NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Tushavin G.V., et al. *Nanosystems: Phys. Chem. Math.*, 2022, **13** (3), 299–307.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2022-13-3-299-307

Original article

Ladder operators approach to representation classification problem for Jordan–Schwinger image of su(2) algebra

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ABSTRACT The eigenvalues of the complete commuting set of self-adjoint operators determine the classification of states. We construct a classification for the image of the Jordan–Schwinger mapping of the $su(2)$ algebra. We use the ladder operator approach to construct a canonical basis of irreducible representations and define the self-adjoint operators of the complete commuting set.

KEYWORDS Ladder operators, su(2), Jordan–Schwinger map, representation theory

FOR CITATION Tushavin G.V., Zaitseva E.V., Trifanov A.I. Ladder operators approach to representation classification problem for Jordan–Schwinger image of su(2) algebra. *Nanosystems: Phys. Chem. Math.*, 2022, **13** (3), 299–307.

1. Introduction

The study of dynamics of some quantum systems can be reduced to the study of the dynamic group of the Hamiltonian. Generators of the dynamical group form an algebra. The structures of invariant spaces of the algebra and the group are similar. Eigenvalues of self-adjoint operators of the complete commuting set are used to the state classification. The ladder operator approach used to build the complete set and obtain eigenbasis. In articles [1–4], ladder operators are constructed for different algebras, which are obtained in consequence of modification of quantum harmonic oscillator model. In our work, we have formulated a general approach to the analysis of such systems.

The Lie algebra of the dynamic group of the Hamiltonian of the quantum harmonic oscillator model is the Heisenberg–Weyl algebra [5, 6] – $w(1)$. Generators of this algebra are Hermitian-conjugate boson creation/annihilation operators – a and $a^\dagger$ which obey the following commutation relations

$$[a,\, a^\dagger] = \hat{I}, \quad [a,\, \hat{I}] = 0 = [a^\dagger,\, \hat{I}]. \tag{1}$$

Here $\hat{I}$ is the identity operator of algebra $w(1)$. By introducing a particle number operator $\hat{N} = a^\dagger a$, the mentioned Hamiltonian can be expressed as

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}). \tag{2}$$

The complete commuting set of operators for this Hamiltonian contains only one operator $\hat{N}$, which spectrum determines the observed energy levels. Operators a and $a^\dagger$ are ladder operators for the operator $\hat{N}$. They satisfy the commutation relations

$$\left[\hat{N}, a^\dagger\right] = a^\dagger, \quad \left[\hat{N}, a\right] = -a. \tag{3}$$

Action of ladder operators a and $a^\dagger$ transforms an eigenvector of operator $\hat{N}$ into another eigenvector

$$\begin{gathered} \hat{N}\,|n\rangle = n\,|n\rangle\,, \quad \hat{N}(a^\dagger\,|n\rangle) = a^\dagger(\hat{N} + \hat{I})\,|n\rangle = (n+1)(a^\dagger\,|n\rangle), \\ \hat{N}(a\,|n\rangle) = a(\hat{N} - \hat{I})\,|n\rangle = (n-1)(a\,|n\rangle), \quad a\,|0\rangle = 0 = \hat{N}\,|0\rangle\,, \end{gathered} \tag{4}$$

The annihilation operator a (unlike the creation operator $a^\dagger$) has a non-trivial kernel corresponding to the vacuum state of the quantum oscillator. The corresponding eigenvector $|0\rangle$ is called vacuum vector. Thus, the spectrum of operator $\hat{N}$ consists of integer non-negative numbers $\mathbb{N} \cup \{0\}$, and an arbitrary eigenvector can be obtained by the action of ladder operators on any particular eigenvector, e.g. the vacuum vector. In the canonical basis of the eigenvectors $\{|n\rangle\}$ the operators a and $a^\dagger$ have the form

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad a\,|n+1\rangle = \sqrt{n+1}\,|n\rangle\,, \quad a\,|0\rangle = 0, \tag{5}$$

and vector $|n\rangle$ is expressed as

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle\,. \tag{6}$$

In this way, dynamics of multidimensional harmonic oscillator can be described by an algebra which generators are represented through the bosonic polynomials resulting from Jordan–Schwinger mapping [7, 8] of generator matrices into $w(1)^{\otimes m}$ for certain m:

$$X = (x_{ij}) \mapsto \breve{X} = \sum_{i,j=1}^{m} x_{i,j} a_i^{\dagger} a_j, \quad [\breve{X}, \breve{Y}] = [X, \breve{Y}]. \tag{7}$$

The image of the identity matrix is the total particle number operator

$$\breve{I} = N = \sum_{\mu=1}^{m} a_{\mu}^{\dagger} a_{\mu} = \sum_{\mu=1}^{m} N_{\mu}. \tag{8}$$

