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Original article

On eigenvalues and virtual levels of a two-particle Hamiltonian on a d-dimensional lattice

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ABSTRACT The two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^d$ (where $\mu > 0$, $\mathbb{T}^d$ is a d-dimensional torus), associated to the Hamiltonian h of the system of two quantum particles moving on a d-dimensional lattice, is considered as a perturbation of free Hamiltonian $h_0(k)$ by the certain 3^d rank potential operator $\mu\mathbf{v}$. The existence conditions of eigenvalues and virtual levels of $h_\mu(k)$, are investigated in detail with respect to the particle interaction μ and total quasi-momentum $k \in \mathbb{T}^d$.

KEYWORDS two-particle Hamiltonian, invariant subspace, orthogonal projector, eigenvalue, virtual level, multiplicity of virtual level.

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1. Introduction

In quantum modeling of interacting many-body systems for the manipulation of ultracold atoms and unique setting, coherent optical fields provide a strong tool because of their high-degree controllable parameters such as optical lattice geometry, dimension, particle mass, two-body potentials, temperature etc. (See [1–4]). The recent experimental and theoretical results show that integrating plasmonic systems with cold atoms, using optical potential fields formed from the near field scattering of light by an array of plasmonic nanoparticles, allows one to considerably increase the energy scales in the realization of Hubbard models and engineer effective long-range interaction in many body dynamics [5–7]. A several of numerical results for the bound state energies of one and two-particle systems in two adjacent 3D layers, connected through a window were presented in [8] and investigated the relation between the shape of a window and energy levels, as well as the number of eigenfunction's nodal domains. In papers [9] and [10], some spectral properties of the discrete Schrödinger operator with zero-range and short range attractive potentials, respectively, were studied.

In general, the Schrödinger operator $h(k)$, $k \in \mathbb{T}^d$, associated with the lattice Hamiltonian h of two arbitrary particles with some dispersion relation and short range potential interaction acts in $L_2(\mathbb{T}^d)$ as [11]

$$h(k) = h_0(k) - \mathbf{v}, \quad k \in \mathbb{T}^d,$$

where $h_0(k)$ is a multiplication operator by $\mathcal{E}_k(p) = \dfrac{1}{m_1}\varepsilon(p) + \dfrac{1}{m_2}\varepsilon(p-k)$ and $\mathbf{v}$ is the integral operator with kernel $v(p,s) = v(p-s)$. In [8], several numerical results for the bound state energies of one and two-particle systems were presented in two adjacent 3D layers, connected through a window. The authors investigated the relation between the shape of the window and the energy levels, as well as the number of eigenfunction's nodal domains.

In [12], the existence conditions and positiveness of eigenvalues of the Schrödinger operator $h(k)$, $k \in \mathbb{T}^d$, were studied with respect to the quasi-momentum k and the virtual level at the lower edge of the essential spectrum.

The existence and absence of eigenvalues of the family $h(k)$ depending on the energy of interaction and quasi-momentum k were investigated in [13] and [14] for the cases $\varepsilon(p) = \sum\limits_{i=1}^{3}(1-\cos 2p_i)$, $v(p-s) = \sum\limits_{\alpha=1}^{3}\mu_\alpha \cos(p_\alpha - s_\alpha)$ and $\varepsilon(p) = \sum\limits_{i=1}^{3}(1-\cos 2np_i)$, $v(p-q) = \sum\limits_{l=1}^{N}\sum\limits_{i=1}^{3}\mu_{li}\cos l(p_i - q_i)$, respectively. The spectral properties of this operator $h(k)$ for the one dimensional case were studied in [15] and more general case in [16]. For general case $\varepsilon(p)$ satisfying some conditions and $v(p-s) = \mu_0 + \sum\limits_{\alpha=1}^{d}\mu_\alpha\cos(p_\alpha - q_\alpha)$ was investigated in [17]. Detailed spectral properties of the

Hamiltonian $\hat{h}_{\mu\lambda}$, $\mu, \lambda \geqslant 0$, describing the motion of one quantum particle on a three-dimensional lattice in an external field and more general case were investigated in the papers [18] and [19], respectively. In [20] a class of potentials is found for which the discrete spectrum of the two-particle Schrödinger operator $h(k)$ is preserved when $h(k)$ is perturbed by a potential from this class.

In the recent paper [21], a two particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^3$, $\mu > 0$, associated to the Hamiltonian h on the three-dimensional lattice is considered. The authors investigated existence conditions of eigenvalues and bound states of $h_\mu(k)$. The investigation is based on the construction of invariant subspaces for the operator $h_\mu(k)$ which allow one to study the compact perturbations of rank one.

This paper is a continuation of the work [21]. We consider a two particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^3$, $\mu > 0$, associated with the Hamiltonian h for a system of two particles on the d-dimensional lattice $\mathbb{Z}^d$ interacting through *attractive short-range potential V*. We investigate the existence conditions of eigenvalues and virtual levels of the two-particle Schrödinger operator $h_\mu(k)$, where $h_\mu(k)$ is considered as a perturbation of free Hamiltonian $h_0(k)$ by the certain potential operator $\mu\mathbf{v}$ with rank 3^d. The main idea of the investigation is to represent $\mu\mathbf{v}$ via the sum of one rank orthogonal projectors $\mu\mathbf{v}_l$. This allows one to represent the corresponding Birman–Schwinger operator $\mathbf{T}(\mu, k; z)$ via the sum of the one rank projectors $\mathbf{T}_l(\mu, k; z)$, $l = 1, 2, \ldots, 3^d$. Moreover, the study of spectral properties of $h_\mu(k)$ reduces to the investigation of 3^d one rank operators $\mathbf{T}_l(\mu, k; z)$. The virtual level of $h_\mu(k)$ is studied as $k = \mathbf{0}$.

