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Original article

Free energy and entropy for the constructive Gibbs measures of the Ising model on the Cayley tree of order three

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ABSTRACT In this paper, we identify non-translation-invariant constructive Gibbs measures for the Ising model on a third-order Cayley tree, which differ from known ones. We provide the conditions for the existence of at least two distinct Gibbs measures, which implies that a phase transition occurs. The free energies and entropies corresponding to the identified measures are calculated. These free energies and entropies are then compared with the known ones and shown to differ from them.

KEYWORDS Cayley tree, Ising model, Gibbs measure, free energy, entropy.

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1. Introduction

One of the main problems for the Ising model is to describe all limiting Gibbs measures corresponding to the model. It is well known that for the Ising model such measures form a nonempty, convex, and compact subset in the set of all probability measures. The problem of completely describing the element of this set is far from being completely solved. Some translation-invariant (see, e.g., [1–3]), periodic [4, 5], and continuum sets of non-periodic Gibbs measures for the Ising model on a Cayley tree have already been described (see, e.g., [2, 3, 6, 7]).

In [8, 9], the notion of weakly periodic Gibbs measure is introduced. This notion generalizes the notion of periodic Gibbs measures and such a measure is non-periodic. In [10], the author constructs new sets of non-periodic Gibbs measures for the Ising model on the Cayley tree of order k, which called (k_0)-translation-invariant, (k_0)-periodic and (k_0)-non-translation-invariant, respectively. In [11], authors investigate p-adic Gibbs measures for the q-state Potts model with an external field and establish the conditions for the existence of a phase transition. In p-adic case, such kind of constructed measures for the Ising model were studied in [12–14]. In [15], the authors construct a very wide class of Gibbs measures. The existence of the Gibbs measures for a given model defines the occurence of a phase transition [16, 17].

Entropy and free energy are fundamental concepts in thermodynamics that provide a powerful framework for understanding the behavior of systems. By considering both the energy content and the disorder of a system, these concepts allow us to predict the direction and extent of spontaneous processes, making them invaluable tools in various fields of science and engineering [18].

The progress in nanoscience and nanotechnology has spurred considerable research into adapting the principle of thermodynamics and statistical mechanics for small systems with a limited number of particles, moving beyond traditional large- scale applications [5].

Nanoscale systems are characterized by their dynamic structures, unlike the static equilibrium of macroscopic phases. Phase coexistence in these small systems shifts from sharp points to ranges of temperature and pressure. This behavior invalidates the Gibbs phase rule and allows for the formation of metastable phases that are unique to the nanoscale [19, 36–39].

Working with nanoscale materials presents difficulties in accurately describing how their properties and phase changes behave. To overcome these hurdles, we need to formulate new thermodynamic and statistical models specifically designed for small-scale systems. This is particularly crucial for understanding molecular self-assembly, a core process in bottom-up nanotechnology, which is fundamentally governed by phase transition phenomena as emphasized by Feynman [20].

In [21], the authors present, for the Ising model on the Cayley tree, some explicit formulae of the free energies (and entropies) according to boundary conditions. They include translation-invariant, periodic as well as those corresponding to (recently discovered) weakly periodic Gibbs states. It is proved that the free energies for the translation-invariant and periodic boundary conditions are equal.

In [22], the authors consider the calculation of the entropy of an Ising ferromagnet with nonmagnetic impurities distributed at random over the lattice sites or bonds. In [23], the authors found the exact free energy of such a chain as a function of the impurity concentration, temperature, and the external magnetic field. In [24], the authors studied Ising-Vannimenus model on a Cayley tree for order two with competing nearest-neighbor and prolonged next-nearest-neighbor interactions. Moreover, the free energies and entropies, associated with translation invariant Gibbs measures, are calculated.

In [25], the author considers the Ising-Vannimenus model on a Cayley tree of order three. In [26], the authors generalized the Ising-Vannimenus model's Gibbs measures on a Cayley tree of any order using the Kolmogorov consistency condition and classified the fixed points. They obtained a new formula to calculate the free energy of the model on the Cayley tree of any order under given boundary conditions. In [27], the author studied thermodynamic properties of mixed-spin (2, 1/2) Ising and Blum-Capel models on the Cayley tree.

In [28], the author constructs the partition function and then calculate the free energy of the Ising model having the prolonged next nearest and nearest neighbor interactions and external field on a two-order Cayley tree using the self-similarity of the semi-infinite Cayley tree. In [29], the author calculated the free energy and entropy for (1,1/2)-MSIM. In [30], the author identified regions where the disordered phases are extreme by means of the tree-indexed Markov chain and satisfied the Kesten-Stigum condition for non-extremality of the disordered phase according to the fixed point.

Recently, in [31], we constructed new Gibbs measures for the Ising model on the Cayley tree of order two and calculated free energies of these measures. We noticed that these free energies are equal to the free energies of the translation-invariant boundary conditions.

In this paper, we construct Gibbs measures for the Ising model on a third-order Cayley tree, which differ from those mentioned above, and calculate the free energies and entropies. Also, we prove that the free energies and entropies corresponding to the obtained measures (which are constructed on the Cayley tree of order three) differ from the free energies of the translation-invariant boundary conditions.

This paper is organized as follows. In Section 2, we present necessary main definitions and formulas. In the next two sections, we construct new Gibbs measures. In Section 5, we calculate the free energy corresponding to these measures. In Section 6, we calculate the entropy corresponding to the measures.

2. Preliminaries

Let $\Gamma^k = (V, L), k \geq 1$ be the Cayley tree of order k, where V and L are the set of vertices and the set of all edges of the tree Γ^k, respectively. Γ^k can be represented as a group G_k, which is the free product of $k+1$ cyclic groups of the second order (see, e.g. [2,7]).

Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them $l = \langle x, y\rangle$. A collection of nearest neighbor pairs $\langle x, x_1\rangle$, $\langle x_1, x_2\rangle, ..., \langle x_m, y\rangle$ is called the path from x to y. By the path, one can define distance d on the tree. The distance between vertices x and y is the number of edges of the shortest path from x to y. Let us fix a vertex $x^0 \in V$ and call it as a root of the tree. Then for any natural number n, we introduce the following set:

$$W_n = \{x \in V \mid d(x, x^0) = n\},\ V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{\langle x, y\rangle \in L :\ x, y \in V_n\},$$

The sets W_n and V_n are called a sphere and a ball with radius n, respectively.

For a given $x \in W_n$, we in troduce

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

which is called a set of direct successors of x.

For $A \subseteq V$, a spin configuration on A is defined as a function

$$x \in A \to \sigma_A(x) \in \Phi = \{-1, 1\}.$$

The set of all configurations coincides with $\Omega_A = \Phi^A$. We denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

We consider the Hamiltonian of the Ising model

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y), \tag{1}$$

where $J \in R$, $\sigma(x) \in \Phi$ and $\langle x, y\rangle$ are the nearest neighbors.