In our paper [10], we consider the image of the algebra $su(2)$ [9] represented by the operators $N,\, J_z,\, J_+,\, J_-$, which are expressed using bosonic operators $a_i,\, a_i^{\dagger}$ by the Jordan–Schwinger mapping of generator matrices of the irreducible representation of dimension $(2s+1)$ of $su(2)$ algebra [8]:

$$J_z = \sum_{\mu=-s}^{s} \mu a_{\mu}^{\dagger} a_{\mu}, \quad J_+ = \sum_{\mu=-s}^{\mu=s-1} \sqrt{(s+\mu+1)(s-\mu)} a_{\mu+1}^{\dagger} a_{\mu} = (J_-)^{\dagger}. \tag{9}$$

We denote this algebra as $su^j(2)$ Bosonic operators for each degree of freedom obey the following commutation relations:

$$[a_i, a_j] = [a_i^{\dagger}, a_j^{\dagger}] = 0, \quad [a_i, a_j^{\dagger}] = \delta_{ij}. \tag{10}$$

The Fock basis, defined by eigenvalues of particle number operators of each degree of freedom, is complete and consists of vectors of the form $|n_{-s}, n_{-s+1}, \ldots, n_s\rangle$.

The Jordan–Schwinger mapping is a Lie algebras homomorphism, thus the matrix images obey the same commutation relations as their pre-images. The operators $J_z,\, J_+,\, J_-$ are generators of the algebra $su(2)$. They satisfy the corresponding commutative relations

$$[J_z,\, J_{\pm}] = \pm J_{\pm}, \quad [J_+,\, J_-] = 2J_z. \tag{11}$$

In $su(2)$ algebra, the Casimir operator commuting with all generators exists and by Schur's lemma, in the space of irreducible representation, such an operator is proportional to the identity operator. Recall that the image of the unit matrix in the Jordan–Schwinger mapping is operator N. Hence,

$$[N,\, J_z] = 0, \quad [N,\, J_{\pm}] = 0. \tag{12}$$

The Casimir operator J^2 for generators $J_z,\, J_+,\, J_-$ is defined as follows

$$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+). \tag{13}$$

The image of the canonical basis of an irreducible representation has standard form

$$\begin{gathered}
J_z \,|j,\, j_z\rangle = j_z \,|j,\, j_z\rangle\,, \quad J^2 \,|j,\, j_z\rangle = j(j+1)\,|j,\, j_z\rangle\,, \\
J_+ \,|j,\, j_z = j\rangle = 0, \quad J_- \,|j,\, j_z = -j\rangle = 0, \\
J_+ \,|j,\, j_z\rangle = \sqrt{(j-j_z)(j+j_z+1)}\,|j,\, j_z+1\rangle\,, \\
J_- \,|j,\, j_z+1\rangle = \sqrt{(j-j_z)(j+j_z+1)}\,|j,\, j_z\rangle\,.
\end{gathered}$$

The commuting set of operators $\{N; J^2, J_z\}$ is complete in the cases $s = \frac{1}{2}$ and $s = 1$. In these cases, the eigenvalues of operators N, J^2, J_z uniquely determine the basis vectors $|n;\, j,\, j_z\rangle$. In other cases within a fixed eigenvalue n of the operator N, the eigenvalues $j(j+1)$ of the operator J^2 are nontrivially degenerate. Note that if s is a non-negative integer, then j is also a non-negative integer. The arbitrary Fock vector will be an eigenvector for the operators $\{N; J^2, J_z\}$, but not for the operator J^2.

The aim of our work is to augment the existing commutative set $N;\, J^2,\, J_z$ to a complete one. In our paper, we propose a method for constructing generalized ladder operators, which are used for classification and construction of the canonical basis.

2. Generalized ladder operators

Let us consider the self-adjoint operator $H = H^{\dagger}$. We will call an operator $p^{\dagger}$ a *right ladder operator* (hereafter, RLO) if there exists a nonzero selfadjoint operator $P = P^{\dagger} \neq 0$ commuting with H, such that one of the following commutation relations is satisfied

$$[H,\, p^{\dagger}] = p^{\dagger} P \quad \text{or} \quad Hp^{\dagger} = p^{\dagger}(P + H). \tag{14}$$

The expression conjugated to (14) is the definition of the *left ladder operator* (LLO)

$$[p,\, H] = Pp. \tag{15}$$

For RLO $p^\dagger$, we will call the operator P a *right function* in the case when the operator P is represented as a function of $P(H, H_1, \ldots, H_n)$ of the commuting set of self-adjoint operators $H,\ H_1,\ \ldots, H_n$.