2. Statement of the main result

A two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^d$, $\mu > 0$, associated to the Hamiltonian h for a system of two particles on the lattice $\mathbb{Z}^d$ interacting via attractive short-range potential, is a self-adjoint operator and acts in $L_2(\mathbb{T}^d)$ as

$$h_\mu(k) = h_0(k) - \mu\mathbf{v}, \quad k = (k_1, k_2, \ldots, k_d) \in \mathbb{T}^d, \quad \mu > 0,$$

where $h_0(k)$ is a multiplication operator by

$$\mathcal{E}_k(p) = \frac{1}{m_1}\varepsilon(p) + \frac{1}{m_2}\varepsilon(p-k), \;\; \varepsilon(p) = \sum_{i=1}^{d}(1 - \cos 2p_i),$$

with $\mathbf{v}$ being an integral operator with kernel

$$v(p-s) = 1 + \sum_{\alpha=1}^{d}\cos(p_\alpha - s_\alpha) + \sum_{\gamma=1}^{d}\cos(p_\alpha - s_\alpha)\cos(p_\beta - s_\beta) + \cdots + \prod_{\alpha=1}^{d}\cos(p_\alpha - s_\alpha),$$

$\alpha, \beta, \gamma \in \{1, 2, \ldots, d\}$, $\alpha < \beta < \gamma < \alpha$.

Note that by the Weyl theorem [22] the essential spectrum $\sigma_{ess}(h_\mu(k))$ of the operator $h_\mu(k)$ coincides with the spectrum of the unperturbed operator $h_0(k)$

$$\sigma_{ess}(h_\mu(k)) = \sigma(h_0(k)) = [m(k), M(k)],$$

where $m(k) = \min\limits_{p\in\mathbb{T}^d} \mathcal{E}_k(p), M(k) = \max\limits_{p\in\mathbb{T}^d} \mathcal{E}_k(p)$.

Since $\mathbf{v} \geqslant 0$ for $\mu > 0$,

$$\sup_{\|f\|=1} (h_\mu(k)f, f) \leqslant \sup_{\|f\|=1} (h_0(k)f, f) = M(k)(f, f), \qquad f \in L_2(\mathbb{T}^d).$$

Hence, $h_\mu(k)$ does not have eigenvalues lying to the right of the essential spectrum, i.e.,

$$\sigma(h_\mu(k)) \cap (M(k), +\infty) = \varnothing.$$

Let $\{\varphi_l\}$ be the orthogonal system in $L_2(\mathbb{T}^d)$, where φ_l is defined as

$$\varphi_l(p) = \prod_{\alpha=1}^{d}\eta_l(p_\alpha), \quad \{\eta_l(p_\alpha)\} \in \{1, \cos p_1, \ldots, \cos p_d, \sin p_1, \ldots, \sin p_d\}.$$

The number of these orthogonal functions is 3^d.

We numerate the elements of the system $\{\varphi_l\}_{l=1}^{3^d}$ to the following rule.

Consider a set of d-tuples $(\alpha_1, \ldots, \alpha_d)$ consisting of 3 digital system. Corresponding for the number zero to 1, 1 to cosine and 2 to sine we construct the following one to one mapping

$$(\alpha_1, \ldots, \alpha_d) \leftrightarrow \eta_l(p_{\alpha_1})\eta_l(p_{\alpha_2}) \cdots \eta_l(p_{\alpha_d}).$$

For example, for $d = 4$ the tuples $(0, 0, 0, 0), (0, 0, 1, 2)$ and $(1, 2, 2, 1)$ correspond to the functions 1, $\cos p_3 \sin p_4$ and $\cos p_1 \sin p_2 \sin p_3 \cos p_4$, respectively. We order and numerate the set of d tuples as

$$\begin{array}{ccccc} (00\cdots00), & (00\cdots01), & (00\cdots10), & \cdots & (11\cdots11), \\ \updownarrow & \updownarrow & \updownarrow & \vdots & \updownarrow \\ \varphi_1 & \varphi_2 & \varphi_3 & \cdots & \varphi_{2^d} \\ (00\cdots02), & (00\cdots12), & (00\cdots20), & \cdots & (22\cdots22), \\ \updownarrow & \updownarrow & \updownarrow & \vdots & \updownarrow \\ \varphi_{2^d+1} & \varphi_{2^d+2} & \varphi_{2^d+3} & \cdots & \varphi_{3^d} \end{array}$$

By construction $\varphi_l(\mathbf{0}) = 1$ for $l = 1, \dots, 2^d$ and $\varphi_l(\mathbf{0}) = 0$ for $l = 2^d + 1, \dots, 3^d$.

The operator $\mathbf{v}$ is expressed via the orthogonal functions φ_l, $l = 1, \dots, 3^d$ in the form

$$(\mathbf{v}f)(p) = \sum_{l=1}^{3^d} (\mathbf{v}_l f)(p), \quad (\mathbf{v}_l f)(p) = (\varphi_l, f)\varphi_l(p),$$

where $(\cdot, \cdot)$ is the inner product in $L_2(\mathbb{T}^d)$.

It follows from the nonnegativity of the operator $\mathbf{v} \geqslant 0$ that the square root $\mathbf{v}^{\frac{1}{2}} \geqslant 0$ exists. The operator $\mathbf{v}^{\frac{1}{2}}$ acts in $L_2(\mathbb{T}^d)$ as

$$(\mathbf{v}^{\frac{1}{2}} f)(p) = \sum_{l=1}^{3^d} \frac{1}{\|\varphi_l\|} (\mathbf{v}_l f)(p).$$

Let $\mathbb{C}$ be the complex plane, and let $\mathbf{r}_0(k; z)$, $z \in \mathbb{C} \backslash [m(k), M(k)]$ be the resolvent of $h_0(k)$.