For every n, we then define a measure μ_n on Ω_{V_n} by

$$\mu_n(\sigma_n) = Z_n^{-1}\exp\{-\beta H(\sigma_n) + \sum_{x\in W_n} h_x\sigma(x)\}, \tag{2}$$

where $\beta = \dfrac{1}{T}$ (T is a temperature, $T>0$), $\sigma_n = \{\sigma(x), x\in V_n\}\in\Omega_{V_n}$, Z_n^{-1} is the normalizing factor, and

$$H(\sigma_n) = -J\sum_{\langle x,y\rangle\in L_n}\sigma(x)\sigma(y).$$

The compatibility condition for the measures $\mu_n, n>1$ is

$$\sum_{\sigma^{(n)}}\mu_n(\sigma_{n-1},\sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \tag{3}$$

where $\sigma^{(n)} = \{\sigma(x), x\in W_n\}$.

Let $\mu_n, n\geq 1$ be a sequence of measures on the sets Ω_{V_n} that satisfy compatibility condition (3). By the Kolmogorov theorem, we then have a unique limit measure μ on Ω such that

$$\mu(\{\sigma\mid_{V_n}=\sigma_n\}) = \mu_n(\sigma_n),$$

for every $n=1,2,...$ and $\sigma_n\in\Phi^{V_n}$. Such a measure is called a splitting Gibbs measure corresponding to the Hamiltonian (1) and function $h_x, x\in V$.

Theorem 2.1 [1–3] The measures $\mu_n(\sigma_n)$, $n=1,2,...$, in (2) are compatible iff for any $x\in V$, the following equation holds:

$$h_x = \sum_{y\in S(x)} f(h_y,\theta), \tag{4}$$

where $f(x,\theta) = \arctan h(\theta\tanh x)$, $\theta = \tanh(J\beta)$.

Definition 2.1 Let K be a subgroup of $G_k,\ k\geq 1$. We say that a function $h=\{h_x\in R: x\in G_k\}$ is K- periodic if $h_{yx}=h_x$ for all $x\in G_k$ and $y\in K$. A G_k-periodic function h is called the translation-invariant one.

Definition 2.2 A Gibbs measure is called K-periodic if it corresponds to K- periodic function h.

Definition 2.3 A set of quantities $h=\{h_x, x\in G_k\}$ is called K- weakly periodic, if $h_x=h_{ij}$, for any $x\in H_i$, $x_\downarrow\in H_j$, where $x_\downarrow$-ancestor of x.

Theorem 2.1 establishes a one-to-one correspondence: a boundary condition solving the functional equation (4) uniquely determines a Gibbs measure and conversely. This means that finding all Gibbs measures is equivalent to finding all solutions to equation (4).

We aim to investigate the relationship between the boundary condition and the resulting free energy, specifically when the free energy exists:

$$F(\beta,h) = -\lim_{n\to\infty}\frac{1}{\beta\mid V_n\mid}\ln Z_n(\beta,h). \tag{5}$$

The authors of [15] define a large variety of Gibbs measures. Studying the Gibbs measures on a k-order Cayley tree is equivalent to solving a system of equations

$$\begin{cases} h = (a_1-a_2)f(h,\theta) + (a_3-a_4)f(l,\theta), \\ l = (b_1-b_2)f(h,\theta) + (b_3-b_4)f(l,\theta), \end{cases} \tag{6}$$

where $a_i, b_i, i=1,2,3,4$ are non-negative integers and

$$a_1+a_2+a_3+a_4 = k, \qquad b_1+b_2+b_3+b_4 = k. \tag{7}$$

Theorem 2.2 [15] Independently of the parameters, there is one Gibbs measure which corresponding to the solution $(0,0)$ of the system of equations (6), and if

$$\mid((a_3-a_4)(b_1-b_2)-(a_1-a_2)(b_3-b_4))\theta^2 + (a_1-a_2+b_3-b_4)\theta\mid > 1$$

then there are at least three distinct Gibbs measures corresponding to the solutions $(0,0)$, $(\pm h_*, \pm l_*)$, of system of equations (6), where $h_*>0$, $l_*>0$.

If an arbitrary edge $\langle x,y\rangle = l\in L$ is deleted from the Cayley tree Γ^k, it splits into two components, i.e., two identical semi-infinite trees Γ_0^k and Γ_1^k (see Fig. 1). In this paper, we consider semi-infinite Cayley tree $\Gamma_0^k = (V^0, L^0)$. The vertex x_0 is considered as a root of tree, the root has k nearest neighbors and all other vertices of Γ_0^k has $k+1$ nearest neighbors.

This work identifies new families of Gibbs measures that differ from those described in the previous studies [5,31–34], as well as those presented in Theorem 2.2.

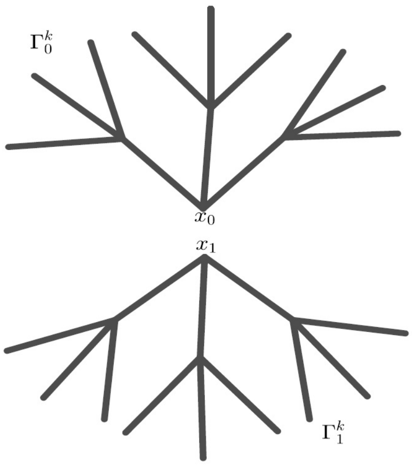

FIG. 1. The Cayley tree of order $k=3$, separated to two semi-infinite sub-trees Γ_0^k and Γ_1^k

3. Constructive Gibbs measures with rule A

This section will detail the construction of Gibbs measures for the Ising model on a Cayley tree of order three. The following definition specifies the set of quantities $h=\{h_x, x\in V\}$:

(A) If $h_x=h_2$, then we put h_1 on all direct successors of x.

If $h_x=h_1$, then we put h_2 and 0 on an arbitrary vertex and the other two vertices of direct successors of x, respectively.

If $h_x=0$, then we put 0 on all direct successors of x (see Fig. 2).

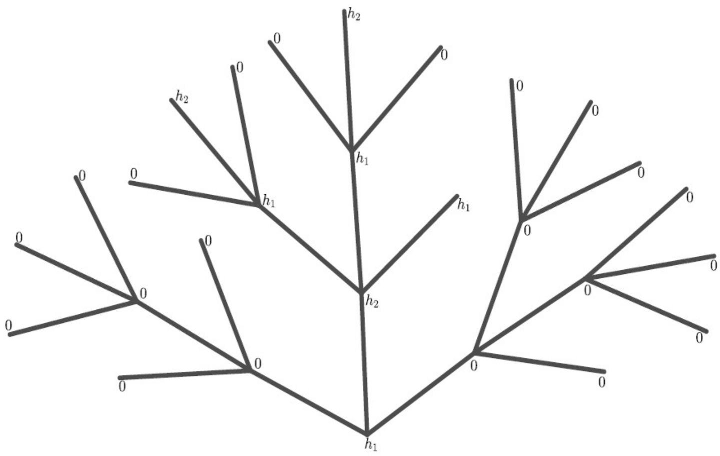

FIG. 2. Set of quantities $h=\{h_x,\, x\in V\}$ corresponding to the rule (A)

The set of the boundary condition $\{0,h_1,h_2\}$ defined by the rule **(A)** must satisfy the boundary condition (4):

$$\begin{cases} h_1=f(h_2,\theta), \\ h_2=3f(h_1,\theta). \end{cases} \tag{8}$$

It is clear that $h_1=h_2=0$ is a solution of (8).

