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Original article

Phase transitions for the “uncle–nephew” model

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ABSTRACT We investigate a problem of phase transition for all possible phases of “uncle–nephew” model on the semi-infinite Cayley tree of second order. It is proved that one can reach the phase transition for this model only in the class of ferromagnetic phase.

KEYWORDS Cayley tree, Ising model, phase transition, periodic Gibbs measures, family tree.

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1. Introduction

Lattice models with competing interactions on the Cayley tree have recently been considered extensively because of the appearance of non-trivial magnetic orderings (see [1–7]). The important point is that statistical mechanics on trees involve non-linear recursion equations which are naturally connected to the rich world of dynamical systems [1].

The Ising model with competing “uncle–nephew” interactions was introduced in [8] as a model on the Cayley tree with competing interactions up to the third nearest-neighbours with spins belonging to different branches of the tree when the Cayley tree is considered as family tree. In this paper, it has been proved that phase diagram of this model consists of four phases, namely, expected ferromagnetic, anti-ferromagnetic, and new paramagnetic and para-modulated phases. Later on in [9], the problem of phase transition in the class of ferromagnetic phases has been studied.

In this paper, we will discuss the problem of phase transition for the Ising model with competing “uncle–nephew” interactions in the classes of anti-ferromagnetic, paramagnetic and para-modulated phases.

2. Preliminary

Below, we consider a semi-infinite Cayley tree $\Gamma_+^2 = (V, \Lambda)$ of the second order, i.e. an infinite graph without cycles with three edges issuing from each vertex except for x^0 which has only two edges, where V is the set of its vertices and Λ is the set of edges.

Two vertices $x, y \in V$ are called nearest-neighbours if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l =< x, y >$. The distance $d(x, y)$ $x, y \in V$ on the Cayley tree is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \{x \in V | d(x, x^0) \leqslant n\},$$

and Λ_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level.

Two vertices $x, y \in V$ are called *third-nearest-neighbors* if $d(x, y) = 3$. The third-nearest-neighbor vertices x and y are called *prolonged third-nearest-neighbors* and is denoted by $< \tilde{x,y} >$ if $x \in W_n$ and $y \in W_{n+3}$ for some n, i.e. they belong to the same branch of the tree.

The third-nearest-neighbor vertices $x, y \in V$, that are not prolonged, are called *two-level third-nearest-neighbours* and are denoted by $< \tilde{x,y} >$. In this case vertices x and y belong to the different branches of the tree.

Considering a semi-infinite Cayley tree Γ_+^2 of second order as a *genealogical* or *family* tree, we can reinterpret neighboring relations as a contiguous relations. If $< x, y >$ are nearest neighbors with $x \in W_n$ and $y \in W_{n+1}$ for some n, a vertex x is called the parent of vertex y, and y is called a child of x. If $< \tilde{x,y} >$ are prolonged third-nearest-neighbors, where $x \in W_n$ and $y \in W_{n+3}$ for some n, then a vertex x is called the great grandparent of vertex y and y is called a great grandchild of x respectively. If $< \tilde{x,y} >_l$ and $< \tilde{x,y} >_r$ are left and right two-level third-nearest-neighbors, then a vertex x is called the *uncle* of vertex y and y is called a *nephew* of x, respectively.

The uncle–nephew Ising model. A model with the following Hamiltonian

$$H(\sigma) = -J_1 \sum_{<x,y>} \sigma(x)\sigma(y) - J_3 \sum_{<\tilde{x,y}>} \sigma(x)\sigma(y) \tag{1}$$

is called the uncle–nephew Ising model (see [8]), where the sum in the first term ranges all nearest-neighbours, the second sum ranges all second-level third-nearest-neighbours. Here $J_1, J_3 \in \mathbb{R}$ are coupling constants.

