On existence conditions for periodic solutions to a differential equation with constant argument

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ABSTRACT   We deal with a linear differential equation with piecewise constant argument. The considered equation with the initial condition has the unique solution. We obtain the sufficient conditions for existence of n-periodic solution for the considered problem and describe the positivity conditions for the solution.

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1.   Introduction

Differential equations with piecewise constant arguments (briefly DEPCA) occur when trying to extend the theory of functional differential equations with continuous arguments to differential equations with discontinuous arguments. In [1] Cooke and Wiener studied a new differential equation alternately of retarded and advanced type. They have shown that all equations with piecewise constant delays have characteristics similar to the equations studied in [2]. These equations are closely related to impulse and loaded equations and, especially, to difference equations of a discrete argument. The equations are similar in structure to those found in certain "sequential-continuous" models of disease dynamics [5]. Differential equations with piecewise constant arguments are usually referred to as a hybrid system. It can model certain harmonic oscillators with almost periodic forcing [3], [4]. For a survey of work on ordinary and partial differential equations with piecewise constant arguments (DEPCA) we refer the reader to [6], [7]. Functional differential equations with deviated argument provide a mathematical model for systems where the changes of state depend upon its past history or its future. DEPCA also arises in the process of replacing some terms of differential equation by their piecewise constant approximations. This point of view has applications in impulsive or loaded differential equations of control theory, and stabilization of systems with discrete (sample) control [7], [8]. Recently published papers [13] and [14] studied DEPCA special form. Authors reduced n-periodic solvable problem to the study of a system of n linear equations. Furthermore, by applying the well-known properties of linear system in algebra, all existence conditions are described for n-periodic solutions that yields explicit formula for the solutions of the equations.

Paper [17] dealt with a non-autonomous piecewise linear difference equation

$$x'(t) = -g(x([t])),$$

where g is a linear signal function that describes a discrete version of a single neuron model. The authors investigated the periodic behavior of solutions relative to the periodic sequence with the period of three internal decay rate. More precisely, they showed that only periodic cycles with period $3k$, $k = 1, 2, 3, ...$ can exist. Further, the considered model is investigated in [18]. The authors proved that the model contains a large quantity of initial conditions that generate eventually periodic solutions.

In inertial neural networks, inertial terms are described by the first-order derivative terms which have important meaning in engineering technology, biology, physics and information systems. For more details, see, e.g., [19–21]. The problems of periodic solutions for neutral-type inertial neural networks with single and multiple variables delays are investigated in [22–24].

In this paper, we consider a linear differential equations with piecewise constant argument of the form

$$T'(t) + a(t)T(t) + b(t)T([t]) = 0, \quad t > 0, \tag{1}$$

with

$$T(0) = T_0, \tag{2}$$

where nonzero functions $a(t), b(t)$ are n-periodic and continuous on $\mathbb{R}_+ = [0, \infty)$, where n is a positive integer number. Note that the general case of the problem $(1, 2)$ was studied in [15], where authors obtain the existence conditions for solutions and proved Gronwall type integral inequality as an application. In this note, we obtain the existence conditions for periodic solution to this initial value problem. We obtain the explicit formula and the positivity conditions for the solutions. As an application of the obtained result, we derive a condition for non-asymptotically stable solution of a certain class of partial differential equations with piecewise constant argument, which claims the correctness of the conditions of asymptotic stability conditions for the solution given in [7].

2. On a solution of equation

Let us define a definition of solution for $(1, 2)$.

Definition 2.1. A function $T(t)$ is called a solution of $(1, 2)$ if the following conditions are satisfied:
   (i) $T(t)$ is continuous on $\mathbb{R}_+$;
   (ii) $T'(t)$ exists and is continuous in $\mathbb{R}_+$, with possible exception at points $[t] \in \mathbb{R}_+$, where one-sided derivatives exist;
   (iii) $T(t)$ satisfies Eq. $(1, 2)$ in $\mathbb{R}_+$, with the possible exception at the points $[t] \in \mathbb{R}_+$.

Example 2.1. Let $a(t) = 1$ and $b(t) = \beta \sin 2\pi t$, where β is a real number satisfying the equation

$$\left(1 - \frac{\beta}{1 + 4\pi^2} (1 - e) \right) - e = 0.$$

For this case the function $T(t)$ on $\mathbb{R}_+$ defined as

$$T(t) = T_0 e^{-t} \left(1 - \frac{\beta}{1 + 4\pi^2} \left(1 + e^t (\sin 2\pi t - \cos 2\pi t) \right) \right), \quad t \in [0, 1).$$

is one periodic solution of $(1, 2)$.

