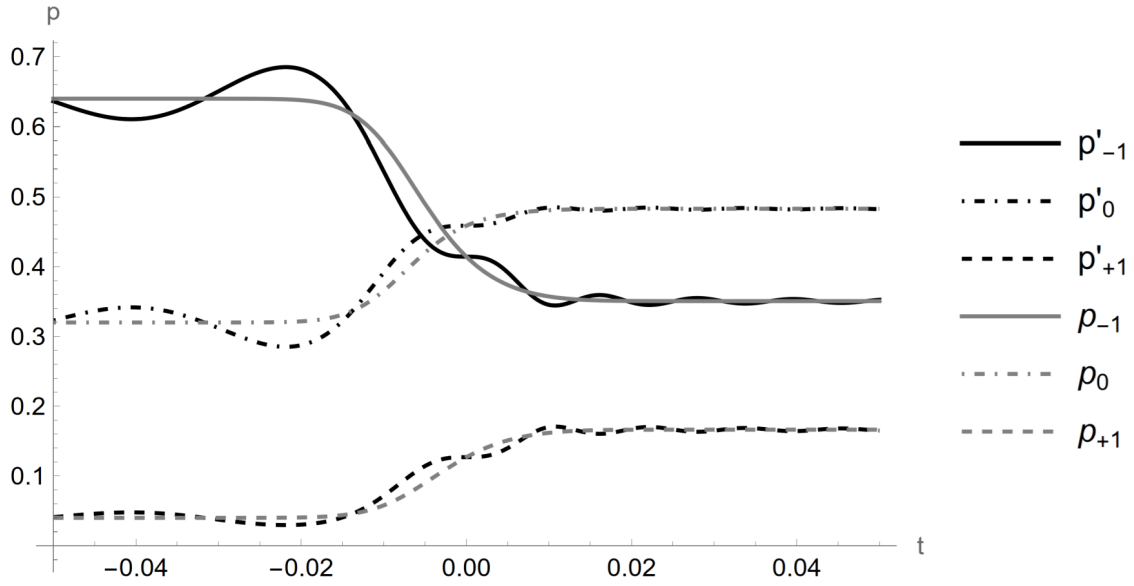


FIG. 1. Time-dependent probabilities, p_j and p'_j , of observing a photon in j^{th} mode with $E \equiv I$ and where E is expressed as in Eq. (29) correspondingly. Here we consider $\gamma(t) = \frac{1 + \text{erf}(t/s)}{2} |\gamma_1 - \gamma_2| + \min(\gamma_1, \gamma_2)$ with $\gamma_1 = 400$, $\gamma_2 = 100$, $s = 0.01$, $\omega = 100$, and $\vec{b}(0) = (1, 0, 0)$

higher than the “frequency” (inverse characteristic time) of transition between γ_1 and γ_2 . In order to specify dependencies, one should consider additional terms for Magnus expansion, which adds multiplicative frequencies of $b(t)$.

5. Generalizations and extensions

As we discussed earlier, the considered case, despite being simple, provides useful analytic transformation in regard to the bounds of (non-) adiabatic considerations’ applicability. In this section two extensions of the model are considered: we examine whether an increase of a number of interacting modes or a higher total number of photons in a Fock subspace may or may not change obtained estimations.

5.1. More interacting modes

Consider a larger amount of interacting modes, so $S = 2$, $V = 1$, $j \in \{-2, -1, 0, 1, 2\}$, and $|\mathbf{v}_1\rangle = |1, 0, 0, 0, 0\rangle$, $|\mathbf{v}_2\rangle = |0, 1, 0, 0, 0\rangle$, $|\mathbf{v}_3\rangle = |0, 0, 1, 0, 0\rangle$, $|\mathbf{v}_4\rangle = |0, 0, 0, 1, 0\rangle$, $|\mathbf{v}_5\rangle = |0, 0, 0, 0, 1\rangle$. In this case

$$\int_0^t W(\tau) d\tau = (-im_{opt}\omega t)I + \begin{pmatrix} 2ig(t) & a(t) & 0 & 0 & 0 \\ -a(t) & ig(t) & \sqrt{\frac{3}{2}}a(t) & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}}a(t) & 0 & \sqrt{\frac{3}{2}}a(t) & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}}a(t) & -ig(t) & a(t) \\ 0 & 0 & 0 & -a(t) & -2ig(t) \end{pmatrix}. \quad (42)$$

Applying the same procedure described above, we obtain

$$|E| = I + \mathcal{O}[r(t)], \quad (43)$$

and the deviation from the identity matrix in this case is proportional to $r(t)$. Thus, the same result as in Eq. (38) can be obtained. Considering the structure of matrix W , extension towards even more interacting modes implies the same proportionality of the deviation from the identity matrix to $r(t)$.