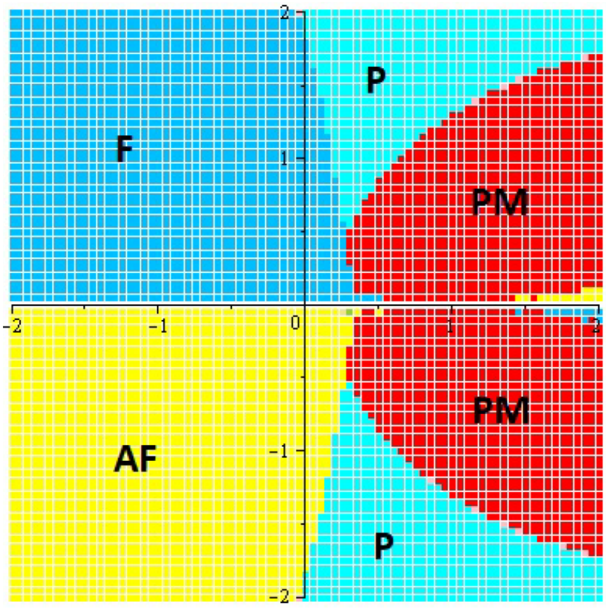

FIG. 1. Phase diagram of the uncle–nephew Ising model

Assume that $a = \exp(2J_1/(k_BT))$, $b = \exp(2J_3/(k_BT))$, where k_B is the Boltzmann constant and T is the absolute temperature.

In [8], the following recurrent relations have been produced by renormalizing

$$\begin{aligned} x' &= \frac{1}{aD}\left[(1 + a^{-2}b^2 + 2bx)^2 - (1 - a^{-2}b^2)^2y^2\right], \\ y' &= \frac{2}{D}(b^2 - a^{-2})(2bx + b^2 + a^{-2})y, \end{aligned} \tag{2}$$

where $D = (2bx + b^2 + a^{-2})^2 + (b^2 - a^{-2})^2y^2$. This recursion relations provide us a numerically exact phase diagram in (α, γ) space, where $\alpha = (k_BT)/J_1$ is on $y-$axis and $\gamma = J_3/J_1$ is on $x-$axis, with $a = \exp(2\alpha^{-1})$ and $b = \exp(2\alpha^{-1}\gamma)$. To produce the phase diagram, one iterates the recurrence relations (2) starting with random initial conditions and looks after their behaviour after a large number of the iterations. In order to investigate the phase diagram, we have to look at local properties, namely, the local magnetization of the root x^0. Then, the average magnetization $m^{(n)}$ for the n-th generation is given by

$$m^{(n)} = \frac{(a-1)(a+1)y^{(n)}}{a^2((1 + 2x^{(n)} + 1}. \tag{3}$$

The resultant phase diagram on the plane (γ, α) is shown in Fig. 1. The diagram consists of four phases, paramagnetic, ferromagnetic, para-modulated with period $p = 2$ and anti-ferromagnetic. That is, for the dynamical system described by equation (2), some of the trajectories are converging, and other trajectories have a cycle of second order. It is evident that one can reach ferromagnetic phase if $J_1 > 0$, $J_3 > 0$, anti-ferromagnetic phase if $J_1 < 0$, $J_3 < 0$, and paramagnetic or para-modulated phases if $J_1J_3 < 0$.

In [9], the problem of phase transition in the class of ferromagnetic phases has been solved, namely the following statement was proved.

Theorem 1. If $1 < a^2b^2 \leq 3$, then there exists a unique translation-invariant Gibbs measure, and if $a^2b^2 > 3$ then there exist three translation-invariant Gibbs measures.

Below we consider the problem of phase transition for this model in the classes of anti-ferromagnetic phases, paramagnetic and para-modulated phases.

3. Functional equations

In [8], the following recurrent equation $t' = f(t)$ has been obtained to investigate the problem of phase transition. Here

$$f(t) = \left(\frac{a^2b^2t + 1}{t + a^2b^2}\right)^2, \tag{4}$$

Where $a = \exp(2\beta J_1)$, $b = \exp(2\beta J_3)$ and $\beta = 1/k_BT$ is the inverse temperature. One can find all details in [9].