Denote

$$M(i, t) = e^{-\int_i^t a(m)dm} \left(1 - \int_i^t b(s) e^{\int_i^s a(r)dr} ds \right), \quad t > i, \ i = 0, 1, 2, \dots.$$

Theorem 2.1. The solution of $(1, 2)$ is well defined for all $t \geqslant 0$ and is given by

$$T(t) = T_0 \prod_{i=0}^{n-1} \Big(M(i, i+1) \Big) M(n, t) \quad \text{for } t \in [n, n+1), \quad n = 0, 1, 2, \dots. \tag{3}$$

Proof. The solution of $(1, 2)$ in $t \in [0, 1)$ is

$$T(t) = T_0 e^{-\int_0^t a(s)ds} \left(1 - \int_0^t b(s) e^{\int_0^s a(r)dr} ds \right) = T_0 M(0, t) \quad \text{for } t \in [0, 1).$$

Then

$$T(1) = \lim_{t \to 1-0} T(t) = T_0 M(0, 1).$$

The solution of $(1, 2)$ in $t \in [1, 2)$ has the form

$$T(t) = T(1) e^{-\int_1^t a(s)ds} \left(1 - \int_1^t b(s) e^{\int_1^s a(r)dr} ds \right)$$

or

$$T(t) = T_0 M(0, 1) M(1, t) \quad \text{for } t \in [1, 2).$$

Let the function

$$T(t) = T_0 \prod_{i=0}^{k-2} \Big(M(i, i+1) \Big) M(k-1, t) \quad \text{for } t \in [k-1, k), \ k = 3, 4, \dots$$

be a solution of $(1, 2)$ in $[k-1, k)$, then the function

$$T(t) = T_0 \prod_{i=0}^{k-1} \Big(M(i, i+1) \Big) M(k, t) \quad \text{for } t \in [k, k+1), \ k = 3, 4, \dots. \tag{4}$$

is a solution of $(1, 2)$ in $t \in [k, k+1)$. $\qquad \square$

3. On periodic solutions

Theorem 3.1. Let $a(t)$ and $b(t)$ be n-periodic continuous functions. Then the solution for $(1, 2)$ is n-periodic iff

$$\prod_{i=0}^{n-1} M(i, i+1) = 1.$$

Proof. Let $T(t)$ be a n-periodic solution for $(1, 2)$. Then $T(n) = T_0$. As $t = n$, by (3), one obtains $1 = \prod_{i=0}^{n-1} \Big(M(i, i+1)\Big)$.

Conversely, let $\prod_{i=0}^{n-1} M(i, i+1) = 1$. We show that $T(n+t) = T(t)$ for all $t \in \mathbb{R}_+$. Let $t + n \in [n+k, n+k+1)$, where k is an integer number. Then

$$T(t+n) = T_0 \prod_{i=0}^{n+k-1} M(i, i+1) \, M(n+k, t+n) \quad \text{for } t + n \in [n+k, n+k+1).$$

Changing the variables $r' = \rho + n$, $s = s' + n$ in the integral

$$M(n+k, t+n) = e^{-\int_{n+k}^{t+n} a(r')dr'} \left(1 - \int_{n+k}^{t+n} b(s)e^{\int_{n+k}^{s} a(r)dr}ds\right)$$

we obtain

$$M(n+k, t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \left(1 - \int_{k}^{t} b(s')e^{\int_{n+k}^{s'+n} a(r)dr}ds'\right).$$

Then, changing variable $r = r' + n$ gives one

$$M(n+k, t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \left(1 - \int_{k}^{t} b(s')e^{\int_{k}^{s'} a(r')dr'}ds'\right),$$

i.e. $M(n+k, t+n) = M(k, t)$ for all $t + n \in [n+k, n+k+1)$. Using this equation, we provide the following calculations

$$\prod_{i=0}^{n+k-1} M(i, i+1) = \prod_{i=0}^{n-1} M(i, i+1) \prod_{i=n}^{n+k-1} M(i, i+1) = \prod_{i=n}^{n+k-1} M(i, i+1)$$

$$= \prod_{j=0}^{k-1} M(j+n, j+n+1) = \prod_{j=0}^{k-1} M(j, j+1).$$

Therefore

$$T(t+n) = T_0 \prod_{i=0}^{n+k-1} M(i, i+1) \, M(n+k, t+n)$$

$$= T_0 \prod_{i=0}^{k-1} M(i, i+1) \, M(k, t) = T(t) \quad \text{for } t \in [k, k+1).$$