Using the formula $f(x,\theta)=\operatorname{arctan}h(\theta\tanh x)=\frac{1}{2}\ln\frac{(1+\theta)e^{2h}+(1-\theta)}{(1-\theta)e^{2h}+(1+\theta)}$, and introducing the notations $\alpha=\frac{1-\theta}{1+\theta}$, $z_i=e^{2h_i}$, $i=1,2$, we have $\alpha\in(0,1)\cup(1,+\infty)$. By the last notations, (8) yields the following system of equations:

$$\begin{cases} z_1 = \dfrac{z_2+\alpha}{\alpha z_2+1}, \\ z_2 = \left(\dfrac{z_1+\alpha}{\alpha z_1+1}\right)^3. \end{cases} \tag{9}$$

The following lemma defines the number of positive solutions of (9).

Lemma 3.1 Let $\mathcal{N}$ be a number of the positive solutions (9). Then the following assertions hold:

$$\mathcal{N} = \begin{cases} 3, \quad \text{if} \quad \alpha \in (0\,;\,2-\sqrt{3}) \cup (2+\sqrt{3}\,;\infty), \\ 1, \quad \text{if} \quad \alpha \in \cup(2-\sqrt{3}\,;\,1) \cup (1\,;2+\sqrt{3}). \end{cases} \tag{10}$$

Proof: See Appendix 1.

Using Lemma 3.1, we obtain the following theorem:

Theorem 3.1 For the Gibbs measures for the Ising model corresponding to $h = \{h_x, x \in V\}$ which defined by rule **(A)** on the Cayley tree of order three the following statements hold:

(1) if $\alpha \in (0; 2-\sqrt{3}) \cup (2+\sqrt{3}; \infty)$, then there are three Gibbs measures, moreover, two of them are non-translation-invariant Gibbs measures;

(2) if $\alpha \in (2-\sqrt{3}; 1) \cup (1; 2+\sqrt{3})$, then there exists a unique translation-invariant Gibbs measure.

Remark 3.1 The translation-invariant measure corresponding to $(z_1, z_2) = (1, 1)$ found in Theorem 3.1 was studied [1, 8, 19].

4. Constructive Gibbs measures with rule B

The values for the set $h = \{h_x, x \in V\}$ are assigned according to the following rules.

(B) If $h_x = h_2$, then we put h_1 on all direct successors of x.

If $h_x = h_1$, then we put 0 and h_2 on an arbitrary vertex and the other two vertices of direct successors of x, respectively.

If $h_x = 0$, then we put 0 on all direct successors of x (see Fig. 3).

The set of quantities $\{0, h_1, h_2\}$ as determined by the constructive rule **(B)** is constrained by the boundary condition specified in equation (4):

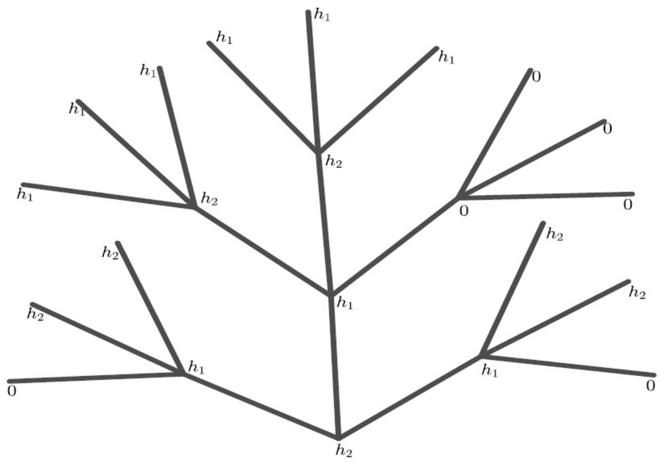

FIG. 3. Set of quantities $h = \{h_x,\, x \in V\}$, which corresponding to the rule (B)

$$\begin{cases} h_1 = 2f(h_2, \theta), \\ h_2 = 3f(h_1, \theta). \end{cases} \tag{11}$$

From (11), we write the following:

$$\begin{cases} z_1 = \left(\dfrac{z_2+\alpha}{\alpha z_2+1}\right)^2, \\ z_2 = \left(\dfrac{z_1+\alpha}{\alpha z_1+1}\right)^3. \end{cases} \tag{12}$$

According to the calculation in based on graphical analysis and the Descartes' theorem we have the following result:

Lemma 4.1 Let $\mathcal{K}$ be a number of the positive solutions of (12). Then the following assertions hold:

$$\mathcal{K} = \begin{cases} 3, \text{ if } \alpha \in (0;\ \alpha_{c_1}] \cup [\alpha_{c_2};\ \infty)\,, \\ 1, \text{ if } \alpha \in (\alpha_{c_1};\ \alpha_{c_2}). \end{cases} \tag{13}$$

where $\alpha_{c_1} \approx 0.44, \alpha_{c_2} \approx 2.3$.

Proof: See Appendix 2.

Using Lemma 4.1 we obtain the following theorem:

Theorem 4.1 Let $k = 3$. For the Ising model constructed with rule **(B)** on the Cayley tree, the following holds true:

(1) if $\alpha \in (0;\ \alpha_{c_1}] \cup [\alpha_{c_2};\ \infty)$, then there are three Gibbs measures, moreover, two of them are non-translation-invariant Gibbs measures;
(2) if $\alpha \in (\alpha_{c_1};\ \alpha_{c_2})$, then there exists unique translation-invariant Gibbs measure.

Remark 4.1 The translation-invariant measure corresponding to $(z_1, z_2) = (1, 1)$ found in Theorem 4.1 was studied [1,8,19].

Remark 4.2 The measures found by rules **(A)** and **(B)** are neither periodic nor weakly periodic, they are new Gibbs measures (see in [5,31–35]).

5. Free energy for the Gibbs measures constructed by rules (A) and (B)

In the section, we calculate the free energy for the measures found in the previous sections. Free energy plays an important role in the fields of chemistry and physics, being used to determine the occurrence of processes and reactions, energy transfer between systems, and their energetic stability.

The following theorem gives a formula of the free energy.

Theorem 5.1 [21] For boundary conditions satisfying (4), the free energy is given by the formula

$$F(\beta, h) = -\lim_{n\to\infty} \frac{1}{\mid V_n \mid} \sum_{x\in V_n} a(x), \tag{14}$$

where

$$a(x) = \frac{1}{2\beta} \ln\left[4\cosh(h_x - \beta J) \cdot \cosh(h_x + \beta J)\right].$$

Theorem 5.2 The free energy of the Gibbs measures which are defined by rule **(A)** for the Ising model on the Cayley tree of order three is defined the following formula:

$$F_A(\beta, h_A) = -\frac{3}{4\beta} \ln(2\cosh(\beta J)), \tag{15}$$

where h_A are constructed according to the rule **(A)**.