To investigate the problem of phase transition in the class of ferromagnetic phases, we have to describe the fixed points of the map $t' = f(t)$ (see [9]). Thus, to investigate the same problem in the class of anti-ferromagnetic phases, one should investigate the recurrent equation $t'' = f(f(t))$ that is, the following equation

$$t'' = \left(\frac{a^2b^2(a^2b^2t+1)^2 + (t+a^2b^2)^2}{(a^2b^2t+1)^2 + a^2b^2(t+a^2b^2)^2}\right)^2. \tag{5}$$

Then, to find the fixed points of (5), we consider the following algebraic equation

$$\begin{aligned} &4a^4b^4t^5 - (a^8b^8 - a^4b^4 - 2a^6b^6 - 2a^2b^2 + 1)t^4 + (2a^6b^6 - 6a^4b^4 + 2a^2b^2)t^3 \\ +\quad &(2a^6b^6 - 6a^4b^4 + 2a^2b^2)t^2 + (a^8b^8 - a^4b^4 - 2a^6b^6 - 2a^2b^2 + 1)t - a^4b^4 = 0. \end{aligned}$$

After factorization, this equation is reduced to the following form:

$$(t-1)[t^2 - (a^4b^4 - 2a^2b^2 - 1)t + 1][a^4b^4t^2 + (a^4b^4 + 2a^2b^2 - 1)t + a^4b^4] = 0. \tag{6}$$

One can see that the solutions of equation $(t-1)[t^2 - (a^4b^4 - 2a^2b^2 - 1)t + 1] = 0$ are the fixed points of the recursive equation

$$t' = \left(\frac{a^2b^2t+1}{t+a^2b^2}\right)^2. \tag{7}$$

Since, the fixed points of (7) describe the translation-invariant (that is, ferromagnetic phases). Only solutions of the quadratic equation

$$a^4b^4t^2 + (a^4b^4 + 2a^2b^2 - 1)t + a^4b^4 = 0 \tag{8}$$

describe the anti-ferromagnetic phases.

The discriminant of this equation is equal to $D = (a^4b^4 + 2a^2b^2 - 1)^2 - 4a^8b^8$. If $D > 0$ and $a^4b^4 + 2a^2b^2 - 1 < 0$ then equation (8) has two positive solutions $\tilde{t_1}$ and $\tilde{t_2}$ such that $\tilde{t_2}' = \tilde{t_1}$ and $\tilde{t_1}' = \tilde{t_2}$. The conditions $D > 0$ and $a^4b^4 + 2a^2b^2 - 1 < 0$ is equivalent to the $a^2b^2 < \frac{1}{3}$. Thus, we have proved the following theorem:

Theorem 2. If $a^2b^2 < \frac{1}{3}$, then there exist two anti-ferromagnetic phases, i.e. periodic phases with period 2.

4. Paramagnetic phases

In [8], it was shown that we reach the paramagnetic phase if $J_1J_3 < 0$ and it is described by the following recurrent equation

$$x' = \frac{1}{a}\left(\frac{1 + a^{-2}b^2 + 2bx}{2bx + a^{-2} + b^2}\right)^2. \tag{9}$$

To describe fixed points of (9), we have to solve the following equation

$$ax = \left(\frac{1 + a^{-2}b^2 + 2bx}{2bx + a^{-2} + b^2}\right)^2. \tag{10}$$

Let us assume that

$$f(x) = \left(\frac{1 + a^{-2}b^2 + 2bx}{2bx + a^{-2} + b^2}\right)^2. \tag{11}$$

Then

$$f'(x) = \frac{4b(a^{-2}-1)(1-b^2)(1 + a^{-2}b^2 + 2bx)}{[2bx + a^{-2} + b^2]^3}. \tag{12}$$

If $J_1J_3 < 0$ then $(a^{-2}-1)(1-b^2) < 0$, thus, $f'(x) < 0, f(0) = \frac{1 + a^{-2}b^2}{a^{-2} + b^2} > 1$ and $\lim_{x\to\infty} f(x) = 1$. Therefore, equation (10) has single root. Thus, we have proved.

Theorem 3. On the set of paramagnetic phases, phase transition does not occur.