$\square$

Example 3.1. Let $a(t) = \pi^2$ and $b(t) = \beta \sin 2\pi t$. For this case, the function $M(i, t)$ is represented as

$$M(i, t) = e^{-\pi^2(t-i)} - \frac{\beta e^i}{\pi^2(1 + 4\pi^2)}\left(e^{-\pi^2(t-i)} + \sin 2\pi t - \cos 2\pi t\right) \quad \text{for } t > i.$$

(a) Let β be a root of

$$M(0, 1)M(1, 2) = 1.$$

Then, the solution for $(1, 2)$ is unique 2-periodic solution defined as

$$T(t) = \begin{cases} T_0 M(0, t), & t \in [0, 1), \\ T_0 M(0, 1)M(1, t), & t \in [1, 2). \end{cases}$$

(b) Let β be a root of
$$M(0,1)M(1,2)M(2,3) = 1.$$
Then, the solution for $(1,2)$ is unique 3-periodic solution defined as
$$T(t) = \begin{cases} T_0 M(0,t), & t \in [0,1), \\ T_0 M(0,1)M(1,t), & t \in [1,2), \\ T_0 M(0,1)M(1,2)M(2,t), & t \in [2,3). \end{cases}$$

Theorem 3.2. Let $a(t)$ and $b(t)$ be n-periodic continuous functions and
$$\prod_{i=0}^{n-1} M(i,i+1) = 1.$$
Then, the problem $(1,2)$ has positive solution if any one if the following hypotheses holds true:
(i) $b(t) \leqslant 0$; or
(ii) $b(t) \geqslant 0$ and
$$\max_{0 \leqslant t \leqslant n} b(t) e^{\int_i^t a(r)dr} < 1.$$

Proof. Let $T(t)$ be a solution of the problem. Then, by Theorem 3.1, $T(t)$ is n-periodic and by Theorem 2.1, it has the form
$$T(t) = T_0 \prod_{i=0}^{k-1} M(i,i+1) \, M(k,t) \quad \text{for } t \in [k,k+1), k = 0,1,...,n-1.$$

(i) If $b(t) \geqslant 0$ then $1 - \int_i^t b(s) e^{\int_i^s a(r)dr} ds) > 0, \quad t > i, \ i = 0,1,...n-1$. Hence, $T(t) > 0, t \in [0,n]$.

(ii) If $b(t) \geqslant 0$ and
$$\max_{0 \leqslant t \leqslant n} b(t) e^{\int_i^t a(r)dr} < 1.$$
Then,
$$\int_i^t b(s) e^{\int_i^s a(r)dr} ds) < \max_{0 \leqslant t \leqslant n} b(t) e^{\int_i^t a(r)dr} (t-i) < 1 \quad \text{for } t \in [i,i+1), i = 0,1,...n-1.$$
This leads to inequality $M(i,t) > 0$ for $t > i, i = 0,1,...n-1$. $\qquad \square$

The case $a(t)$ and $b(t)$ are constant

We analyze periodic solutions of $(1),(2)$ for the case when the functions $a(t)$ and $b(t)$ are constant functions, i.e. $a(t) = a$ and $b(t) = b$. In this case the solution (3) has the form
$$T(t) = T_0 \left(e^{-a} - \frac{b}{a}(1 - e^{-a}) \right)^n \left(e^{-a(t-n)} - \frac{b}{a}(1 - e^{-a(t-n)}) \right), \quad t \in [n, n+1). \tag{5}$$
Note that for this case if $b = -a$ then the Cauchy problem $(1),(2)$ has only constant solution $T(t) = T_0$.