Proof Let $\mathcal{A} \subset V$. Denote by $\mid \mathcal{A}(h_{A_i}) \mid$ the number of h_{A_i} in the set $\mathcal{A}$, where $i = 0, 1, 2$ and h_{A_i} is defined by rule **(A)**. Firstly, we calculate $\mid W_{A_n}(h_{A_0}) \mid$, $\mid W_{A_n}(h_{A_1}) \mid$, $\mid W_{A_n}(h_{A_2}) \mid$ (see Fig. 1). W_{A_n} are constructed according to the rule **(A)**. After some calculations, we have

$$\begin{aligned} \mid W_{A_n}(h_{A_0}) \mid &= \begin{cases} 3^n - 3^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 3^n - 3^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases} \\ \mid W_{A_n}(h_{A_1}) \mid &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ 3^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases} \\ \mid W_{A_n}(h_{A_2}) \mid &= \begin{cases} 3^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

It is clear that

$$\mid V_{A_n}(h_{A_i}) \mid = \sum_{k=0}^{n} \mid W_{A_k}(h_{A_i}) \mid, \ \ i = 1, 2. \tag{16}$$

Using (16) we find the following:

$$
\begin{aligned}
\mid V_{A_n}(h_{A_0}) \mid &= \begin{cases} A_n, \ n \ \text{ is even}, \\ B_n, \ n \ \text{ is odd}, \end{cases} \\
\mid V_{A_n}(h_{A_1}) \mid &= \begin{cases} C_n, \text{if } \ n \ \text{ is even}, \\ D_n, \ \text{if } \ n \ \text{ is odd}, \end{cases} \\
\mid V_{A_n}(h_{A_2}) \mid &= \begin{cases} E_n, \text{if } \ n \ \text{ is even}, \\ F_n, \ \text{if } \ n \ \text{ is odd}, \end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
A_n &= \frac{9^{\frac{n+2}{2}}-1}{8} - \frac{3^{\frac{n+2}{2}}-1}{2}, \\
B_n &= \frac{3\cdot(9^{\frac{n+1}{2}}-1)}{8} - \frac{3^{\frac{n+1}{2}}-1}{2}, \\
C_n &= \frac{3}{2}(3^{\frac{n}{2}}-1)+1, D_n = 0, E_n = 0, \\
F_n &= \frac{1}{2}\cdot(3^{\frac{n+1}{2}}-1).
\end{aligned}
$$

According to (14), we have

$$
\begin{aligned}
F_A(\beta, h_A) &= \\
&= -\frac{1}{2\beta}\lim_{n\to\infty}\frac{2}{3^{n+1}-1}\begin{cases} A_n \ln[4\cosh(-\beta J)\cdot\cosh(\beta J)], \ \text{if } \ n \text{ is even} \\ B_n \ln[4\cosh(-\beta J)\cdot\cosh(\beta J)], \text{if } \ n \text{ is odd} \end{cases} \\
&- \frac{1}{2\beta}\lim_{n\to\infty}\frac{2}{3^{n+1}-1}\begin{cases} C_n \ln[4\cosh(h_{A_1}-\beta J)\cdot\cosh(h_{A_1}+\beta J)], \ \text{if } \ n \text{ is even} \\ D_n \ln[4\cosh(h_{A_1}-\beta J)\cdot\cosh(h_{A_1}+\beta J)], \ \text{if } n \text{ is odd} \end{cases} \\
&- \frac{1}{2\beta}\lim_{n\to\infty}\frac{2}{3^{n+1}-1}\begin{cases} E_n \ln[4\cosh(h_{A_2}-\beta J)\cdot\cosh(h_{A_2}+\beta J)], \ \text{if } \ n \text{ is even} \\ F_n \ln[4\cosh(h_{A_2}-\beta J)\cdot\cosh(h_{A_2}+\beta J)], \ \text{if } n \text{ is odd} \end{cases} \\
&= -\frac{1}{2\beta}\cdot\frac{3}{4}\ln[4\cosh(-\beta J)\cdot\cosh(\beta J)] = -\frac{3}{4\beta}\ln[2\cosh(\beta J)].
\end{aligned}
$$

Similarly, we can prove the following theorem for the Gibbs measures which are defined by rule **(B)**.

Theorem 5.3 The free energy of the Gibbs measures for the Ising model, defined by rule **(B)** on the Cayley tree of order three, equals

$$
F_B(\beta, h_B) = -\frac{3}{4\beta}\ln(2\cosh(\beta J)), \tag{17}
$$

where h_B is constructed according to the rule **(B)**.

Remark 5.1 a) Due to Theorems 5.2 and 5.3, we have

$$
F_A(\beta, h_A) = F_B(\beta, h_B).
$$

Moreover, we also show that

$$
F_A(\beta, h_A) = F_B(\beta, h_B) \neq F_{TI}(\beta, 0),
$$

where $F_{TI}(\beta, 0) = -\frac{1}{\beta}\ln(2\cosh(\beta J))$ is the free energy of the translation-invariant (TI) boundary condition [21].

b) In [21] it is shown that

$$
F_{WP}(\beta, h) < F_{TI}(\beta, 0).
$$

In the present paper, we show that

$$
F_{TI}(\beta, 0) < F_A(\beta, h_A) = F_B(\beta, h_B).
$$

It implies that the free energy corresponding to the rule **(A)** differs from the free energy corresponding to the weakly periodic b.c., more precisely,

$$
F_{WP}(\beta, h) < F_{TI}(\beta, 0) < F_A(\beta, h_A) = F_B(\beta, h_B).
$$

Corollary 5.1 The free energy of the Gibbs measures which are defined by rules **(A)** and **(B)** for the Ising model on the Cayley tree of order three is defined as follows::

$$F_A(\beta,h_A)=F_B(\beta,h_B)=\frac{3}{4}F_{TI}(\beta,0).$$

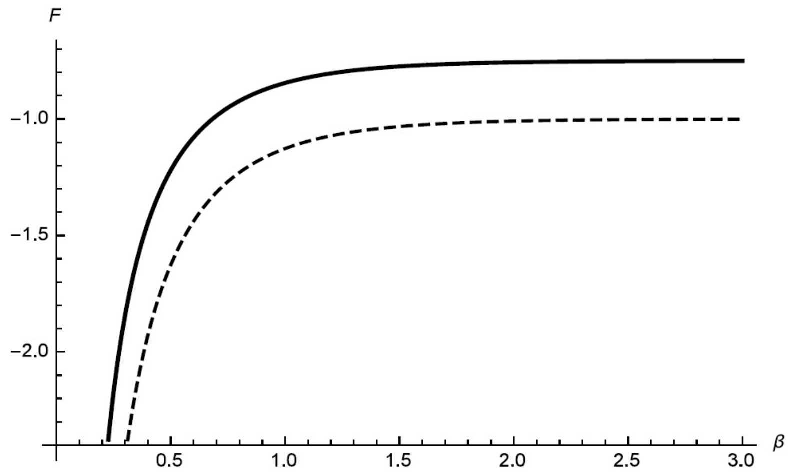

FIG. 4. Free energy $F_{TI}(\beta,0)$ (dotted line) and free energy built according to rules **(A)** and **(B)** i.e $F_A(\beta,h_A)$ = $F_B(\beta,h_B)$(solid line). Here $J=1$ and $k=3$

6. Entropy for the Gibbs measures constructed by rules (A) and (B)

In this section, we calculate the entropy for rules **(A)** and **(B)**. The concept of entropy is crucial to understanding processes in both physics and chemistry. In physics, it expresses the energetic state and disorder of systems, while in chemistry, it is used to determine whether reactions are spontaneous or non-spontaneous. The increase in entropy is the main trend observed in natural processes, indicating changes in energy and the efficiency of movements between systems.