For an arbitrary polynomial of the operator H the operator p is a ladder operator. In view of the bilinearity of the commutator it suffices to show that for any degree of H the following property holds

$$[H^n,\, p^\dagger] = p^\dagger((H+P)^n - H^n) \quad \text{or} \quad H^n p^\dagger = p^\dagger (H+P)^n. \tag{16}$$

Proof. To prove this statement it is enough to use the recurrent property

$$[H^n,\, p^\dagger] = [H,\, p^\dagger]H^{n-1} + H[H^{n-1}, p^\dagger] = p^\dagger P H^{n-1} + [H^{n-1}, p^\dagger](H+P), \tag{17}$$

which can be proved by applying the method of mathematical induction, where the base of induction is the definition of the ladder operator.

Also, one can show that the result of multiplying the RLO by the self-adjoint operator A which commutes with operators H and P is again the RLO of operator H:

$$[H,\, pA] = pAP. \tag{18}$$

2.1. Ladder operators construction

Let the system of the self-adjoint operator H and the set of operators $\{T_\mu\}_{\mu=1}^n$ have the following properties

$$[H,\, T_\eta] = \sum_{\mu=1}^n T_\mu \alpha_{\mu\eta}, \quad \alpha_{\mu\eta}^\dagger = \alpha_{\mu\eta}, \quad [\alpha_{\mu\eta}, H] = 0. \tag{19}$$

We are looking for a nontrivial set of self-adjoint operators $\sigma_\eta = {\sigma_\eta}^\dagger$ which commute with H and $\{\alpha_{\mu\eta}\}$ and the operator $\sum_{\eta=1}^n T_\eta \sigma_\eta$ is the RLO for H again:

$$[H,\, \sum_{\eta=1}^n T_\eta \sigma_\eta] = \sum_{\eta=1}^n T_\eta \sigma_\eta P, \quad [P,\, \sigma_\eta] = 0.$$

Substituting (19) into the previous expression, we obtain the following equation

$$\sum_{\mu=1}^n T_\mu (\sum_{\eta=1}^n \alpha_{\mu\eta}\sigma_\eta - \sigma_\mu P) = 0, \tag{20}$$

which can be represented in matrix form

$$\begin{pmatrix} T_1 & T_2 & \ldots & T_n \end{pmatrix} (A-P) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \ldots \\ \sigma_n \end{pmatrix} = 0, \tag{21}$$

where we use $(A-P)$ instead of matrix

$$(A-P) = \begin{pmatrix} \alpha_{11}-P, & \alpha_{12}, & \ldots, & \alpha_{1n} \\ \alpha_{21}, & \alpha_{22}-P, & \ldots, & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}, & \alpha_{n2}, & \ldots, & \alpha_{nn}-P \end{pmatrix}.$$

One of many solutions (21) is the solution to the equation

$$(A-P) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \ldots \\ \sigma_n \end{pmatrix} = 0. \tag{22}$$

Since all elements of the matrix $(A-P)$ commute with each other, we can consider the determinant of the matrix $(A-P)$, which must be equal to zero, since the coefficients $\{\sigma^\eta\}$ are in the nontrivial kernel of the matrix $(A-P)$. Hence, the equation for the right functions of RLO arises

$$\det(A-P) \equiv 0. \tag{23}$$

The determinant is a polynomial of the operator P of degree n, and its roots are various right-hand functions of RLO. Then, by substituting the obtained roots into equation (21), we can find their corresponding coefficients $\{\sigma_\eta\}$ of the RLO's.

3. Irreducible representations of the $su^J(2)$ algebra

3.1. Ladder operators of the Casimir operator

In this section, we construct ladder operators for the Casimir operator in the case of integer s. If s is integer, then irreducible representations with integer weights only are implemented, and the kernel of operator J_z is always nontrivial for any number of particles. Thus, any irreducible representation can be recovered by ladder operators of the algebra $su(2)$ from its state lying in the kernel of operator J_z, Thus, it suffices to solve the classification problem within the kernel of J_z. When s is half-integer, the irreducible representations of all possible weights are realized. However, the proposed approach can be easily applied to this case: the only difference is that the resulting ladder operators will not commute with the operator J_z but its ladder operators will be .