Consider the operator $\tilde{h}_\mu(k)$ acting in $L_2(\mathbb{T}^d)$ in accordance with the formula

$$\tilde{h}_\mu(k) = \tilde{h}_0(k) - \mu \mathbf{v},$$

where $\tilde{h}_0(k)$ is the operator of multiplication by the function $\tilde{\mathcal{E}}_k(\cdot)$,

$$\tilde{\mathcal{E}}_k(p) = \sum_{i=1}^{d} \Big(\frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2} \cos 2k_i + \frac{1}{m_2^2}} \cos 2p_i \Big).$$

The operator $h_\mu(k)$ is unitary equivalent to the operator $\tilde{h}_\mu(k)$ (See Lemma 2 in [14]). The equivalence is performed by the unitary operator $U : L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ as $\tilde{h}_\mu(k) = U^{-1} h_\mu(k) U$, where

$$(Uf)(p) = f(p - \frac{1}{2}\theta(k)),$$

$$\theta(k) = (\theta_1(k_1), \dots, \theta_d(k_d)), \quad \theta_i(k_i) = \arccos \frac{\frac{1}{m_1} + \frac{1}{m_2} \cos 2k_i}{\sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2} \cos 2k_i + \frac{1}{m_2^2}}}, \quad i = 1, 2, \dots, d.$$

For any $z \in \mathbb{C} \backslash [m(k), M(k)]$, we define a Birman–Schwinger integral operator $\mathbf{T}(\mu, k; z) = \mu \mathbf{v}^{\frac{1}{2}} \mathbf{r}_0(k; z) \mathbf{v}^{\frac{1}{2}}$. The rank of $\mathbf{T}(\mu, k; z)$ is equal to 3^d and it represents via one rank orthogonal projectors $\mathbf{T}_l(\mu, k; z)$ as

$$\mathbf{T}(\mu, k; z)\psi = \sum_{l=1}^{3^d} \mathbf{T}_l(\mu, k; z)\psi,$$

$$\mathbf{T}_l(\mu, k; z)\psi = \frac{\mu}{\|\varphi_l\|^2} \big(\varphi_l, \mathbf{r}_0(k; z)\varphi_l\big)(\varphi_l, \psi)\varphi_l,$$

where

$$\big(\varphi_l, \mathbf{r}_0(k; z)\varphi_l\big) = \int_{\mathbb{T}^d} \frac{\varphi_l^2(s)ds}{\tilde{\mathcal{E}}_k(s) - z}, \quad l = 1, 2, \dots, 3^d, \quad z \in \mathbb{C} \backslash [m(k), M(k)]. \tag{1}$$

A nonzero eigenvalue of the operator $\mathbf{T}_l(\mu, k; z)$ is $\lambda_l(z) = \mu\big(\varphi_l, \mathbf{r}_0(k; z)\varphi_l\big)$, $l = 1, \dots, 3^d$ and φ_l is an eigenfunction corresponding to $\lambda_l(z)$. Moreover,

$$\sigma(\mathbf{T}(\mu, k; z)) = \{0 \cup \lambda_1(z) \cup \dots \cup \lambda_{3^d}(z)\}.$$

For each $z \in \mathbb{C} \setminus [m(k), M(k)]$ and $k \in \mathbb{T}^d$, denote by $\Delta_l(\mu, k; z)$ and $\Delta(\mu, k; z)$ the Fredholm determinants of the operators $I - \mathbf{T}_l(\mu, k; z)$ and $I - \mathbf{T}(\mu, k; z)$, respectively. Then

$$\Delta_l(\mu, k; z) = 1 - \mu \int\limits_{\mathbb{T}^d} \frac{\varphi_l^2(s)ds}{\tilde{\mathcal{E}}_k(s) - z}, \quad l = 1, 2, \ldots, 3^d$$

and the equality

$$\Delta(\mu, k; z) = \prod_{l=1}^{3^d} \Delta_l(\mu, k; z).$$

holds.

The following lemma is a consequence of the Fredholm theorem.

Lemma 2.1. A number z, $z \in \mathbb{C} \setminus [m(k), M(k)]$, is an eigenvalue of $h_\mu(k)$ if and only if $\Delta(\mu, k; z) = 0$. Moreover, the multiplicity of a zero of the function $\Delta(\mu, k; \cdot)$ then coincides with the multiplicity of an eigenvalue of the operator $h_\mu(k)$.

Remark 2.1. Clearly, the operator $h_\mu(k)$ has an eigenvalue $z < m(k)$, i.e., $\mathrm{Ker}(h_\mu(k) - zI) \neq 0$ if and only if the compact operator $\mathbf{T}(\mu, k; z)$ in $L_2(\mathbb{T}^d)$ has an eigenvalue equal to 1 and there is a function $\psi \in \mathrm{Ker}\big(\mathbf{T}(\mu, k; z) - I\big)$ such that

$$f(\cdot) = \frac{\mu \mathbf{v}^{\frac{1}{2}} \psi(\cdot)}{\tilde{\mathcal{E}}_k(\cdot) - z} \in L_2(\mathbb{T}^d).$$

In this case, $f \in \mathrm{Ker}(h_\mu(k) - zI)$. Moreover, if $z < m(k)$, then

$$\dim \mathrm{Ker}(h_\mu(k) - zI) = \dim \mathrm{Ker}(\mathbf{T}(\mu, k; z) - I),$$

$$\mathrm{Ker}(h_\mu(k) - zI) = \Big\{ f : \, f(\cdot) = \frac{\mu \mathbf{v}^{\frac{1}{2}} \psi(\cdot)}{\tilde{\mathcal{E}}_k(\cdot) - z}, \, \psi \in \mathrm{Ker}\big(\mathbf{T}(\mu, k; z) - I\big) \Big\}.$$