The entropy is found by the following expression (see in [21]):

$$S(\beta,h)=-\frac{dF(\beta,h)}{dT}. \tag{18}$$

Theorem 6.1 The entropy of the Gibbs measures which are given by rule **(A)** for the Ising model on the Cayley tree of order three is defined the following formula:

$$S_A(\beta,h_A)=\frac{3}{4}\cdot\left(\ln(2\cosh(J\beta))-J\beta\tanh(J\beta)\right). \tag{19}$$

Proof We calculate the entropy for the Gibbs measures constructed according to rule **(A)**

$$\begin{aligned} S_A(\beta,h_A)&=-\frac{dF(\beta,h_A)}{dT}=-\frac{d(-\frac{3}{4}\ln 2\cosh\beta J)}{dT}=\\ &=\frac{3}{4}\ln(e^{\frac{J}{T}}+e^{-\frac{J}{T}})-\frac{3}{4}\cdot\frac{J}{T}\cdot\frac{e^{\frac{J}{T}}-e^{-\frac{J}{T}}}{e^{\frac{J}{T}}+e^{-\frac{J}{T}}}=\\ &=\frac{3}{4}\cdot\left(\ln(2\cosh(J\beta))-J\beta\tanh(J\beta)\right). \end{aligned} \tag{20}$$

Remark 6.1 In [21], the entropy of the (TI) boundary comdishihe the following formula for $h=0$

$$S_{TI}(\beta,0)=\ln(2\cosh(J\beta))-J\beta\tanh(J\beta). \tag{21}$$

We have the following expression according to Theorem 6.1

$$S_A(\beta,h_A)=\frac{3}{4}\cdot\left(\ln(2\cosh(J\beta))-J\beta\tanh(J\beta)\right),$$

i.e. the entropy of Gibbs measures constructed according to rule **(A)** differs from $S_{TI}(\beta,0)$.

The entropy calculation for Gibbs measures associated with rule **(B)** follows the same procedure as for rule **(A)**.

Theorem 6.2 The entropy of the Gibbs measures for the Ising model defined by rule **(B)** on the Cayley tree of order three, equals

$$S_B(\beta,h_B)=\frac{3}{4}\cdot\left(\ln(2\cosh(J\beta))-J\beta\tanh(J\beta)\right). \tag{22}$$

Remark 6.2 In [21] and according to Theorem 6.2, we get the following

$$S_B(\beta,h_B)=\frac{3}{4}\cdot\left(\ln(2\cosh(J\beta))-J\beta\tanh(J\beta)\right),$$

i.e. the entropy of the calculated according to the **(B)** is different from the Gibbs measures $S_{TI}(\beta, 0)$.

Corollary 6.1 The following relation is valid between the entropy of the Gibbs measures constructed according to **(A)** and **(B)** and the entropy $S_{TI}(\beta, 0)$ of the Gibbs measures:

$$S_A(\beta, h_A) = S_B(\beta, h_B) = \frac{3}{4} S_{TI}(\beta, 0).$$

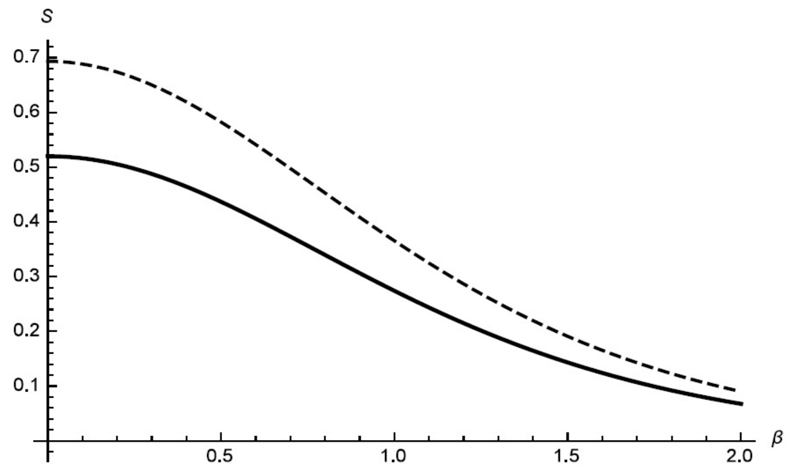

FIG. 5. Entropy $S_{TI}(\beta, 0)$ (dotted line) and entropy built according to rules **(A)** and **(B)**, i.e $S_A(\beta, h_A)$ = $S_B(\beta, h_B)$(solid line). Here $J = 1$ and $k = 3$

Appendix 1

Let us now explain how to calculate the proof of Lemma 3.1.

Proof Solving system (9) reduces to analyzing the following equation

$$z_1 = \frac{\left(\frac{z_1+\alpha}{\alpha\cdot z_1+1}\right)^3 + \alpha}{\alpha \cdot \left(\frac{z_1+\alpha}{\alpha\cdot z_1+1}\right)^3 + 1}. \tag{23}$$

Simplifying (23), we have

$$(z_1 - 1) \cdot (z_1 + 1) \cdot ((\alpha^3 + \alpha) \cdot {z_1}^2 + (-\alpha^4 + 6 \cdot \alpha^2 - 1) \cdot z_1 + \alpha^3 + \alpha) = 0. \tag{24}$$

It follows $z_1^{(0)} = 1$ or

$$(\alpha^3 + \alpha) \cdot {z_1}^2 + (-\alpha^4 + 6 \cdot \alpha^2 - 1) \cdot z_1 + \alpha^3 + \alpha = 0. \tag{25}$$

According to (9), every positive value of z_1 corresponds to a positive value of z_2. Therefore, we only need to focus on the positive solutions of equation (25). Equation (25) can be rearranged as follows:

$$\alpha \cdot (\alpha^2 + 1) z_1^2 - (\alpha^4 - 6 \cdot \alpha^2 + 1) z_1 + \alpha \cdot (\alpha^2 + 1) = 0. \tag{26}$$

Denote by D the discriminant of the equation (26), i.e.

$$D = (\alpha^4 - 6 \cdot \alpha^2 + 1)^2 - 4 \cdot (\alpha^3 + \alpha)^2.$$

Case 1. Equation (26) have two positive solutions, i.e.

$$\begin{cases} D > 0, \\ \alpha^4 - 6 \cdot \alpha^2 + 1 > 0. \end{cases} \tag{27}$$

After solving the inequalities, we find that

$$\alpha \in (0; 2 - \sqrt{3}) \cup (2 + \sqrt{3}; \infty),$$

the equation (26) has two positive solutions

$$z_1^{(1,2)} = \frac{\alpha^4 - 6 \cdot \alpha^2 + 1 \pm \sqrt{\alpha^8 - 16 \cdot \alpha^6 + 30 \cdot \alpha^4 - 16 \cdot \alpha^2 + 1}}{2 \cdot \alpha \cdot (\alpha^2 + 1)},$$

Inserting these solutions into system (9), we obtain two corresponding solutions

$$z_2^{(1,2)} = \left(\frac{3\alpha^4 - 4\alpha^2 + 1 \pm \sqrt{\alpha^8 - 16\alpha^6 + 30\alpha^4 - 16\alpha^2 + 1}}{\alpha(\alpha^4 - 4\alpha^2 + 3 \pm \sqrt{\alpha^8 - 16\alpha^6 + 30\alpha^4 - 16\alpha^2 + 1})}\right)^3.$$

It is clear that solutions $z_1^{(1,2)}$ and $z_2^{(1,2)}$ are positive.