5. Para-modulated phases

It was shown in [8] that one reaches the para-modulated phases if $J_1J_3 < 0$. As shown above, the paramagnetic phases are described by the following equation

$$x' = \frac{1}{a}\left(\frac{1 + a^{-2}b^2 + 2bx}{2bx + a^{-2} + b^2}\right)^2. \tag{13}$$

For brevity, let us introduce notations $A = a^{-2}b + b^{-1}$ and $B = a^{-2}b^{-1} + b$. Then equation (13) can be rewritten as follows

$$x' = \frac{1}{a}\left(\frac{A + 2x}{2x + B}\right)^2. \tag{14}$$

Then, modulated and para-modulated phases are described by the following recurrent equation

$$x'' = \frac{1}{a}\left[\frac{4(A+2)x^2+4A(B+2)x+AB^2+2A^2}{4(B+2)x^2+4(B^2+2A)x+B^3+2A^2}\right]^2. \tag{15}$$

Let us consider the fixed points

$$x = \frac{1}{a}\left[\frac{4(A+2)x^2+4A(B+2)x+AB^2+2A^2}{4(B+2)x^2+4(B^2+2A)x+B^3+2A^2}\right]^2 \tag{16}$$

for paramagnetic and para-modulated phases and the fixed points

$$x = \frac{1}{a}\left(\frac{A+2x}{2x+B}\right)^2 \tag{17}$$

for paramagnetic phases. To find fixed points for para-modulated phases we rewrite equations (16) and (17) as polynomials of fifth and third degree, divide the first one by the second one and obtain the following quadratic equation.

$$4(B+2)^2x^2 - 16[A^2+3A-(2-a)B^3-(5-4a)B^2-4B]x+(B+2A)^2 = 0. \tag{18}$$

This equation has two positive roots if

$$A^2+3A-(2-a)B^3-(5-4a)B^2-4B > 0$$

and its discriminant D is positive. It is evident that $D > 0$ if

$$A^2-A-(2-a)B^3-(6-4a)B^2-2AB-6B > 0. \tag{19}$$

As shown in Fig. 1, one can reach a para-modulated phase if $-J_3/J_1 \geq 1/3$ and $T/J_1 < 2$, that is $a > e$ and $b^3 < e^{-1}$. Thus, one can rewrite this inequality as

$$A^2-A+(a-2)B^3+(4a-6)B^2-2AB-6B > 0. \tag{20}$$

Substituting values of A and B, one comes to the following form of the inequality

$$\begin{aligned} &a^6(a-2)b^6+a^2(4a^5-6a^4-2a^2+1)b^5-a^4(6a^2-3a+14)b^4-2a^2(a^4+5a^2+2)b^3 \\ -\ &a^2(a^4-8a^3+6a^2+6)b^2+a^2(a^4-2a^2+4a-6)b+a-2 > 0. \end{aligned}$$

Correspondingly, the paramagnetic- para-modulated transition is given by the following condition

$$A^2-A+(a-2)B^3+(4a-6)B^2-2AB-6B = 0. \tag{21}$$

Here $a = \varphi(b)$ with $a = \exp(J_1/T)$, $b = \exp(J_3/T)$. If $A^2-A+(a-2)B^3+(4a-6)B^2-2AB-6B > 0$ then equation (21) has two positive solutions $\tilde{x_1}$ and $\tilde{x_2}$ such that $\tilde{x_2}' = \tilde{x_1}$ and $\tilde{x_1}' = \tilde{x_2}$. Thus, we have proved the following theorem:

Theorem 4. If $A^2-A+(a-2)B^3+(4a-6)B^2-2AB-6B > 0$ then there exist two para-modulated phases with period 2.

6. Conclusion

In this paper, we discussed the problem of phase transition for the Ising model with competing "uncle–nephew" interactions. In the previous paper [8], for $J_1 > 0$, $J_3 > 0$, it was proved that there exists phase transition in the class of ferromagnetic phases.

In this paper, we proved that for $J_1 < 0$, $J_3 < 0$, a phase transition occurs in the class of anti-ferromagnetic phases.

Also we investigated the problem of phase transition for remaining two classes of phases – paramagnetic and para-modulated phases. We proved that in the class of paramagnetic phase there is no phase transition. As for the class of para-modulated phases, it was shown that phase transition can occur. Thus, we investigated the problem of phase transition for all phase classes of the considered model.

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Conflict of interest: the authors declare no conflict of interest.