Since the minimum points of $\tilde{\mathcal{E}}_k(\cdot)$ are non-degenerate, the operator $\mathbf{T}_l(\mu, k; z)$ in $L_2(\mathbb{T}^d)$ is well defined as $z = m(k)$ for any $d \geqslant 3$ and $l = 1, \ldots, 3^d$. The equality $\varphi_l(\mathbf{0}) = 0$, $l = 2^d + 1, \ldots, 3^d$ provides well defined of $\mathbf{T}_l(\mu, k; z)$ in $L_2(\mathbb{T}^d)$ as $z = m(k)$ for any $d = 1, 2$ and $l = 2^d + 1, \ldots, 3^d$. According to (1), the following limits

$$\big(\varphi_l, \mathbf{r}_0(k; m(k))\varphi_l\big) := \lim_{z \nearrow m(k)} \big(\varphi_l, \mathbf{r}_0(k; z)\varphi_l\big), \quad l = 1, 2, \ldots, 3^d,$$

exist (finite or infinite). We set

$$\mu_l(k) := \frac{1}{\big(\varphi_l, \mathbf{r}_0(k; m(k))\varphi_l\big)}, \quad l = 1, 2, \ldots, 3^d.$$

Assumption 2.1. Assume that $m = m_1 = m_2$ and $k \in \Pi$, where

$$\Pi = \Big\{ k = (k_1, k_2, \ldots, k_d) \in \mathbb{T}^d : \text{ at least } d - 2 \, (d \geqslant 3) \text{ coordinates are equal to } -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \Big\}.$$

If the Assumption 2.1 is not fulfilled, then $\mu_l(k) = 0$ for $d = 1, 2$, $l = 1, \ldots, 2^d$ and $0 < \mu_l(k) < \infty$ for $d \geqslant 3$, $l = 1, \ldots, 2^d$ or for $d \geqslant 1$, $l = 2^d + 1, \ldots, 3^d$.

Definition 2.1. Let $d = 3, 4$ $\big(d = 1, 2\big)$. We say that the operator $h_\mu(\mathbf{0})$ has a virtual level at $z = 0$ (lower edge of the essential spectrum) if 1 is an eigenvalue of $\mathbf{T}(\mu, \mathbf{0}; 0)$ $\Big($of $\mathbf{T}_l(\mu, \mathbf{0}; 0)$ for some $l > 2^d\Big)$ with some associated eigenfunction ψ satisfying the condition

$$\frac{\mathbf{v}^{\frac{1}{2}} \psi(\cdot)}{\tilde{\mathcal{E}}_{\mathbf{0}}(\cdot)} \notin L_2(\mathbb{T}^d).$$

The number of such linearly independent eigenvectors ψ of the operator $\mathbf{T}(\mu, \mathbf{0}; 0), d \geqslant 3$ $\Big($of all operators $\mathbf{T}_l(\mu, \mathbf{0}; 0)$ for $d = 2\Big)$, is called the multiplicity of the virtual level of the operator $h_\mu(\mathbf{0})$.

Note that, if the number 1 is an eigenvalue of the operator $\mathbf{T}(\mu, k; m(k))$, and the corresponding eigenfunction ψ with

$$\frac{\mathbf{v}^{\frac{1}{2}} \psi(\cdot)}{\tilde{\mathcal{E}}_k(\cdot) - m(k)} \in L_2(\mathbb{T}^d), \quad d \geqslant 3,$$

then the function $\dfrac{\mathbf{v}^{\frac{1}{2}} \psi(\cdot)}{\tilde{\mathcal{E}}_k(\cdot) - m(k)}$ is the eigenfunction of $h_\mu(k)$ corresponding to the eigenvalue $z = m(k)$.

Theorem 2.1. Suppose that the Assumption 2.1 are not fulfilled. Then for any $k \in \mathbb{T}^d$, the following statements are true

1. Let $d = 1, 2$ and $\mu \in (0, \mu_*(k)),\ \mu_*(k) = \min\limits_{2^d < l \leqslant 3^d} \mu_l(k)$. Then the operator $h_\mu(k)$ has 2^d eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum.

2. Let $d \geqslant 3$ and $\mu \in (0, \mu_*(k)),\ \mu_*(k) = \min\limits_{1 \leqslant l \leqslant 3^d} \mu_l(k)$. Then the operator $h_\mu(k)$ has no eigenvalues lying to the left of the essential spectrum.

3. Let $d \geqslant 1$ and $\mu \in (\mu^*(k), +\infty),\ \mu^*(k) = \max\limits_{1 \leqslant l \leqslant 3^d} \mu_l(k)$. Then the operator $h_\mu(k)$ has 3^d eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum.

We split the set Π into three subsets Π_n, $n = 0, 1, 2$, of $k \in \Pi$, whose $d - n$ coordinates are only equal to $-\frac{\pi}{2}$ or $\frac{\pi}{2}$.

Theorem 2.2. Let the Assumption 2.1 be fulfilled and let $d \geqslant 3$. Then for any $\mu > 0$ and $k \in \Pi_n$, $n = 0, 1, 2$ the operator $h_\mu(k)$ has at least

$$s_n = 2^d + \sum_{i=1}^{d-n} 2^{d-i} \cdot 3^{i-1}$$

eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum. Moreover, if $n = 0$, then $h_\mu(k)$ has $s_0 = 3^d$ eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum.

Let r_s be a positive integer number defined as

$$r_s = C_d^0 + C_d^1 + \cdots + C_d^s, \quad s = 0, 1, \ldots, d. \tag{2}$$

We split the numbers $\{1, 2, \ldots, 2^d\}$ into $d + 1$ as

$$\{1, 2, \ldots, 2^d\} = D_0 \cup \cdots \cup D_d,$$

where $D_s = \{1 + r_s - C_d^s, \ldots, r_s\}$.

