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Original article

Fast forward of adiabatic quantum dynamics: an application to planar Dirac systems

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ABSTRACT We study a scheme of fast-forward adiabatic quantum dynamics of a $(2+1)$ Dirac particle. This scheme was originally proposed by Masuda and Nakamura. In this scheme, we include the adiabatic parameter that maintains the adiabatic motion of the particle and fast forward its motion by introducing a time scaling parameter. The fast forward adiabatic state is obtained by determining the regularization term and driving potential. We introduce the proposed method to the system with the Dirac particle using a $(2+1)$ dimension-time-dependent Dirac equation and obtain the regularization term, the driving scalar potential V_{FF} and the driving vector potential A_{FF}. By tuning the driving electric field, this method can accelerate the adiabatic dynamics of an electron as a Dirac particle trapped in the ground state in the plane xy and an electric field in the x direction and a constant magnetic field in the y direction. This acceleration will preserve the ground state of the wave function from the initial time to the final time.

KEYWORDS fast forward, adiabatic, planar Dirac

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1. Introduction

Attempting to control the dynamics of quantum systems is one of humanity's biggest dreams in this millennium. To achieve this, some theoretical frameworks have been developed. The theory currently being developed and widely used is an adiabatic quantum theorem [1–6], which allows us to maintain the quantum state during the evolution of the system. Adiabatic quantum dynamics is crucial in understanding how quantum systems evolve, especially in quantum computing and quantum annealing [7]. However, challenges such as small energy gaps and slow evolution times are significant hurdles that must be overcome for practical implementation. It takes a very long time to obtain the desired target state, which is less beneficial when manufacturing a massive product or designing large macroscopic systems. Some methods have been developed to solve this problem. The fast-forward method is one of the promising methods developed in many aspects of quantum systems [8–10]. Another method called a shortcut to adiabaticity (STA) has also attracted much attention [11–14]. A wide variety of techniques have been developed in terms of STA. Transitionless quantum driving [15, 16], and the STA method from the classical point of view [17–19]. The STA method has also been considered in term of open quantum systems [20]. The fast-forward scheme is developed to accelerate the adiabatic quantum dynamics by modifying the original Hamiltonian by introducing the adiabatic time parameter that goes very slow and tends to zero. Then, by obtaining the additional Hamiltonian to satisfy the Schrodinger equation, the adiabatic time parameter is replaced by the time scaling magnification factor that goes to infinity. Fast forward methods recently have been developed to accelerate adiabatic quantum dynamics of quantum spin systems [21–24], quantum tunneling [25, 26], stochastic heat engine [27] and also on how to accelerate the dynamics of particle which include non-equilibrium steady states of charged particle [28]. The fast-forward technique and STA method has a strong correlation and it has been also reported [29]. Some methods are considering the relativistic quantum dynamics that governed by the Dirac equation [30–32]. The technique of STA on a relativistic quantum system has been found by using time-rescaling methods [33]. Key features of Dirac dynamics include its relativistic nature, prediction of antiparticles, spin-1/2 particle behavior, and Lorentz invariance, making it essential for understanding the quantum dynamics of relativistic systems [34, 35]. In this study, we design a method to accelerate relativistic quantum dynamics using a fast-forward scheme on $(2+1)$ dimensional using the Dirac equation. By applying the fast forward method to the $(2+1)$ Dirac particle, we can derive the expression of driving energy in terms of driving scalar potential, driving vector potential, and driving electric field. These driving

energy can accelerate the adiabatic dynamics of an electron as a Dirac particle trapped in the ground state of the xy plane. The adiabatic acceleration that preserve the ground state of the wave function from the initial time to the final time, is essential to improve the unique properties of such two dimensional material like graphene. These techniques aim to tailor graphene's unique characteristics to specific applications, such as improving electrical conductivity, and electron mobility. Recently, this study has developed and attracted a lot of attention due to the ability to describe quasi-particles in graphene, a revolutionary material with a host of exceptional and remarkable properties [36, 37]. Graphene, a single layer of carbon atoms arranged in a two-dimensional honeycomb lattice. One of its most intriguing features is its ability to host massless Dirac fermions. This connection between graphene and Dirac particles arises due to the special nature of its electronic band structure [38]. Application of fast-forward approach gives one useful protocol to understand and control this two-dimensional honeycomb lattice which is important for their applications in nanoscale electronic sensors, heat conductive material, and energy harvesting devices [39]. In Section 2, we shall introduce a time-dependent $(2+1)$ Dirac equation. Section 3 considers about adiabatic dynamics of the Dirac equation. Sections 4 and 5 consider the fast forward of adiabatic dynamics of the Dirac particle and the example respectively. Section 6 is devoted to the conclusion.