Theorem 3.3. Let $b = \dfrac{(e^a + 1)a}{e^a - 1}$. Then, the Cauchy problem $(1),(2)$ has unique 2-periodic solution
$$T(t) = \begin{cases} T_0 \left(e^{-at} - \frac{b}{a}(1 - e^{-at}) \right), & t \in [0,1), \\ -T_0 \left(e^{-a(t-1)} - \frac{b}{a}(1 - e^{-a(t-1)}) \right), & t \in [1,2). \end{cases}$$

Proof. Let $T(t)$ be a solution of $(1),(2)$ defined as (5). We show that $T(t) = T(t+2)$ for $t \in [0,\infty)$. Note that $e^{-a} - \frac{b}{a}(1 - e^{-a}) = -1$ for $b = \dfrac{(e^a + 1)a}{e^a - 1}$. Hence, by (5), one has
$$T(t) = T_0(-1)^n \left(e^{-a(t-n)} - \frac{b}{a}(1 - e^{-a(t-n)}) \right), \quad t \in [n, n+1).$$
Hence,
$$T(t+2) = T_0(-1)^{n+2} \left(e^{-a(t-n)} - \frac{b}{a}(1 - e^{-a(t-n)}) \right), \quad t+2 \in [n+2, n+3),$$
i.e. $T(t) = T(t+2)$. $\qquad \square$

Remark that equation $(1,2)$ has no non-integer w-periodic solution. However, if the solution $T(t)$ of this equation is periodic then $T(t)$ is constant function or 2-periodic function.

4. Application

In this section, we consider the partial differential equation (PDE) with piecewise constant arguments

$$u_t(x,t) = a^2(t)u_{xx}(x,t) - b(t)u(x,[t]), \tag{6}$$

$$u(0,t) = u(1,t) = 0, \tag{7}$$

$$u(x,0) = v(x), \tag{8}$$

where v is a continuous function on $[0,1]$, $a(t)$ and $b(t)$ are n-periodic functions on $[0,\infty)$.

It has been shown in [12] that PDEs (6-8) with piecewise constant time naturally arise in the process of approximating PDEs using piecewise constant arguments. It is important to investigate boundary value problems (BVP) and initial-value problems for EPCA in partial derivatives and explore the influence of certain discontinuous delays on the behavior of solutions to some typical problems of mathematical physics. For example, the measuring of the lateral heat change at discrete moments of time leads to the equation with piecewise continuous delay [11].

Definition 4.1. A function $u(x,t)$ is called a solution of (6-8) if the following conditions are satisfied:

(i) $u(x,t)$ is continuous on $\Omega = [0,1] \times \mathbb{R}_+$, $\mathbb{R}_+ = [0,\infty)$;

(ii) u_t and u_{xx} exist and are continuous in Ω, with possible exception at points $(x,[t]) \in \Omega$, where one-sided derivatives exist with respect to second argument;

(iii) $u(x,t)$ satisfies Eq. (6) in Ω, with the possible exception at the points $(x,[t]) \in \Omega$ and conditions (7) and (8).

Definition 4.2. (see [7]). If any solution $u(x,t)$ of (6) satisfies

$$\lim_{t \to \infty} u(x,t) = 0, \quad x \in [0,1],$$

then the zero solution of (6) is called asymptotically stable.

By using separation of variables, the formal solution $u(x,t)$ in (6-8) can be represented as (see [9])

$$u(x,t) = \sum_{j=1}^{\infty} T_j(t)\sin(j\pi x), \tag{9}$$

where $T_j(t)$ is the solution of the initial equation

$$T_j'(t) = -a^2(t)\pi^2 j^2 T_j(t) - b(t)T_j([t]),$$
$$T_j(0) = v_j. \tag{10}$$

Here

$$v_j = 2\int_0^1 v(x)\sin(j\pi x)dx,$$

$$v(x) = u(x,0) = \sum_{j=1}^{\infty} v_j \sin(j\pi x).$$

Note that by Theorem 3.1, if $\prod_{i=0}^{n-1} M_j(i,i+1) = 1$ for some $j \in \mathbb{N}$, where

$$M_j(i,t) = e^{-j^2\pi^2 \int_i^t a^2(m)dm}\left(1 - \int_i^t b(s)e^{j^2\pi^2 \int_i^s a^2(r)dr}ds\right),$$

(10) has n-periodic solution

$$T_j(t) = T_j(0)\prod_{i=0}^{n-1}\Big(M_j(i,i+1)\Big)M_j(n,t), \quad t \in [n,n+1).$$

Since $T_j(t)$ is a periodic function, we can conclude the following theorem.

Theorem 4.1. Let $a(t)$ and $b(t)$ be n-periodic continuous functions. Assume that there exists $j \in \mathbb{N}$ such that

$$\prod_{i=0}^{n-1} M_j(i,i+1) = 1.$$

Then, the zero solution of (6) is not asymptotically stable.