Remark that for any $d \geqslant 3$ the following assertions

$$\mu_{r_s}(\mathbf{0}) = \mu_r(\mathbf{0}), \quad r \in D_s,\ s = 0, 1, \ldots, d,$$
$$\mu_{r_0}(\mathbf{0}) < \mu_{r_1}(\mathbf{0}) < \mu_{r_2}(\mathbf{0}) < \cdots < \mu_{r_d}(\mathbf{0}),$$
$$\mu_{r_d}(\mathbf{0}) < \mu_l(\mathbf{0}),\ \ r_d = 2^d,\ \ l = 2^d + 1, \ldots, 3^d$$

hold (see Lemma 4.2 below).

Theorem 2.3. Suppose that the Assumption 2.1 are not fulfilled. Then the following statements are true

1. If $d = 1$ and $\mu = \mu_3(\mathbf{0})$, then $h_\mu(\mathbf{0})$ has a virtual level at $z = 0$ and two simple negative eigenvalues.

2. If $d = 2$ and $\mu = \mu_*(\mathbf{0}) = \min\limits_{4 < l \leqslant 9} \mu_l(\mathbf{0})$, then $h_\mu(\mathbf{0})$ has three negative eigenvalues, two of them simple and one of them two-fold, and a two-fold virtual level at $z = 0$.

3. Let $d = 3, 4$ $\big(d > 4\big)$ and $\mu = \mu_{r_s}(\mathbf{0})$ for some $s \in \{0, 1, \ldots, d\}$. Then the operator $h_\mu(\mathbf{0})$ has s eigenvalues λ_l, $l = 0, 1, \ldots, s-1$, with multiplicity C_d^l and $\lambda_0 < \cdots < \lambda_{s-1} < 0$. Additionally, the operator $h_\mu(\mathbf{0})$ has a virtual level $\big($an eigenvalue$\big)$ at $z = 0$ with multiplicity C_d^s.

Moreover, if $d \geqslant 3$ and $\mu_{r_d}(\mathbf{0}) < \mu < \min\limits_{l > 2^d} \mu_l(\mathbf{0})$, then the operator $h_\mu(\mathbf{0})$ has $d + 1$ eigenvalues λ_l, $l = 0, 1, \ldots, d$, lying to the left of the essential spectrum, with $\lambda_0 < \cdots < \lambda_d < 0$ and $C_d^0 + C_d^1 + \cdots + C_d^d = 2^d$, where C_d^r is the multiplicity of λ_r.

4. Let $d \geqslant 3$ and $\mu = \mu_l(\mathbf{0})$ for some $l \in \{2^d + 1, \ldots, 3^d\}$. Then the number $z = 0$ is an eigenvalue of the operator $h_\mu(\mathbf{0})$ and this operator has $2^d + q$ eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum, where q is the number of elements of the set $\{\mu_n :\ \mu_n > \mu_l(\mathbf{0}), n > 2^d\}$.

Remark 2.2. A similar Theorems 2.1, 2.2 and 2.3 describe the dependence of the number of eigenvalues and their arrangement on the parameter μ for all $\mu \in \mathbb{R}$. In this case, the eigenvalues of $h_\mu(k)$ are located both to the left and to the right of the essential spectrum. In the case $\mu < 0$, the eigenvalues of $h_\mu(k)$ are only to the right of the essential spectrum.

3. The eigenvalues of $h_\mu(k)$

In this section, we prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Remark that the integral

$$\int\limits_{\mathbb{T}^d} \frac{\varphi_l^2(s)ds}{\tilde{\mathcal{E}}_k(s) - m(k)}$$

converges for any $\varphi_l \in \mathcal{H}_l$ for $l = 1, \ldots, 3^d$, $d \geqslant 3$ and for $l = 2^d + 1, \ldots, 3^d$, $d = 1, 2$. The function $\Delta_l(\mu, k; \cdot)$ is continuous and monotonically decreasing on $z \in (-\infty, m(k))$ for any fixed $\mu > 0$ and $k \in \mathbb{T}^d$.

1. Let $d = 1, 2$. The following equalities

$$\lim_{z\to-\infty} \Delta_l(\mu, k; z) = 1,$$

$$\lim_{z\nearrow m(k)} \Delta_l(\mu, k; z) = -\infty \quad \text{for} \quad l = 1, 2, \ldots, 2^d$$

hold. Then there is a unique number $z_l(\mu, k) < m(k)$, $l = 1, 2, \ldots, 2^d$ such that $\Delta_l(\mu, k; z_l(\mu, k)) = 0$. According to Lemma 2.1, the operator $h_\mu(k)$ has 2^d eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum.

Since

$$\lim_{z\nearrow m(k)} \Delta_l(\mu, k; z) = \Delta_l(\mu, k; m(k)) < \infty, \quad \text{for} \quad l = 2^d + 1, \ldots, 3^d$$

and the function $\Delta_l(\mu, k; \cdot)$ $\Big(\Delta_l(\cdot, k; z)\Big)$ is monotonically decreasing on $z \in (-\infty, m(k))$ $\Big($on $\mu \in (0, \infty)\Big)$ for any fixed $\mu > 0$ $\Big(z \in (-\infty, m(k))\Big)$, the inequalities

$$\Delta_l(\mu, k; z) > \Delta_l(\mu, k; m(k)) > \Delta_l(\mu_*(k), k; m(k)) = 0 \quad \text{for all} \quad \mu \in (0, \mu_*(k))$$

hold. Then by the Lemma 2.1 the operator $h_\mu(k)$ has only 2^d eigenvalues (taking into account the multiplicity) lying to the left of the essential spectrum.

2. For the case when $d \geqslant 3$ by similar way we can show that $\Delta_l(\mu, k; z) > 0$ for all $\mu \in (0, \mu_*(k))$. This proves the required assertion.