Case 2. Equation (26) has a unique positive solution, i.e.

$$\begin{cases} D = 0, \\ \alpha^4 - 6 \cdot \alpha^2 + 1 > 0. \end{cases} \tag{28}$$

The equation $D = 0$ yields that $\alpha_{1,2} = \pm 1$, $\alpha_{3,4} = 2 \pm \sqrt{3}$ and $\alpha_{5,6} = -2 \pm \sqrt{3}$. Since $\alpha \in (0,\, 1) \cup (1,\, +\infty)$, we consider the cases $\alpha_{3,4} = 2 \pm \sqrt{3}$. According to the inequality in (28), it is sufficient to consider the case $\alpha_{3,4} = 2 \pm \sqrt{3}$. If $\alpha_{3,4} = 2 \pm \sqrt{3}$ then we get the translation-invariant solution $(z_1, z_2) = (1, 1)$.

Case 3. Let us assume that (26) does not have any positive solution, i.e.

$$D < 0, \tag{29}$$

or

$$\begin{cases} D \geq 0, \\ \alpha^4 - 6 \cdot \alpha^2 + 1 \leq 0. \end{cases} \tag{30}$$

It means that if $\alpha \in (2 - \sqrt{3}; 1) \cup (1; 2 + \sqrt{3})$, then the equation (26) does not have any positive solution.

Appendix 2

The proof of Lemma 4.1 is found using this basic idea using the following algebraic substitutions.

Proof Solving system (12) reduces to analyzing the following equation

$$z_2 = \left(\frac{\left(\frac{z_2+\alpha}{\alpha \cdot z_2+1}\right)^2 + \alpha}{\alpha \cdot \left(\frac{z_2+\alpha}{\alpha \cdot z_2+1}\right)^2 + 1} \right)^3. \tag{31}$$

Simplifying the last equation, we receive

$$\begin{aligned}
&(z_2 - 1) \cdot (\alpha + 1)^3 \cdot (\alpha^3 \cdot z_2^6 - (\alpha^6 - 3 \cdot \alpha^5 + 6 \cdot \alpha^4 - 14 \cdot \alpha^3 + 6 \cdot \alpha^2 - 3 \cdot \alpha + 1) \cdot z_2^5 - \\
&-(\alpha^6 + 3 \cdot \alpha^5 - 9 \cdot \alpha^4 - 5 \cdot \alpha^3 - 9 \cdot \alpha^2 + 3 \cdot \alpha + 1) \cdot z_2^4 - \\
&-(\alpha^6 + 6 \cdot \alpha^5 - 15 \cdot \alpha^4 - 4 \cdot \alpha^3 - 15 \cdot \alpha^2 + 6 \cdot \alpha + 1) \cdot z_2^3 - \\
&-(\alpha^6 + 3 \cdot \alpha^5 - 9 \cdot \alpha^4 - 5 \cdot \alpha^3 - 9 \cdot \alpha^2 + 3 \cdot \alpha + 1) \cdot z_2^2 - \\
&-(\alpha^6 - 3 \cdot \alpha^5 + 6 \cdot \alpha^4 - 14 \cdot \alpha^3 + 6 \cdot \alpha^2 - 3 \cdot \alpha + 1) \cdot z_2 + \alpha^3) = 0.
\end{aligned} \tag{32}$$

It follows that $z_2^{(0)} = 1$ or

$$\begin{aligned}
&\alpha^3 \cdot z_2^6 - (\alpha^6 - 3 \cdot \alpha^5 + 6 \cdot \alpha^4 - 14 \cdot \alpha^3 + 6 \cdot \alpha^2 - 3 \cdot \alpha + 1) \cdot z_2^5 - \\
&-(\alpha^6 + 3 \cdot \alpha^5 - 9 \cdot \alpha^4 - 5 \cdot \alpha^3 - 9 \cdot \alpha^2 + 3 \cdot \alpha + 1) \cdot z_2^4 - \\
&-(\alpha^6 + 6 \cdot \alpha^5 - 15 \cdot \alpha^4 - 4 \cdot \alpha^3 - 15 \cdot \alpha^2 + 6 \cdot \alpha + 1) \cdot z_2^3 - \\
&-(\alpha^6 + 3 \cdot \alpha^5 - 9 \cdot \alpha^4 - 5 \cdot \alpha^3 - 9 \cdot \alpha^2 + 3 \cdot \alpha + 1) \cdot z_2^2 - \\
&-(\alpha^6 - 3 \cdot \alpha^5 + 6 \cdot \alpha^4 - 14 \cdot \alpha^3 + 6 \cdot \alpha^2 - 3 \cdot \alpha + 1) \cdot z_2 + \alpha^3 = 0.
\end{aligned} \tag{33}$$

Due to (12), each positive solution z_2 defines a positive solution z_1. Thus, it is sufficient to consider positive solution of (33). We make a substitution $\xi = z_2 + \dfrac{1}{z_2}$ and form the following equation

$$\begin{aligned}
&\alpha^3 \cdot \xi^3 + (-\alpha^6 + 3 \cdot \alpha^5 - 6 \cdot \alpha^4 + 14 \cdot \alpha^3 - 6 \cdot \alpha^2 + 3 \cdot \alpha - 1) \cdot \xi^2 + \\
&+(-\alpha^6 - 3 \cdot \alpha^5 + 9 \cdot \alpha^4 + 2 \cdot \alpha^3 + 9 \cdot \alpha^2 - 3 \cdot \alpha + 1) \cdot \xi + \\
&+(\alpha^2 + 1) \cdot (\alpha^4 - 12 \cdot \alpha^3 + 26 \cdot \alpha^2 - 12 \cdot \alpha + 1) = 0.
\end{aligned}$$

