

Entropy for two weakly coupled quantum waveguides: asymptotic approach

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ABSTRACT Electron in a system of two plane quantum waveguides coupled through small window is considered. The method of matching the asymptotic expansions of the boundary problem solution is used. The dependence of entropy on the window width is described. for the case of small window in the framework of the asymptotic approach.

KEYWORDS quantum waveguides, entropy, asymptotics

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1. Introduction

A possibility of using coupled quantum (optical) waveguides for quantum computations (see, e.g., [1–4]) stimulates the interest to investigation of electron (photon) behaviour in the system. It can be useful for many quantum computing applications (see, e.g., [5–7]). An additional interest to the subject is inspired by widely discussed idea of topological quantum computations [8]. The study of spectral properties of waveguide systems with point defects and coupling apertures is one of the classical directions in mathematical physics and quantum microscopic. Bound states and resonances are of special interest [9, 10]. It was shown that small perturbation of the waveguide boundary (coupling window or deformation) can cause the appearance of the discrete spectrum (bound states) or resonances (quasibound states) (see, e.g., [11–16]). In the present paper, we consider the entropy of the electron bound state in a system of two 2D quantum waveguides coupled through small window.

2. Description of the system

We consider a system of two planar waveguides Ω^+ and Ω^- with widths $d_{\pm}$ ($d_+ > d_-$), respectively, coupled through an aperture of small diameter $2a$ (Fig. 1). The wave function satisfies the Helmholtz equation with the Dirichlet conditions on the external boundaries and the Neumann condition on the internal boundary:

$$\begin{aligned}\Delta\psi + k^2\psi &= 0, \\ \psi|_{x_2=-d_-, x_2=d_+} &= 0, \\ \frac{\partial\psi}{\partial x_2}|_{x_2=0} &= 0.\end{aligned}\tag{1}$$

It was previously proven in [11] that such a system has an eigenvalue k_a close to the spectral threshold. In [15, 16], the asymptotics (as $a \rightarrow 0$) of this eigenvalue k_a were obtained, which can be conveniently written as

$$\gamma_a = \sqrt{\frac{\pi^2}{d_+^2} - k_a^2} = \frac{\pi^3}{d_+^3} a^2 + o(a^2).\tag{2}$$

Far from the aperture, the wave function is expressed through the normal derivative of Green's function (as $a \rightarrow 0$):

$$\psi_a^{\pm}(x) = \mp\gamma_a \frac{\partial}{\partial n_y} G^{\pm}(x, 0, k) + o(1),\tag{3}$$

where the following expansion is known for the derivative of Green's function:

$$\frac{\partial G^{\pm}}{\partial n_y}(x, 0, k_a) = \sum_{n=1}^{\infty} \frac{\pi n}{d_{\pm}^2 \sqrt{\frac{\pi^2 n^2}{d_{\pm}^2} - k_a^2}} \sin\left(\frac{\pi n x_2}{d_{\pm}}\right) \exp\left(-|x_1| \sqrt{\frac{\pi^2 n^2}{d_{\pm}^2} - k_a^2}\right).\tag{4}$$

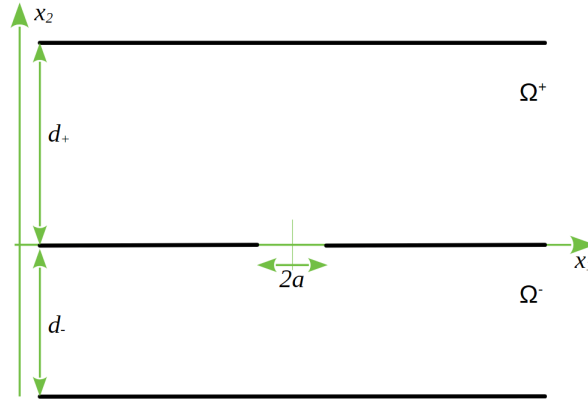


FIG. 1. The system of two planar waveguides

Extracting the first term and taking into account the known asymptotics, we obtain

$$\begin{aligned} -\psi_a^+(x) &\approx \frac{\pi}{d_+} \sin \frac{\pi x_2}{d_+} \exp \left(-\frac{\pi^3}{d_+^3} a^2 |x_1| \right) + \frac{\pi^3}{d_+^4} \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^2-1}} \sin \left(\frac{\pi n x_2}{d_+} \right) \exp \left(-\frac{\pi}{d_+} \sqrt{n^2-1} |x_1| \right), \\ \psi_a^-(x) &\approx \frac{\pi^3}{d_+^3 d_-^2} a^3 \sum_{n=1}^{\infty} \frac{n}{\sqrt{\frac{n^2}{d_-^2} - \frac{1}{d_+^2}}} \sin \left(\frac{\pi n x_2}{d_-} \right) \exp \left(-\pi |x_1| \sqrt{\frac{n^2}{d_-^2} - \frac{1}{d_+^2}} \right). \end{aligned} \quad (5)$$