3. Note that for the case $d = 1, 2$ and $l = 1, 2, \ldots, 2^d$

$$\lim_{z\nearrow m(k)} \Delta_l(\mu, k; z) = -\infty \qquad \text{for all} \quad \mu > 0 \tag{3}$$

holds. Let $\mu \in (\mu^*(k), +\infty)$, $\mu^*(k) = \max\limits_{1\leqslant l\leqslant 3^d} \mu_l(k)$. Then for the cases $l = 2^d+1, \ldots, 3^d$, $d = 1, 2$ and $l = 1, 2, \ldots, 3^d$, $d \geqslant 3$ we have

$$\lim_{z\nearrow m(k)} \Delta_l(\mu, k; z) = \Delta_l(\mu, k; m(k)) = 1 - \frac{\mu}{\mu_l(k)} < 0. \tag{4}$$

Since $\Delta_l(\mu, k; \cdot)$ is a continuous monotonic function on $(-\infty, m(k))$ and

$$\lim_{z\to-\infty} \Delta_l(\mu, k; z) = 1,$$

according to (3), (4), there exists a unique $z_l(\mu, k) \in (-\infty, m(k))$ such that

$$\Delta_l(\mu, k; z_l(\mu, k)) = 0 \quad \text{for all} \quad l = 1, 2, \ldots, 3^d.$$

Hence by Lemma 2.1 the operator $h_\mu(k)$ has 3^d eigenvalues (taking into account the multiplicity) lying to the left $m(k)$. □

Proof of Theorem 2.2. The case $n = 0$. Let $k \in \Pi_0$, i.e. $k_i = \pm\frac{\pi}{2}$, $i = 1, 2, \ldots, d$. The function $\tilde{\mathcal{E}}_k(\cdot)$ is a constant function. Therefore, we obtain

$$\lim_{z\nearrow m(k)} \big(\varphi_l, \mathbf{r}_0(k; z)\varphi_l\big) = \lim_{z\nearrow m(k)} \int\limits_{\mathbb{T}^d} \frac{\varphi_l^2(s)ds}{\tilde{\mathcal{E}}_k(s) - z} = +\infty, \quad l = 1, 2, \ldots, 3^d,$$

which implies

$$\lim_{z\nearrow m(k)} \Delta_l(\mu, k; z) = -\infty$$

for any $\mu > 0$. Since

$$\lim_{z\to-\infty} \Delta_l(\mu, k; z) = 1,$$

there exists unique $z_l(\mu, k) \in (-\infty, m(k))$ such that $\Delta(\mu, k; z_l(\mu, k)) = 0$ for any $\mu > 0$ and $l = 1, 2, \ldots, 3^d$. Hence by the Lemma 2.1 the operator $h_\mu(k)$ has 3^d eigenvalues (taking into account the multiplicity) lying to the left $m(k)$, $k \in \Pi_0$.

The case $n = 1$. We prove theorem for the case $k \in \Pi_1$ with $k_i = \pm\frac{\pi}{2}$, $i = 1, 2, \ldots, d-1$. The function $\tilde{\mathcal{E}}_k(\cdot)$ does not depend on $p_1, p_2, \cdots p_{d-1}$ and is expressed as

$$\tilde{\mathcal{E}}_k(p) = \frac{2d}{m} - \frac{1}{m}\sqrt{2 + 2\cos 2k_d}\cos 2p_d.$$

Then there exist n_1 functions $\xi_m(k; \cdot) := (\varphi_{l_m}\mathbf{r}_0(k; \cdot), \varphi_{l_m})$ with $\varphi_{l_m}(p_1, \cdots, p_{d-1}, 0) \neq 0$, $m = 1, 2, \ldots, n_1$, where $n_1 = 2^d + \sum\limits_{i=1}^{d-1} 2^{d-i} \cdot 3^{i-1}$. Since $(\tilde{\mathcal{E}}_k(p) - m(k)) = O(p_d^2)$ as $p_d \to 0$ and $\tilde{\mathcal{E}}_k(\cdot)$ does not depend on $p_1, p_2, \cdots p_{d-1}$, we have

$$\lim_{z\nearrow m(k)} \xi_{l_r}(k; z) = +\infty.$$

This gives one

$$\lim_{z \nearrow m(k)} \Delta_{l_r}(\mu, k; z) = -\infty, \quad l_r = 1, 2, \ldots, n_1.$$

Hence there exists unique $z_{l_r}(\mu, k) \in (-\infty, m(k))$ such that

$$\Delta_l(\mu, k; z_{l_r}(\mu, k)) = 0, \quad l_r = 1, 2, \ldots, n_1.$$

According to Lemma 2.1, we obtain the required assertion for the case $k \in \Pi_1$ with $k_i = \pm\frac{\pi}{2}$, $i = 1, 2, \ldots, d-1$. The proofs for the remaining cases with $k \in \Pi_1$ can be constructed in a similar way.

The case $n = 2$ can be proven analogously. □

4. Virtual level and eigenvalues of the operator $h_\mu(\mathbf{0})$

In this section, we prove Theorem 2.3. According to the definition of a virtual level of $h_\mu(0)$, we study the equation

$$\mathbf{T}(\mu, \mathbf{0}; 0)\psi = \psi.$$

We note that $\Delta_l(\mu, \mathbf{0}; \cdot)$ is well defined at $z = 0$ for the cases $l = 2^d + 1, \ldots, 3^d$, $d = 1, 2$ and $l = 1, 2, \ldots, 3^d$, $d \geqslant 3$. According to Lemma 2.1, we can prove the following assertion.