Let s be a non-negative integer. Consider the set of operators $\{p_k^\dagger\}_{k=0}^s$ and $\{m_k^\dagger\}_{k=1}^s$ commuting with operator J_z

$$\begin{gathered} p_0^\dagger = 2a_0^\dagger, \\ p_k^\dagger = \frac{1}{\prod_{i=1}^k \sqrt{(s+i)(s-i+1)}}\left(a_{-k}^\dagger J_+^k + a_k^\dagger J_-^k\right), \\ m_k^\dagger = \frac{1}{\prod_{i=1}^k \sqrt{(s+i)(s-i+1)}}\left(a_{-k}^\dagger J_+^k - a_k^\dagger J_-^k\right). \end{gathered} \tag{24}$$

All operators from the sets $\{p_k^\dagger\}$ and $\{m_k^\dagger\}$ are ladder operators of operator N

$$[N, p_k^\dagger] = p_k^\dagger, \quad [N, m_k^\dagger] = m_k^\dagger.$$

The operators $p_k^\dagger$ and $m_k^\dagger$ are closed with respect to the action of the Casimir operator J^2 in the sense of definition (19)

$$\begin{gathered} [J^2, p_0^\dagger] = s(s+1)p_0^\dagger + 2s(s+1)p_1^\dagger, \\ [J^2, p_k^\dagger] = ((s+k+1)(s-k) - k(k-1))p_k^\dagger + (s+k+1)(s-k)p_{k+1}^\dagger + \\ + p_{k-1}^\dagger((\hat{j}+J_z+1)(\hat{j}-J_z) - k(k-1)) + 2k(m_k^\dagger + m_{k-1}^\dagger)J_z, \\ [J^2, m_k^\dagger] = ((s+k+1)(s-k) - k(k-1))m_k^\dagger + (s+k+1)(s-k)m_{k+1}^\dagger + \\ + m_{k-1}^\dagger((\hat{j}+J_z+1)(\hat{j}-J_z) - k(k-1)) + 2k(p_k^\dagger + p_{k-1}^\dagger)J_z, \end{gathered} \tag{25}$$

where the operator $\hat{j}$ is defined as

$$\hat{j} = \frac{1}{2}\left(\sqrt{\hat{I} + 4J^2} - \hat{I}\right). \tag{26}$$

Let us find the right-hand functions of the ladder operators from equation (21). We will construct ladder operators for the kernel J_z since the whole basis of the irreducible representation can be restored by the action of operators $J_\pm$. For this reason, we can replace the operator J_z in equation (25) by zero $J_z = 0$

$$\begin{gathered} [J^2, p_0^\dagger] = s(s+1)p_0^\dagger + 2s(s+1)p_1^\dagger, \\ [J^2, p_k^\dagger] = ((s+k+1)(s-k) - k(k-1))p_k^\dagger + (s+k+1)(s-k)p_{k+1}^\dagger + p_{k-1}^\dagger((\hat{j}+1)\hat{j} - k(k-1)), \\ [J^2, m_k^\dagger] = ((s+k+1)(s-k) - k(k-1))m_k^\dagger + (s+k+1)(s-k)m_{k+1}^\dagger + m_{k-1}^\dagger((\hat{j}+1)\hat{j} - k(k-1)). \end{gathered} \tag{27}$$

Let us construct matrix $(A - P)$. The matrix A is a block-diagonal one

$$A = \begin{pmatrix} P & 0 \\ 0 & M \end{pmatrix},$$

consisting of two tridiagonal matrices P and M of dimensions $\dim P = s+1$ and $\dim M = s$, respectively,

$$P = \begin{pmatrix} s(s+1) & \hat{j}(\hat{j}+1), & 0, & \dots & 0 & 0 \\ 2s(s+1) & s(s+1)-4 & (\hat{j}-1)(\hat{j}+2) & \dots & 0 & 0 \\ 0 & (s-1)(s+2) & s(s+1)-8 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -s^2+5s-2 & \hat{j}(\hat{j}+1)-s^2+s \\ 0 & 0 & 0 & \dots & 2s & -s^2+s \end{pmatrix}.$$

Matrix M is obtained from matrix P by crossing out the first row and column.

The choice of coefficients at $\{p_k^\dagger\}_{k=0}^s$ and $\{m_k^\dagger\}_{k=1}^s$ makes it symmetric, hence, the matrices P and M have different eigenvalues, which are expressed using the operator $\hat{j}$. The traces of matrices P and M are equal to the sum of their eigenvalues. The matrix A corresponds to the following set of eigenvalues

$$\left\{\theta(\theta + 2\hat{j} + 1)\hat{I}\right\}_{\theta=-s}^{s},$$

and the matrix P is matched by θ of the same parity as s, and the matrix M by all others.