Note that for the symmetric system ($d_+ = d_- = d$), the above formulas are not valid and need to be modified. Namely, instead of formulas (2) and (4), we have [15, 16]

$$\gamma_a = \sqrt{\frac{\pi^2}{d^2} - k_a^2} = \frac{2\pi^3}{d^3} a^2 + o(a^2), \quad (6)$$

$$\mp \psi_a^{\pm}(x) \approx \frac{\pi}{d} \sin \frac{\pi x_2}{d} \exp \left(-\frac{2\pi^3}{d^3} a^2 |x_1| \right) + \frac{2\pi^3}{d^4} \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^2-1}} \sin \left(\frac{\pi n x_2}{d} \right) \exp \left(-\frac{2\pi}{d} \sqrt{n^2-1} |x_1| \right). \quad (7)$$

3. Results and discussion

We are interested in the Shannon entropy, which is defined by the following integral (see [17–19]):

$$S = - \int_{\Omega^+ \cup \Omega^-} |\psi|^2 \ln |\psi|^2 dx. \quad (8)$$

Figure 2 presents the results of numerical calculations for the entropy using formula (8) for the asymmetric system at various width ratios of the waveguides.

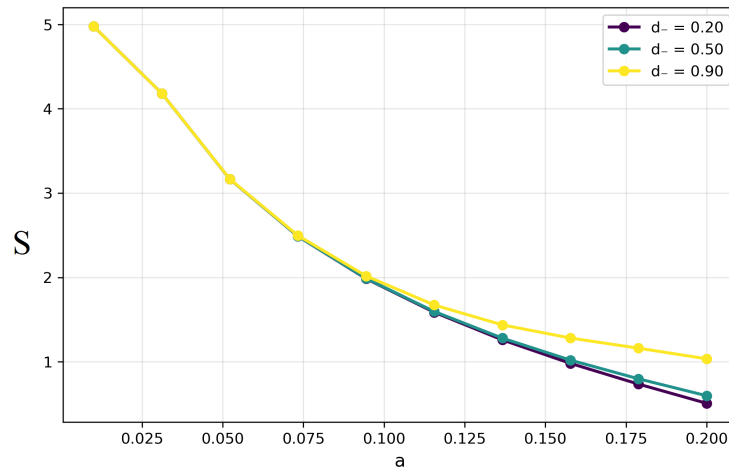


FIG. 2. Entropy of the system of waveguides of different widths ($d_+ > d_-$) as a function of the aperture width, $d_+ = 1$

Similar calculations were performed for the symmetric system. Fig. 3 shows the results of numerical calculations (blue line) and the results taken from the article [17] (red line).

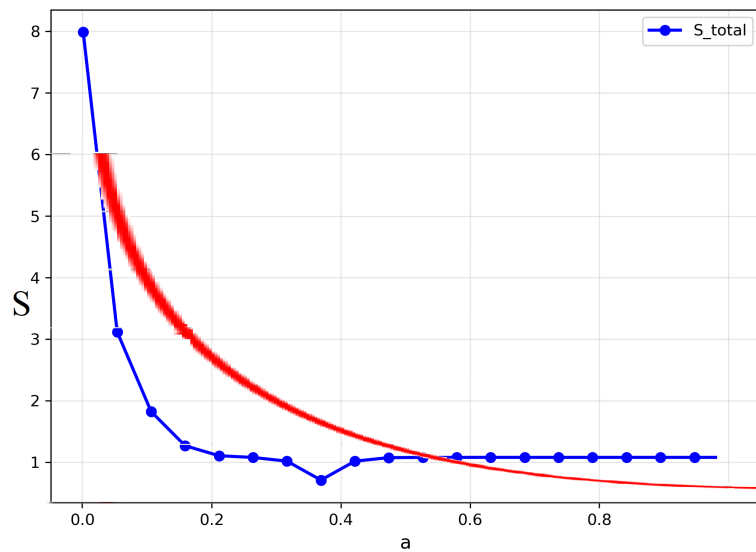


FIG. 3. Entropy of the system of waveguides of equal width ($d_+ = d_- = 1$) as a function of the aperture width: red – from [17], blue – the present work

It should be mentioned that direct numerical approach [17] gives one good results for sufficiently large window and cannot describe the behaviour for $a \rightarrow 0$ (see Fig. 2). As for our asymptotic result, it covers this gap - it presents good description for small window (see Fig. 2).

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