Lemma 4.1. Let $l = 1, 2, \ldots, 3^d$ for $d \geqslant 3$ $\big(l = 2^d + 1, \ldots, 3^d$ for $d = 1, 2\big)$. Then the number $\lambda = 1$ is an eigenvalue of the operator $\mathbf{T}(\mu, \mathbf{0}; 0)$ $\Big(\mathbf{T}_l(\mu, \mathbf{0}; 0)\Big)$ if and only if

$$\prod_{l=1}^{3^d} \Delta_l(\mu, \mathbf{0}; 0) = 0, \quad \Big(\Delta_l(\mu, \mathbf{0}; 0) = 0\Big).$$

Lemma 4.2. For any $d \geqslant 3$, the following assertions are true

$$\mu_{r_s}(\mathbf{0}) = \mu_r(\mathbf{0}), \quad r \in D_s,\ s = 0, 1, \ldots, d,$$
$$\mu_{r_0}(\mathbf{0}) < \mu_{r_1}(\mathbf{0}) < \mu_{r_2}(\mathbf{0}) < \cdots < \mu_{r_d}(\mathbf{0}),$$
$$\mu_{r_d}(\mathbf{0}) < \mu_l(\mathbf{0}),\ \ r_d = 2^d,\ \ l = 2^d + 1, \ldots, 3^d,$$

where $r_0, \ldots, r_d$ are defined by (2).

Proof of lemma 4.2. Since $\tilde{\mathcal{E}}_{\mathbf{0}}(p) = \dfrac{m_1 + m_2}{m_1 m_2} \displaystyle\sum_{i=1}^{d} (1 - \cos 2p_i)$ is symmetric under permutations of p_α and p_β, the equality

$$\int\limits_{\mathbb{T}^d} \frac{\cos^2 s_1 \cdots \cos^2 s_r ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} = \int\limits_{\mathbb{T}^d} \frac{\cos^2 s_{j_1} \cdots \cos^2 s_{j_r} ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)}, \quad r \leqslant d$$

holds.

Hence, the following inequalities

$$\int\limits_{\mathbb{T}^d} \frac{\varphi_1^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} > \int\limits_{\mathbb{T}^d} \frac{\varphi_{j_1}^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} > \int\limits_{\mathbb{T}^d} \frac{\varphi_{j_2}^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} > \cdots > \int\limits_{\mathbb{T}^d} \frac{\varphi_{j_d}^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)},$$

hold, where $j_s \in D_s$, $D_s = \{1 + r_s - C_d^s, \ldots, r_s\}$, $r_s = C_d^0 + C_d^1 + \cdots + C_d^s$, $s = 0, 1, 2, \ldots, d$.

Therefore, the following inequalities

$$\mu_{r_0}(\mathbf{0}) < \mu_{r_1}(\mathbf{0}) < \mu_{r_2}(\mathbf{0}) < \cdots < \mu_{r_d}(\mathbf{0})$$

hold, where

$$\mu_r(\mathbf{0}) = \left(\int\limits_{\mathbb{T}^d} \frac{\varphi_r^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} \right)^{-1}, \quad r = 1, 2, \ldots, d+1.$$

Note that

$$\mu_{r_s}(\mathbf{0}) = \mu_r(\mathbf{0}), \quad r \in D_s, \quad s = 0, 1, 2, \ldots, d. \tag{5}$$

We can easily verify the equality (see Lemma 1, [13])

$$\int\limits_{-\pi}^{\pi} \frac{\cos 2s ds}{a - b \cos 2s} = \frac{2\pi}{b} \frac{a - \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}}$$

for $0 < b < a$. Therefore,

$$\int\limits_{-\pi}^{\pi} \frac{\cos^2 s ds}{a - b \cos 2s} - \int\limits_{-\pi}^{\pi} \frac{\sin^2 s ds}{a - b \cos 2s} > 0$$

for all $0 < b < a$. Applying this inequality, we obtain

$$\int\limits_{\mathbb{T}^d} \frac{\varphi_m^2(s)ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} > \int\limits_{\mathbb{T}^d} \frac{\varphi_l^2(s)ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} \quad \text{for} \quad m \leqslant 2^d < l.$$

This gives one $\mu_{r_s}(\mathbf{0}) < \mu_l(\mathbf{0})$, $l = 2^d + 1, \ldots, 3^d$, where $r_d = 2^d$. □

Proof of Theorem 2.3. Let $d = 1$ and $\mu = \mu_3(\mathbf{0})$. Then according to assertion 1 of Theorem 2.1, for any $\mu > 0$, the operator $h_\mu(\mathbf{0})$ has two simple eigenvalues $z_1(\mu, \mathbf{0}) < z_2(\mu, \mathbf{0}) < 0$ and the corresponding eigenfunctions have the form

$$f_1(p) = \frac{1}{\tilde{\mathcal{E}}_{\mathbf{0}}(p) - z_1(\mu, \mathbf{0})} \quad \text{and} \quad f_2(p) = \frac{\cos p}{\tilde{\mathcal{E}}_{\mathbf{0}}(p) - z_2(\mu, \mathbf{0})}$$

respectively.

Since $\mu = \mu_3(\mathbf{0})$, it follows from Lemma 4.1 that $\lambda = 1$ is an eigenvalue of $\mathbf{T}_3(\mu, \mathbf{0}; 0)$ and $\varphi_3(p) = \sin p$ is the corresponding eigenfunction of $\mathbf{T}_3(\mu, \mathbf{0}; 0)$. One can see that $f_3 \notin L_2(\mathbb{T})$, where $f_3(p) = \sin p / \tilde{\mathcal{E}}_{\mathbf{0}}(p)$, i.e., $z = 0$ is a virtual level of the operator $h_\mu(\mathbf{0})$.