In the last cubic equation, $D < 0$, we know that when $D < 0$, the cubic equation has one real root. Let us find this real root. In order to simplify this solution, we make the following substitution

$$t = (8 \cdot \alpha^{18} - 72 \cdot \alpha^{17} + 360 \cdot \alpha^{15} - 1380 \cdot \alpha^{15} + 4320 \cdot \alpha^{14} -$$
$$-11016 \cdot \alpha^{13} + 22944 \cdot \alpha^{12} - 39132 \cdot \alpha^{11} + 54288 \cdot \alpha^{10} -$$
$$-60712 \cdot \alpha^{9} + 54504 \cdot \alpha^{8} - 39564 \cdot \alpha^{7} + 23952 \cdot \alpha^{6} - 11448 \cdot \alpha^{5} +$$
$$+12 \cdot (-15 \cdot \alpha^{24} + 252 \cdot \alpha^{23} - 2088 \cdot \alpha^{22} + 11568 \cdot \alpha^{21} - 48798 \cdot \alpha^{20} +$$
$$+167328 \cdot \alpha^{19} - 480252 \cdot \alpha^{18} + 1165176 \cdot \alpha^{17} - 2395593 \cdot \alpha^{16} +$$
$$+4179936 \cdot \alpha^{15} - 62049242 \cdot \alpha^{14} + 7861968 \cdot \alpha^{13} - 8527860 \cdot \alpha^{12} +$$
$$+7931520 \cdot \alpha^{11} - 6324768 \cdot \alpha^{10} + 4318848 \cdot \alpha^{9} - 2521269 \cdot \alpha^{8} +$$
$$+1254636 \cdot \alpha^{7} - 527532 \cdot \alpha^{6} + 187272 \cdot \alpha^{5} - 54954 \cdot \alpha^{4} + 13080 \cdot \alpha^{3} -$$
$$-2412 \cdot \alpha^{2} + 288 \cdot \alpha - 15)^{\frac{1}{2}} + 4536 \cdot \alpha^{4} - 1452 \cdot \alpha^{3} + 360 \cdot \alpha^{2} - 72 \cdot \alpha + 8)^{\frac{1}{3}}.$$

The result is the following expression:

$$\xi = (t + 4 \cdot \alpha^{3} \cdot (\alpha^{12} - 6 \cdot \alpha^{11} + 21 \cdot \alpha^{10} - 61 \cdot \alpha^{9} + 141 \cdot \alpha^{8} - 237 \cdot \alpha^{7} +$$
$$+282 \cdot \alpha^{6} - 237 \cdot \alpha^{5} + 141 \cdot \alpha^{4} - 67 \cdot \alpha^{3} + 21 \cdot \alpha^{2} - 6 \cdot \alpha + 1))/$$
$$(6 \cdot \alpha^{6} \cdot t + 2 \cdot (\alpha^{6} - 3 \cdot \alpha^{5} + 6 \cdot \alpha^{4} - 14 \cdot \alpha^{3} + 6 \cdot \alpha^{2} - 3 \cdot \alpha + 1)).$$

By the substitution, we take

$$z_2 + \frac{1}{z_2} = (t + 4 \cdot \alpha^{3} \cdot (\alpha^{12} - 6 \cdot \alpha^{11} + 21 \cdot \alpha^{10} - 61 \cdot \alpha^{9} + 141 \cdot \alpha^{8} - 237 \cdot \alpha^{7} +$$
$$+282 \cdot \alpha^{6} - 237 \cdot \alpha^{5} + 141 \cdot \alpha^{4} - 67 \cdot \alpha^{3} + 21 \cdot \alpha^{2} - 6 \cdot \alpha + 1))/$$
$$(6 \cdot \alpha^{6} \cdot t + 2 \cdot (\alpha^{6} - 3 \cdot \alpha^{5} + 6 \cdot \alpha^{4} - 14 \cdot \alpha^{3} + 6 \cdot \alpha^{2} - 3 \cdot \alpha + 1)).$$

The last one yields that

$$z_2^2 + ((-4 \cdot \alpha^{15} + 24 \cdot \alpha^{14} - 84 \cdot \alpha^{13} + 244 \cdot \alpha^{12} - 564 \cdot \alpha^{11} +$$
$$+948 \cdot \alpha^{10} - 1128 \cdot \alpha^{9} + 948 \cdot \alpha^{8} - 564 \cdot \alpha^{7} + 268 \cdot \alpha^{6} -$$
$$-84 \cdot \alpha^{5} + 24 \cdot \alpha^{4} - 4 \cdot \alpha^{3} - t)/(2 \cdot \alpha^{6} - 6 \cdot \alpha^{5} + 12 \cdot \alpha^{4} - 28 \cdot \alpha^{3} +$$
$$+6 \cdot \alpha^{2} \cdot t + 12 \cdot \alpha^{2} - 6 \cdot \alpha + 2)) \cdot z_2 + 1 = 0.$$

We find the discriminant of the last quadratic equation

$$D = ((-4 \cdot \alpha^{15} + 24 \cdot \alpha^{14} - 84 \cdot \alpha^{13} + 244 \cdot \alpha^{12} -$$
$$-564 \cdot \alpha^{11} + 948 \cdot \alpha^{10} - 1128 \cdot \alpha^{9} + 948 \cdot \alpha^{8} -$$
$$-564 \cdot \alpha^{7} + 268 \cdot \alpha^{6} - 84 \cdot \alpha^{5} + 24 \cdot \alpha^{4} - 4 \cdot \alpha^{3} -$$
$$-t)/(2 \cdot \alpha^{6} - 6 \cdot \alpha^{5} + 12 \cdot \alpha^{4} - 28 \cdot \alpha^{3} + 6 \cdot \alpha^{2} \cdot t +$$
$$+12 \cdot \alpha^{2} - 6 \cdot \alpha + 2))^{2} - 4.$$

We introduce the notation

$$p = (-4 \cdot \alpha^{15} + 24 \cdot \alpha^{14} - 84 \cdot \alpha^{13} + 244 \cdot \alpha^{12} - 564 \cdot \alpha^{11} +$$
$$+948 \cdot \alpha^{10} - 1128 \cdot \alpha^{9} + 948 \cdot \alpha^{8} - 564 \cdot \alpha^{7} + 268 \cdot \alpha^{6} -$$
$$-84 \cdot \alpha^{5} + 24 \cdot \alpha^{4} - 4 \cdot \alpha^{3} - t)/(2 \cdot \alpha^{6} - 6 \cdot \alpha^{5} + 12 \cdot \alpha^{4} - 28 \cdot \alpha^{3} +$$
$$+6 \cdot \alpha^{2} \cdot t + 12 \cdot \alpha^{2} - 6 \cdot \alpha + 2).$$

As a result

$$z_2^2 + p \cdot z_2 + 1 = 0. \tag{34}$$

The discriminant of (34) is as follows:

$$D = p^2 - 4.$$

Case 1. Let (34) have two positive solutions, i.e.

$$\begin{cases} D > 0, \\ p < 0. \end{cases} \tag{35}$$

Case 2. Let (34) have a unique positive solution, i.e.

$$\begin{cases} D = 0, \\ p < 0. \end{cases} \tag{36}$$

Case 3. Assume that (34) does not have any positive solution, i.e.

$$D < 0, \tag{37}$$

or

$$\begin{cases} D \geq 0, \\ p \geq 0. \end{cases} \tag{38}$$

References

[1] Bleher P.M. and Ganikhodjaev N.N. On pure phases of the Ising model on the Bethe lattice. *Theory Probab. Appl.*, 1990, **35**, P. 216–227.

[2] Rozikov U.A. *Gibbs measures on Cayley trees*, (Singapore: World Scientific), 2013.