We will look for the coefficients recurrently, starting with σ_s. For matrices P and M, the equations on the ladder operator will be almost identical. It allows us to obtain a common result for them. Let us find the solution of the following equation

$$\left(P - \theta(\theta + 2\hat{j} + 1)\hat{I}\right)\begin{pmatrix}\sigma_0^\theta \\ \sigma_1^\theta \\ \vdots \\ \sigma_s^\theta = \hat{I}\end{pmatrix} = 0.$$

The coefficient σ_{s-1}^θ is found at $\sigma_s^\theta = \hat{I}$:

$$\sigma_{s-1}^\theta = \hat{j}\frac{\theta}{s} + \frac{\theta^2 + \theta + s^2 - s}{2s}. \tag{28}$$

Consider the k-th string:

$$(s-k)(s+k+1)\sigma_{k-1}^\theta + ((s-k)(s+k+1) - k(k-1) - \theta(\theta + 2\hat{j} + 1))\sigma_k^\theta + (\hat{j} - k)(\hat{j} + k + 1)\sigma_{k+1}^\theta = 0$$

and express σ_{k-1}^θ through σ_k^θ and σ_{k+1}^θ:

$$\sigma_{k-1}^\theta = \frac{\theta^2 + \theta + k^2 + k}{(s+k)(s-k+1)}\sigma_k^\theta + \hat{j}\frac{2\theta\sigma_k^\theta}{(s+k)(s-k+1)} - \sigma_k^\theta - \frac{(\hat{j} + k + 1)(\hat{j} - k)}{(s+k)(s-k+1)}\sigma_{k+1}^\theta, \tag{29}$$

where σ_k^θ is a polynomial of the operator $\hat{j}$ of degree $(s-k)$.

Denote the obtained ladder operators as $\{\tau_\theta^\dagger\}_{\theta=-s}^s$

$$\tau_\theta^\dagger = \begin{cases} \sum\limits_{k=0}^{s} p_k^\dagger \sigma_k^\theta, & \text{for } \theta \text{ of the same parity as } s, \\ \sum\limits_{k=1}^{s} m_k^\dagger \sigma_k^\theta, & \text{otherwise.} \end{cases} \tag{30}$$

Ladder operators have the following commutative relations with the J^2 operator

$$[J^2, \tau_\theta^\dagger] = \tau_\theta^\dagger \theta(\theta + 2\hat{j} + 1). \tag{31}$$

Since the Casimir operator J^2 is represented as a polynomial $J^2 = \hat{j}(\hat{j} + 1)$ of operator $\hat{j}$, we obtain commutation relations between $\hat{j}$ and $\{\tau_\theta^\dagger\}$ from the solution of the following equation

$$[\hat{j}^2 + \hat{j}, \tau_\theta^\dagger] = \tau_\theta^\dagger\, X(X + 2\hat{j} + 1) = \tau_\theta^\dagger\, \theta(\theta + 2\hat{j} + 1).$$

Hence, $X = \theta\hat{I}$ and

$$[j, \tau_\theta^\dagger] = \theta\tau_\theta^\dagger. \tag{32}$$

Operators $\{\tau_\theta^\dagger\}$ are also ladder operators for operators $\dfrac{1}{2\hat{j} + (2k+1)\hat{I}}$, where k is non-negative:

$$\left[\frac{1}{2\hat{j} + (2k+1)\hat{I}}, \tau_\theta^\dagger\right] = \tau_\theta^\dagger\left(\frac{1}{2\hat{j} + (2k+1)\hat{I}} - \frac{1}{2\hat{j} + (2(k-\theta)+1)\hat{I}}\right). \tag{33}$$

There is similar expression with the left-hand function:

$$\left[\frac{1}{2\hat{j} + (2k+1)\hat{I}}, \tau_\theta^\dagger\right] = \left(\frac{1}{2\hat{j} + (2k+1)\hat{I}} - \frac{1}{2\hat{j} + (2(k-\theta)+1)\hat{I}}\right)\tau_\theta^\dagger. \tag{34}$$

For an arbitrary polynomial of functions $\{\hat{j}^k\}_{k=0}^n$ and $\left\{\left(\dfrac{1}{2\hat{j} + (1+2k)\hat{I}}\right)^k\right\}_{k=0}^n$ commutative relations with $\{\tau_\theta\}$ or $\{\tau_\theta^\dagger\}$ can be obtained.

Single-particle Fock states belongs to the irreducible representation of the algebra $su(2)$ corresponding to the eigenvalue $s(s+1)$ of the Casimir operator J^2

$$J^2 \left|0,\, 0,\, \ldots,\, n_k = 1,\, \ldots, 0\right\rangle = s(s+1) \left|0,\, 0,\, \ldots,\, n_k = 1,\, \ldots, 0\right\rangle,$$

and the kernel J_z is one-dimensional and consists of the following vector

$$J_z \left|0,\, 0,\, \ldots,\, n_0 = 1,\, \ldots, 0\right\rangle = 0.$$

The action of the ladder operators $\{\tau_\theta^\dagger\}$ allows us to construct the canonical basis of the kernel J_z. At these, it is easy to show that

$$[\tau_\theta^\dagger \tau_\theta,\, J^2] = 0 = [\tau_\theta^\dagger \tau_\theta,\, J_z] = [\tau_\theta^\dagger \tau_\theta,\, N].$$

Thus, the commuting set $\left\{N; J^2, j_z\right\}$ can be augmented to complete set of commuting operators by some self-adjoint polynomials of ladder operators.