2. Let $d = 2$ and $\mu = \min\limits_{4 < l \leqslant 9} \mu_l(\mathbf{0})$. According to statement 1 of Theorem 2.1, for any $\mu > 0$, the operator $h_\mu(\mathbf{0})$ has four eigenvalues (taking into account the multiplicity) $z_1(\mu, \mathbf{0}) < z_2(\mu, \mathbf{0}) = z_3(\mu, \mathbf{0}) < z_4(\mu, \mathbf{0}) < 0$ and the corresponding eigenfunctions have the form

$$f_1(p) = \frac{1}{\tilde{\mathcal{E}}_{\mathbf{0}}(p) - z_1(\mu, \mathbf{0})}, \quad f_i(p) = \frac{\cos p_i}{\tilde{\mathcal{E}}_{\mathbf{0}}(p) - z_2(\mu, \mathbf{0})}, \ i = 2, 3, \quad f_4(p) = \frac{\cos p_1 \cos p_2}{\tilde{\mathcal{E}}_{\mathbf{0}}(p) - z_4(\mu, \mathbf{0})},$$

respectively.

Observe that the inequalities

$$\int\limits_{\mathbb{T}^2} \frac{\sin^2 s_i ds}{\tilde{\mathcal{E}}_k(s) - z} > \int\limits_{\mathbb{T}^2} \frac{\cos^2 s_i \sin^2 s_j ds}{\tilde{\mathcal{E}}_k(s) - z} > \int\limits_{\mathbb{T}^2} \frac{\sin^2 s_i \sin^2 s_j ds}{\tilde{\mathcal{E}}_k(s) - z}, \quad i, j = 1, 2$$

show that

$$\min_{4 < l \leqslant 9} \mu_l(k) = \min_r \left(\int\limits_{\mathbb{T}^2} \frac{\sin^2 s_r ds}{\tilde{\mathcal{E}}_k(s) - m(k)} \right)^{-1}.$$

For the case when $k = \mathbf{0}$ the equalities

$$\int\limits_{\mathbb{T}^2} \frac{\sin^2 s_1 ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} = \int\limits_{\mathbb{T}^2} \frac{\sin^2 s_2 ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)}$$

holds.

This gives one

$$\mu = \left(\int\limits_{\mathbb{T}^2} \frac{\sin^2 s_1 ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} \right)^{-1} = \left(\int\limits_{\mathbb{T}^2} \frac{\sin^2 s_2 ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s)} \right)^{-1}.$$

Hence, by Lemma 4.1, the number $\lambda = 1$ is an eigenvalue of $\mathbf{T}_l(\mu, \mathbf{0}; 0)$, $l = 5, 6$ and $\varphi_5(p) = \sin p_1$, $\varphi_6(p) = \sin p_2$ are the corresponding eigenfunctions. Since $f_5, f_6 \notin L_2(\mathbb{T}^2)$, where $f_5(p) = \dfrac{\sin p_1}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)}$, $f_6(p) = \dfrac{\sin p_2}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)}$, the number $z = \mathbf{0}$ is a two-fold virtual level of $h_\mu(\mathbf{0})$.

3. Let $d = 3, 4$, and $\mu = \mu_{r_s}(\mathbf{0})$ for some $s \in \{0, 1, \ldots, d\}$. Then, as shown in items 1) and 2) of Theorem 2.3, and by lemma 4.2, the operator $h_\mu(\mathbf{0})$ has s eigenvalues $z_l(\mu, \mathbf{0}) < 0$, $l \in \{0, 1, \ldots, s-1\}$ with multiplicity C_d^s and $z_0(\mu, \mathbf{0}) < \cdots < z_{s-1}(\mu, \mathbf{0}) < 0$.

Since $\mu = \mu_{r_s}(\mathbf{0})$, according to the equality (5) and Lemma 4.1 the number $\lambda = 1$ is an eigenvalue of $\mathbf{T}(\mu, \mathbf{0}; 0)$ with multiplicity C_d^s, where $\varphi_l(p)$, $l \in \{1 + r_s - C_d^s, \ldots, r_s\}$ are the corresponding eigenfunctions. Since $f_l \notin L_2(\mathbb{T}^d)$, where $f_l(p) = \dfrac{\varphi_l}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)}$, the number $z = \mathbf{0}$ is virtual level with multiplicity C_d^s of $h_\mu(\mathbf{0})$.

Let $\mu_{r_d}(\mathbf{0}) < \mu < \min\limits_{l > 2^d} \mu_l(\mathbf{0})$. Then, according to Lemma 4.1, the operator $h_\mu(\mathbf{0})$ has $d + 1$ eigenvalues λ_l with multiplicity C_d^l, $l = 0, 1, \ldots, d$, lying to the left of the essential spectrum, with $\lambda_0 < \cdots < \lambda_d < 0$. Therefore $h_\mu(\mathbf{0})$ has $2^d = C_d^0 + C_d^1 + \cdots + C_d^d$ eigenvalues (taking into account the multiplicity).

The prove of the part 4 can be proven similarly. □

5. Conclusion

We investigate the existence conditions for eigenvalues and virtual levels of the two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^d$, $\mu > 0$ corresponding to the Hamiltonian of the two-particle system on the d-dimensional lattice, where $h_\mu(k)$ is considered as a perturbation of free Hamiltonian $h_0(k)$ by the certain potential operator $\mu\mathbf{v}$ with rank 3^d. The main idea of the study was to represent $\mu\mathbf{v}$ via the sum of one-rank orthogonal projectors $\mu\mathbf{v}_l$. This allowed us to represent the corresponding Birman–Schwinger operator $\mathbf{T}(\mu, k; z)$ via the sum of one-rank projectors $\mathbf{T}_l(\mu, k; z)$, $l = 1, 2, \ldots, 3^d$. Moreover, the study of the spectral properties of $h_\mu(k)$ is reduced to the study of 3^d one-rank projectors $\mathbf{T}_l(\mu, k; z)$. The existence conditions of a virtual level of $h_\mu(k)$ is studied at $k = \mathbf{0}$. The study of the virtual levels of $h_\mu(k)$ for the case when $k \neq \mathbf{0}$ is omitted, since analogous results and existence conditions can be described with respect to k.

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