[3] Zachary S. Countable state space Markov random fields and Markov chains on trees, *Ann.Prob.*, 1983, **11**, P. 894–903.

[4] Ganikhodjaev N.N., Rozikov U.A. A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree. *Theor. Math. Phys.*, 1997, **111**, P. 480–486.

[5] Rozikov U.A. Partition structures of the group representation of the Cayley tree into cosets by finite-index normal subgroups and their applications to the description of periodic Gibbs distributions. *Theor. Math. Phys.*, 1997, **112**, P. 929–933.

[6] Rozikov U.A. Describtion uncountable number of Gibbs measures for inhomogeneous Ising model. *Theor. Math. Phys.*, 1999, **118**, P. 95–104.

[7] Bleher P.M., Ganikhodjaev N.N. On pure phases of the Ising model on the Bethe lattice. *Theory Probab.Appl.*, 1990, **35**, P. 216–227.

[8] Rozikov U.A., Rahmatullaev M.M. Description of weakly periodic Gibbs measures of the Ising model on the Cayley tree. *Theor. Math. Phys.*, 2008, **156**, P. 1218–1227.

[9] Rozikov U.A., Rahmatullaev M.M. On weak periodic Gibbs measures of the Ising model on a Cayley tree, Dokl. AN Uzbekistan, 2008, **4**, P. 12–15.

[10] Rahmatullaev M.M. Ising model on trees: (k_0)- non translation-invariant Gibbs measures. *Journal of Physics: Conference Series*, 2017, **819**, P. 012019.

[11] Rakhmatullaev M.M., Samijonova N.D. Translation-invariant p-adic quasi Gibbs Measures for the Potts Model with an external field on the Cayley Tree. *Nanosystems: Physics, Chemistry, Mathematics*, 2025, **16**(2), P. 164–175.

[12] Rahmatullaev M.M., Khakimov O.N., Tukhtabaev A.M. A p-adic generalized Gibbs measure for the Ising model on a Cayley tree. *Theoretical and Mathematical Physics*, 2019, **201**(1), P. 1521–1530.

[13] Rahmatullaev M.M., Tukhtabaev A.M. Non periodic p-adic generalized Gibbs measure for Ising model, p-Adic Numbers, Ultrametric Analysis and Applications, 2019, **11**(4), P. 319–327.

[14] Rahmatullaev M.M., Tukhtabaev A.M. Some non-periodic p-adic generalized Gibbs measures for the Ising model on a Cayley tree of order k. *Mathematical Physics, Analysis and Geometry*, 2023, **26**(22).

[15] Rozikov U.A., Rakhmatullaev M.M. Ising model on Cayley tree: a new class of Gibbs measures and their comparison with known ones. *J. Stat. Mech.*, 2017, P. 093205.

[16] Mavlonov I.M., Sattarov A.M., Karimova S.A., Haydarov F.H. On solutions to nonlinear integral equation of the Hammerstein type and its applications to Gibbs measures for continuous spin systems. *Nanosystems: Physics, Chemistry, Mathematics*, 2024, **15**(1), P. 23–30.

[17] Rahmatullaev M.M., Egamov D.O. Translation-invariant Gibbs measures for the mixed spin-1/2 and spin-1 Ising model with an external field on a Cayley tree. *Nanosystems: Physics, Chemistry, Mathematics*, 2024, **15**(5), P. 576–585.

[18] Friedli S., Velenik Y. A Concrete Mathematical Introduction, Statistical Mechanics of Lattice Systems, Cambridge University Press, 2017.

[19] Georgii H.O. Gibbs Measures and Phase Transitions.de Gryuter,Berlin (1998)

[20] Feynman R.P. There's plenty of room at the bottom, Engineering and Science, 1960, **5**(23), P. 22–36.

[21] Gandolfo D., Rahmatullaev M.M., Rozikov U.A. Ruiz J. On Free Energies of the Ising Model on the Cayley Tree. *J. Stat Phys*, 2013, **150**, P. 1201–1217.

[22] Semkin S.V., Smagin V.P., Tarasov V.S. Frustrations in a Diluted Ising Magnet on the Bethe Lattice. *J. Exp. Theor. Phys.*, 2022, **134**, P. 713–719.

[23] Semkin S.V., Smagin V.P. Free energy, entropy, and magnetization of a onedimensional Ising model of a diluted magnet. *Theor. Math. Phys.*, 2023, **217**, P. 17881794.

[24] Mukhamedov F., Akin H., and Khakimov O. Gibbs measures and free energies of IsingVannimenus model on the Cayley tree. *Journal of Statistical Mechanics: Theory and Experiment*, 2017.

[25] Akin H. A novel computational method of the free energy for the Ising model on Cayley tree of order three. *Chin. J. Phys.*, 2022, **77**, P. 2276–2287.

[26] Akin H. Calculation of some thermodynamic quantities for the Ising model on a kth order Cayley tree. *Physica B*, 2023, **662**.

[27] Akin H. Investigation of thermodynamic properties of mixed-spin (2,1/2) Ising and Blum-Capel models on a Cayley tree. Chaos, Solitons, Fractals, 2024, **184**, P. 114980.

[28] Akin H. Calculation of the free energy of the Ising model on a Cayley tree via the self-similarity method. *Axioms*, 2022, **11**, P. 703.

[29] Akin H. Quantitative behavior of (1,1/2)-MSIM on a Cayley tree. *Chin. J. Phys.*, 2023, **83**, P. 501–514.

[30] Akin H. The classification of disordered phases of mixed spin (2,1/2) Ising model and the chaoticity of the corresponding dynamical system, Chaos, Solitons, Fractals, 2023, **167**.

[31] Rahmatullaev M.M., Burxonova Z.A. Constructive Gibbs measures for the Ising model on the Cayley tree. *Reports of Mathematical Physics*, 2024, **93**(3), P. 361–370.

[32] Rozikov U.A., Rahmatullaev M.M. Weakly periodic ground states and Gibbs measures for the Ising model with competing interactions on the Cayley tree. *Theor. Math. Phys.*, 2009, **160**, P. 1291–1299.

[33] Rahmatullaev M.M., Dekhkanov J. Existence weakly periodic Gibbs measures for the Ising model on the Cayley tree of order two. *Vladikavkaz Mathematical Journal*, 2021, **23**(4), P. 77–88.

[34] Rahmatullaev M.M. On weakly periodic states and Gibbs measures of the Ising model on the Cayley tree, Fan, Tashkent, 2024. (in Russian)

[35] Preston C. Gibbs States on Countable Sets. Cambridge University Press, London, 1974.

[36] Esfarjani K., Mansoori G.A. Statistical mechanical modeling and its application to nanosystems. *Handbook of Theoretical and Computational Nanotechnology*, 2006, II(14), P. 1-45.

[37] Hill L. *Thermodynamics of Small Systems*, W. A. Benjamin, New York, 1963.

[38] Hill T.L. A Different Approach to Nanothermodynamics. *Nano Letters*, **1**, 2001, P. 273–275.

[39] Mansoori G.A. Proceedings of the first Conference on Nanotechnology-The Next Industrial Revolution, 2002, **2**, P. 345.

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