2. Dirac equation

In this section, we briefly introduce and review the literature on the time-dependent Dirac equation. The (2+1) dimension Dirac equation is written as [40]

$$i\hbar\frac{\partial\Psi(x,y,t)}{\partial t} = H_D\Psi(x,y,t), \tag{1}$$

with Hamiltonian is defined as

$$H_D = (p_x + A_x)\sigma_x + (p_y + A_y)\sigma_y + mc^2\sigma_z + V_0\mathbb{I}, \tag{2}$$

where $\sigma_{x,y,z}$, $p_{x(y)}$, $A_{x(y)}$, V_0 are the Pauli matrices, momenta, vector potentials and scalar potentials, respectively. By substituting the above Hamiltonian, the time dependent Dirac equation can be rewritten as

$$i\hbar\frac{\partial\Psi(x,y,t)}{\partial t} = \left[(-i\hbar\partial_x + A_x)\sigma_x + (-i\hbar\partial_y + A_y)\sigma_y + mc^2\sigma_z + V_0\mathbb{I}\right]\Psi(x,y,t). \tag{3}$$

For convenience in notation, we denote prior wave functions (Ψ_1, Ψ_2). The matrix form of the Dirac equation is written as

$$i\hbar\begin{pmatrix}\frac{\partial\Psi_1}{\partial t}\\ \frac{\partial\Psi_2}{\partial t}\end{pmatrix} = \begin{pmatrix}mc^2 + V_0 & \pi_-\\ \pi_+ & -mc^2 + V_0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}, \tag{4}$$

where $\pi_\pm = \pi_x \pm i\pi_y$, and

$$\pi_{x(y)} = -i\hbar\frac{\partial}{\partial x(y)} + A_{x(y)}. \tag{5}$$

3. Regularization scheme

We begin by considering a Hamiltonian H_0 for the massless Dirac system in (2 + 1) dimensions. The Hamiltonian of this system is written as

$$H_0 = v_F\left(p_x + A_x\right)\sigma_x + v_F\left(p_y + A_y\right)\sigma_y - V_0\mathbb{I}, \tag{6}$$

with the effective Fermi velocity $v_F = c/300$, and p_x, p_y are the components of momentum, A_x, A_y are the components of the vector potential, and V_0 is the scalar potential. Meanwhile, there is the corresponding spinor Ψ_0 for the system that satisfies the adiabatic dynamics where there is a time-dependent parameter $R(t)$ that is slowly changing over time such that

$$R(t) = R_0 + \epsilon t. \tag{7}$$

Here $\epsilon \ll 1$. Such a spinor can be written as

$$\Psi_0 = e^{i\delta_n(t)}\begin{pmatrix}\phi_{1,n}(x,y,R(t))\\ \phi_{2,n}(x,y,R(t))\end{pmatrix}, \tag{8}$$

where δ_n is the dynamical phase.

To accelerate the adiabatic evolution with a slowly changing time-dependent parameter $R(t,\epsilon)$, a huge time scaling factor, denoted by α is necessary. Thus, we must create a scheme to regularize the system so that $\alpha\epsilon$ is finite. This procedure is used by modifying the wave function into an adiabatic wave function Ψ^{reg} as written in Eq. (8), and modifying

the potentials as

$$v_F A_x(x,y,R(t))\sigma_x \to v_F A_x(x,y,R(t))\sigma_x + \epsilon v_F W, \tag{9}$$
$$v_F A_y(x,y,R(t))\sigma_y \to v_F A_y(x,y,R(t))\sigma_y + \epsilon v_F W, \tag{10}$$
$$-V_0(x,y,R(t))\mathbb{I} \to -V_0(x,y,R(t))\mathbb{I} - \epsilon W, \tag{11}$$

where W is a 2×2 matrix, and its components $W_{ij} = W_{ij}(x,y,R(t))$.