Example 4.1. Let $a(t) = \sin 2\pi t$ and $b(t) = \beta \cos 2\pi t$. Consider the problem

$$u_t(x,t) = a^2(t)u_{xx}(x,t) - b(t)u(x,[t]),$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = \sin(\pi x).$$

The solution of this problem is as follows

$$u(x,t) = T_1(t)\sin \pi x,$$

where $T_1(t)$ is a solution of

$$T_1'(t) = -a^2(t)\pi^2 T_1(t) - b(t)T_1([t]),$$
$$T_1(0) = 1.$$

By Theorem 2.1, one has

$$T_1(t) = \prod_{i=0}^{n-1}\Big(M_1(i,i+1)\Big)M_1(n,t), \quad t \in [n,n+1),$$

where

$$M_1(i,t) = e^{-\pi^2\left(\frac{t-i}{2} - \frac{1}{8\pi}(\sin 4\pi t - \sin 4\pi i)\right)}\left(1 - \beta\int_i^t \cos 2\pi s\, e^{\pi^2\left(\frac{s-i}{2} - \frac{1}{8\pi}(\sin 4\pi s - \sin 4\pi i)\right)}ds\right), \quad t > i.$$

Let β be a root of

$$M_1(0,1)M_1(1,2)M_1(2,3) = 1.$$

Then, the function $T_1(t)$ is a 3-periodic function defined as

$$T_1(t) = \begin{cases} M(0,t), & t \in [0,1), \\ M(0,1)M(1,t), & t \in [1,2), \\ M(0,1)M(1,2)M(2,t), & t \in [2,3). \end{cases}$$

Moreover, the solution $u(x,t)$ is not asymptotically stable, i. e.

$$\lim_{t\to\infty} u(x,t) \neq 0, x \in [0,1].$$

Example 4.2. Let $a(t) = 1$ and $b(t) = \beta \sin 2\pi t$. Consider the problem

$$u_t(x,t) = a^2(t)u_{xx}(x,t) - b(t)u(x,[t]),$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = \sum_{j=1}^5 \frac{1}{j}\sin(j\pi x).$$

The solution of this problem is as follows

$$u(x,t) = \sum_{j=1}^5 T_j(t)\sin j\pi x,$$

where $T_j(t)$ is a solution of

$$T_j'(t) = -a^2(t)\pi^2 j^2 T_j(t) - b(t)T_j([t]),$$
$$T_j(0) = \frac{1}{j} \tag{11}$$

defined as

$$T_j(t) = \frac{1}{j}\prod_{i=0}^{n-1}\Big(M_j(i,i+1)\Big)M_j(n,t), \quad t \in [n,n+1),$$

$$M_j(i,t) = e^{-\pi^2 j^2(t-i)}\left(1 - \frac{\beta e^i}{\pi^2 j^2(1+4\pi^2)}\left(1 + e^{\pi^2 j^2(t-i)}(\sin 2\pi t - \cos 2\pi t)\right)\right), \quad t > i.$$

Let β be a root of

$$M_1(0,1)M_1(1,2) = 1.$$

Then, $T_1(t)$ is a 2-periodic function defined as (See the Example 3.1 (a))

$$T_1(t) = \begin{cases} M_1(0,t), & t \in [0,1), \\ M_1(0,1)M_1(1,t), & t \in [1,2). \end{cases}$$

5. Conclusion

In this note, we studied periodic solutions to the initial value problem (1, 2), where nonzero functions $a(t)$, $b(t)$ are assumed to be n-periodic and continuous on $\mathbb{R}_+ = [0, \infty)$.

The condition for possibility of the periodic solutions of the problem and explicit formula for the solutions are obtained. The positivity conditions for the periodic solutions is described, which claims non-oscillatory character for the solution (See [16]). Remark that the obtained results are true for the case when $a(t)$, $b(t)$ are one periodic functions, or satisfied the conditions $a(t) = a(t + n)$, $b(t) = b(t + n)$. As an example, the example is constructed when $a(t)$, $b(t)$ are one-periodic, the equation have 2-periodic or 3-periodic solutions .

As an application of the obtained result, it is given a non-asymptotically stable condition for solution a certain class of partial differential equations with a piecewise constant argument, which claims the correctness of the conditions of asymptotically stable conditions for the solution given in [7].

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