3.2. The annihilated states of the ladder operators of the Casimir operator

The geometry of the Fock space allows us to find annihilated states of ladder operators $\left\{\tau_\theta^\dagger,\, \tau_\theta\right\}_{\theta=-s}^{s}$. Consider the vectors of the operators $\hat{j}$ and N lying in the kernel of the operator J_z

$$\left|n,\, j,\, , j_z = 0\right\rangle,$$

Given n the eigenvalues of the operator $\hat{j}$ are in the range $0 \leqslant j \leqslant ns$. The action of operators within the J_z kernel can be represented by the following scheme for $\omega = 1 \ldots s$:

$$\begin{aligned} &\tau_\omega^\dagger \left|n,\, j\right\rangle \Rightarrow \left|n+1,\, j+\omega\right\rangle, && \tau_{-\omega}^\dagger \left|n,\, j+\omega,\right\rangle \Rightarrow \left|n+1,\, j\right\rangle, \\ &\tau_\omega \left|n+1,\, j+\omega,\right\rangle \Rightarrow \left|n,\, j\right\rangle, \quad \text{è} && \tau_{-\omega} \left|n+1,\, j,\right\rangle \Rightarrow \left|n,\, j+\omega\right\rangle. \end{aligned} \tag{35}$$

Operators $\tau_\omega^\dagger$ have a trivial kernel if ω is the same parity, as s. If ω differs in parity from s, then all one-particle state lies in the kernel of $\tau_\omega^\dagger$. This is due to the antisymmetric definition of the operator $\tau_\omega^\dagger$ for ω other than s parity.

The operators τ_ω transforms all states $j < \omega$ and the vacuum state $n = 0$ into zero, Thus, implementing ω of different representations of the algebra. The algebra of the pair of operators $\tau_\omega^\dagger$ and τ_ω itself is a deformation of the Weyl algebra $w(1)$. Its different representations are defined by the number $r_\theta = j \mod \omega$ and the eigenvalues of the self-adjoint operators $\tau_\omega^\dagger \tau_\omega$.

The operators $\tau_{-\omega}^\dagger$ annihilate all states $j < \omega$, and the operators $\tau_{-\omega}$ annihilate all states $j > ns - \omega$ and vacuum state $n = 0$. Thus, we can say that the operators $\tau_{-\omega}^\dagger$ and $\tau_{-\omega}$ represent a deformation of the algebra $su(2)$, where the representations differ by the number $r_\theta = j \mod \omega$ and the eigenvalues of the self-adjoint operators

$$L_z^\omega = [\tau_{-\omega}^\dagger, \tau_{-\omega}], \quad L_\omega^2 = (L_z^\omega)^2 + \frac{1}{2}\left(\tau_{-\omega}^\dagger \tau_{-\omega} + \tau_{-\omega}\tau_{-\omega}^\dagger\right).$$

The operators $\tau_0^\dagger$ and τ_0 do not change the eigenvalues of the Casimir operator J^2

$$\tau_0^\dagger \left|n,\, j\right\rangle \Rightarrow \left|n+1,\, j\right\rangle, \quad \tau_0 \left|n+1,\, j,\right\rangle \Rightarrow \left|n,\, j\right\rangle. \tag{36}$$

4. Case $s = 1$

In this case, the classification problem is of small interest because of all subspaces of kernel J_z are one-dimensional and the set of commuting operators $N;\ J^2,\ J_z$ is complete. However, the use of ladder operators can be well demonstrated by the following example. For the case of $s = 1$, the generators of the $su(2)$ algebra are represented as follows

$$J_z = \sum_{\mu=-1}^{1} \mu a_\mu^\dagger a_\mu, \quad J_+ = (J_-)^\dagger = \sum_{\mu=-1}^{\mu=0} \sqrt{(\mu+2)(1-\mu)}\, a_{\mu+1}^\dagger a_\mu. \tag{37}$$

Consider the following operators

$$p_0^\dagger = 2a_0^\dagger, \quad \sqrt{2}p_1^\dagger = a_1^\dagger J_- + a_{-1}^\dagger J_+, \tag{38}$$

with the following commutation relations

$$\begin{gathered} [p_0,\, p_0^\dagger] = 4, \quad [p_1,\, p_1^\dagger] = 2\hat{j}(\hat{j}+1) - J_z(2J_z+1) + (N - N_0)(J_z - 2), \\ [p_1,\, p_0^\dagger] = 2(N - N_0), \quad [p_0,\, p_1^\dagger] = 2(N - N_0), \end{gathered} \tag{39}$$

where $N_0 = a_0^\dagger a_0$.