5.2. More photons

Consider a larger amount of photons in interacting modes associated with the corresponding Fock subspace, so $S = 1$, $V = 2$, $j \in \{-1, 0, 1\}$, and $|\mathbf{v}_1\rangle = |0, 1, 1\rangle$, $|\mathbf{v}_2\rangle = |1, 0, 1\rangle$, $|\mathbf{v}_3\rangle = |1, 1, 0\rangle$, $|\mathbf{v}_4\rangle = |2, 0, 0\rangle$, $|\mathbf{v}_5\rangle = |0, 2, 0\rangle$,

$|\mathbf{v}_6\rangle = |0, 0, 2\rangle$. In this case

$$\int_0^t W(\tau) d\tau = (-im_{opt}\omega t)I + \begin{pmatrix} ig(t) & -\frac{a(t)}{\sqrt{2}} & 0 & 0 & -a(t) & a(t) \\ \frac{a(t)}{\sqrt{2}} & 0 & -\frac{a(t)}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{a(t)}{\sqrt{2}} & -ig(t) & -a(t) & a(t) & 0 \\ 0 & 0 & a(t) & 2ig(t) & 0 & 0 \\ a(t) & 0 & -a(t) & 0 & 0 & 0 \\ -a(t) & 0 & 0 & 0 & 0 & -2ig(t) \end{pmatrix}. \quad (44)$$

Applying the same procedure described above, we obtain

$$|E\rangle = I + \mathcal{O}[r(t)]. \quad (45)$$

Considering the structure of matrix W , extension towards even more photons in modes implies the same proportionality of the deviation from the identity matrix to $r(t)$.

6. Discussion and conclusion

A non-adiabatic solution for the non-stationary Schrödinger equation with the non-periodic part of the Hamiltonian (quasi-energy operator), $Q(t)$, see Eq. (1), which describes phase modulation, obtained by $su(2)$ algebraic methods, see App. A, is considered in the article. Expressed in Eq. (9) eigenvectors and eigenvalues of $Q(t)$ allows us constructing the non-stationary solution, described in Eq. (22), in terms of a time-ordered exponentiated matrix W (23).

As a rather simple case for initial analysis, in Section 4, we consider three interacting modes, $S = 1$, a Fock subspace with a single photon, $V = 1$, and only three eigenvectors. The condition that separates adiabatic and non-adiabatic considerations is derived in Eq. (32) and is specified in Eq. (41). An increase in the number of interacting modes, $S > 1$ or a number of photons in the corresponding Fock subspace, $V > 1$, does not affect the condition that is shown in Section 5. At this point we conclude that the provided condition can be applied also for an arbitrary number of interacting modes and an arbitrary amount of photons in quantum optical states to be modulated.

In [3] it was shown that consideration of an infinite number of interacting modes restores classical phase modulation theory as the Wigner d-function transforms to a Bessel function of the first kind as a limiting case. A resonance, $\omega \rightarrow 0$, can also be considered for a stationary problem, providing significant simplifications. However, as it was shown, a non-stationary problem cannot be directly approached in this case; it requires rather sophisticated methods addressing high degeneracy. For now we left the degenerate case beyond the scope of the article for future research.

A. Appendix: Diagonalization of $Q(t)$

Commutation relations of operators that form $su(2)$ algebra are given by

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_0, \quad (46)$$

$$J_{\pm} = J_x \pm iJ_y, \quad J_y = \frac{J_+ - J_-}{2i}, \quad J_x = \frac{J_+ + J_-}{2}, \quad [J_y, J_0] = iJ_x, \quad [J_y, J_x] = -iJ_0. \quad (47)$$

And derived from the Baker-Campbell-Hausdorff formula, expressions for adjoint actions are as follows:

$$e^{aY} X e^{-aY} = X + a[Y, X] + \frac{a^2}{2}[Y, [Y, X]] + \dots, \quad (48)$$

$$e^{iaJ_0} J_{\pm} e^{-iaJ_0} = J_{\pm} e^{\pm ia}, \quad e^{i\varphi J_0} Q(t) e^{-i\varphi J_0} = m_{opt}\omega N + \omega J_0 + \frac{4\gamma(t)}{2S+1} J_x, \quad (49)$$

$$e^{iaJ_y} J_0 e^{-iaJ_y} = \cos(a)J_0 - \sin(a)J_x, \quad e^{iaJ_y} J_x e^{-iaJ_y} = \sin(a)J_0 + \cos(a)J_x, \quad (50)$$

$$e^{i\beta(t)J_y} e^{i\varphi J_0} Q(t) e^{-i\varphi J_0} e^{-i\beta(t)J_y} = m_{opt}\omega N + \Gamma(t)J_0, \quad (51)$$

$$\Gamma(t) = \sqrt{\omega^2 + \frac{16[\gamma(t)]^2}{(2S+1)^2}}, \quad \cos[\beta(t)] = \frac{\omega}{\Gamma(t)}. \quad (52)$$

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