The $m_1^\dagger$ operator annihilates the J_z kernel (this is trivially checked) if we consider the action of the $m_1^\dagger$ operator on arbitrary Fock state $\left|n_{-1} = m,\, n_0 = k,\, n_1 = m\right\rangle$. However, outside the kernel J_z the operator $m_1^\dagger$ acts nontrivially, which is important in the construction of ladder operators on the whole Fock space. In our case it is important to obtain canonical basis inside the kernel of J_z, because the whole basis can be reconstructed by the action of the operators J_+ and J_-.

Consider commutative relations of operators $p_i^\dagger$ with operator J^2

$$[J^2,\, p_0^\dagger] = 2p_0^\dagger + 4p_1^\dagger, \quad [J^2,\, p_1^\dagger] = p_0^\dagger J_- J_+ = p_0^\dagger(\hat{j} - J_z)(\hat{j} + J_z + 1).$$

Assume $J_z \equiv 0$ and rewrite the commutation relations

$$[J^2,\, p_0^\dagger] = 2p_0^\dagger + 4p_1^\dagger, \quad [J^2,\, p_1^\dagger] = p_0^\dagger \hat{j}(\hat{j} + 1). \tag{40}$$

The solution of the equation for the right functions of the ladder operators is given by the operators $-2\hat{j}$ and $2(\hat{j} + 1)$.

Denote the ladder operators $\tau_{-1}^\dagger$ and $\tau_1^\dagger$. They have the following commutative relations with the operator J^2

$$[J^2,\, \tau_{-1}^\dagger] = -\tau_{-1}^\dagger 2\hat{j}, \quad [J^2,\, \tau_1^\dagger] = \tau_1^\dagger 2(\hat{j} + 1). \tag{41}$$

They are expressed using the operators $p_0^\dagger$ and $p_1^\dagger$ as follows

$$\tau_{-1}^\dagger = p_0^\dagger \hat{j} - 2p_1^\dagger, \quad \tau_1^\dagger = p_0^\dagger(\hat{j} + 1) + 2p_1^\dagger. \tag{42}$$

Commutative relations for the operator $\hat{j}$ $\hat{j}$:

$$[\hat{j},\, \tau_{-1}^\dagger] = -\tau_{-1}^\dagger, \quad [\hat{j},\, \tau_1^\dagger] = \tau_1^\dagger. \tag{43}$$

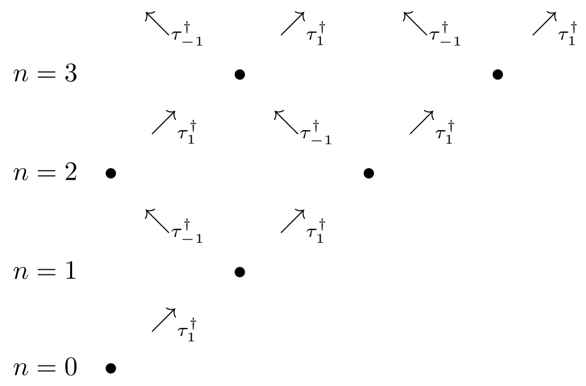
From the Jacobi relation we also obtain that the commutator $[\tau_1^\dagger,\, \tau_{-1}^\dagger]$ is a ladder operator $\hat{j}$:

$$[\hat{j},\, [\tau_1^\dagger,\, \tau_{-1}^\dagger]] = 2[\tau_1^\dagger,\, \tau_{-1}^\dagger].$$

Any vector of the canonical basis can be obtained by the joint action of the ladder operators

$$|n,\, j,\, j_z\rangle_{su2} = \alpha(n, j, j_z) \begin{cases} J_+^{j_z} (\tau_{-1}^\dagger)^{\frac{n-j}{2}} (\tau_1^\dagger)^{\frac{n+j}{2}} |000\rangle_F, & j_z > 0 \\ (\tau_{-1}^\dagger)^{\frac{n-j}{2}} (\tau_1^\dagger)^{\frac{n+j}{2}} |000\rangle_F, & j_z = 0 \\ J_-^{j_z} (\tau_{-1}^\dagger)^{\frac{n-j}{2}} (\tau_1^\dagger)^{\frac{n+j}{2}} |000\rangle_F, & j_z < 0. \end{cases}$$

The action of ladder operators and the structure of irreducible representations of the algebra $su(2)$ can be visualized by the following scheme for $j_z = 0$:



$j =$	0	1	2	3	...
dim	1	3	5	7	...

Now, consider again the operators $p_0^\dagger$ and $p_1^\dagger$ which can be expressed using the operators $\tau_{-1}^\dagger$ and $\tau_1^\dagger$

$$p_0^\dagger = \left(\tau_1^\dagger + \tau_{-1}^\dagger\right) \frac{1}{2\hat{j} + 1}, \quad p_1^\dagger = \frac{1}{4}\left((\tau_1^\dagger - \tau_{-1}^\dagger) - (\tau_1^\dagger + \tau_{-1}^\dagger)\frac{1}{2\hat{j} + 1}\right),$$

where we find commutative relations with the operator $\hat{j}$. We obtain

$$[\hat{j},\, p_0^\dagger] = (p_0^\dagger + 4p_1^\dagger)\frac{1}{2\hat{j} + 1}, \quad [\hat{j},\, p_1^\dagger] = (p_0^\dagger J^2 - p_1^\dagger)\frac{1}{2\hat{j} + 1}. \tag{44}$$

Define new operators, which will also be RLOs of the operator $\hat{K}$

$$A^\dagger = \tau_1^\dagger \frac{1}{2\sqrt{\hat{j}+1}} \frac{1}{\sqrt{(N+1)+\hat{j}+1+1}} \frac{\sqrt{2(\hat{j}+1)+1}}{\sqrt{2(\hat{j}+1)-1}}, \tag{45}$$

$$L_+ = \tau_{-1}^\dagger \frac{1}{2\sqrt{2}\sqrt{\hat{j}+1}} \frac{\sqrt{2(\hat{j}+1)-1}}{\sqrt{2(\hat{j}+1)+1}}. \tag{46}$$

The operators A and $A^\dagger$, defined above, satisfy the commutation relations of the Weyl algebra $w(1)$

$$[A,\, A^\dagger] = \hat{I}.$$

Self-adjoint operator $A^\dagger A$ has the same eigenvalues as the operator $\hat{j}$. The action of the operator on the state $|n,\, j\, j_z\rangle$ is defined by the formula

$$A^\dagger A\, |n,\, j\, j_z\rangle = j\, |n,\, j\, j_z\rangle\, .$$

Using the operators $L_\pm$ as self-adjoint polynomials, the operators L_z and L^2 are defined as follows

$$L_z = \frac{1}{2}[L_+,\, L_-], \quad L^2 = L_z^2 + L_z + L_- L_+.$$

They satisfy the commutation relations of the algebra $su(2)$. Their action on the eigenstates is given by the following expression

$$L_z\, |n,\, j\, j_z\rangle = \left(\frac{n-j}{2} - \frac{n+j}{4}\right) |n,\, j\, j_z\rangle\, ,$$
$$L^2\, |n,\, j\, j_z\rangle = \frac{n+j}{4}\left(\frac{n+j}{4}+1\right) |n,\, j\, j_z\rangle\, .$$

The operators $A^\dagger A$, L_z and L^2 form a complete commutative set and can be used to classify the states of such a system on a par with the sets N_{-1}, N_0, N_1 and K, J_z, N.

By constructing left-hand ladder operators for J^2, we obtain another form of ladder operators

$$\begin{aligned} \bar{\tau}_1 &= [a_0, \hat{j}] + a_0, \quad \bar{\tau}_{-1} = -[a_0, \hat{j}] + a_0, \\ \bar{\tau}_1^\dagger &= [\hat{j}, a_0^\dagger] + a_0^\dagger, \quad \bar{\tau}_{-1} = -[\hat{j}, a_0^\dagger] + a_0^\dagger. \end{aligned} \tag{47}$$

From which, in particular, an interesting expression emerges

$$[\hat{j}[\hat{j},\, a_0^\dagger]] = a_0^\dagger.$$

5. Conclusion

A method of classification and construction of invariant spaces corresponding to various irreducible representations of the $su(2)$ algebra is proposed for $su^j(2)$ algebra. We obtained a set of the ladder operators for the Casimir operator of the $su^j(2)$ algebra, which is used to find the canonical basis of the algebra. Algebras formed by ladder operators are deformations of known algebras, which eigenvalues determine persistent states of the Hamiltonian. In this paper we considered the simplest case for the algebra $su^j(2)$ and applied the ladder operator approach to demonstrate the method. The ladder operator approach is based on commutative algebra relations and can be applied to the analysis of irreducible representations of various Lie algebras. In this paper, we obtained an infinite basis of a complex structure which can be recovered from any chosen element of basis by the action of the ladder operators.

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Submitted 1 May 2022; revised 12 June 2022; accepted 13 June 2022

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Conflict of interest: the authors declare no conflict of interest.