Then, after modifying the potentials of system, we have a regularized Hamiltonian H^{reg} written as

$$H^{reg} = H_0(R(t)) + \epsilon\mathcal{H}, \tag{12}$$

where

$$\mathcal{H} = (2v_F - 1)W. \tag{13}$$

The governing dynamical equation for regularized system is

$$i\partial_t \Psi^{reg} = H^{reg}\Psi^{reg}. \tag{14}$$

By using Eq. (14) and collecting the $O(1)$, we have

$$H_0\Psi^{reg} = -\partial_t\delta_n\Psi^{reg}, \tag{15}$$

and $O(\epsilon)$ terms

$$i\partial_R\Psi^{reg} = (2v_F - 1)W\Psi^{reg}. \tag{16}$$

Here in the limit $\epsilon \to 0$, $\alpha \to \infty$, and $\epsilon\alpha \sim 1$, we can suppress the terms of $O(\epsilon^2)$. Eq. (15) results in the eigenvalue equation for the Hamiltonian H_0. This implies $E_n = -\partial_t\delta_n$. Meanwhile, we solve Eq. (16) to obtain the matrix W.

The components of the regularized spinor Ψ^{reg} and the matrix W are complex numbers. Expanding them into real and imaginary parts, we have

$$au_{11} + cu_{12} - bv_{11} - dv_{12} = -(2v_F - 1)^{-1}\partial_R b, \tag{17}$$
$$au_{21} + cu_{22} - bv_{21} - dv_{22} = -(2v_F - 1)^{-1}\partial_R d, \tag{18}$$
$$bu_{11} + du_{22} + av_{11} + cv_{12} = (2v_F - 1)^{-1}\partial_R a, \tag{19}$$
$$bu_{21} + du_{22} + av_{21} + cv_{22} = (2v_F - 1)^{-1}\partial_R c, \tag{20}$$

where

$$W = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + i\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \tag{21}$$
$$\Psi^{reg} = \begin{pmatrix} a \\ c \end{pmatrix} + i\begin{pmatrix} b \\ d \end{pmatrix}. \tag{22}$$

Eq. (21) is the main equation for the present work. Here, W has a role as a regularization term to fulfill the Dirac equation. By obtaining W, we can preserve the spinor state of the systems during the evolution of time. However, this will result in four freedoms of choice of solutions for W.

3.1. The solution of regularization matrix

The system of equations of our regularization matrix is given by

$$\begin{pmatrix} a & c & 0 & 0 & -b & -d & 0 & 0 \\ 0 & 0 & a & c & 0 & 0 & -b & -d \\ b & d & 0 & 0 & a & c & 0 & 0 \\ 0 & 0 & b & d & 0 & 0 & a & c \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} = \frac{1}{2v_F - 1}\begin{pmatrix} -\partial_R b \\ -\partial_R d \\ \partial_R a \\ \partial_R c \end{pmatrix}. \tag{23}$$

We find the row echelon form of the equation is

$$\begin{pmatrix} 1 & \frac{c}{a} & 0 & 0 & -\frac{b}{a} & -\frac{d}{a} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & \frac{c}{a} & 0 & 0 & -\frac{b}{a} & -\frac{d}{a} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} = \frac{1}{2v_F-1} \begin{pmatrix} -\frac{\partial_R b}{a} \\ \frac{a\partial_R a + b\partial_R b}{ad-bc} \\ -\frac{\partial_R d}{a} \\ \frac{a\partial_R c + b\partial_R d}{ad-bc} \end{pmatrix}. \quad (24)$$

From the above equation, we see that there are four free variables, which are $\{v_{11}, v_{12}, v_{21}, v_{22}\}$.

To solve the algebraic equation in Eq. (24), we should pick up only four unknown variables so that the number of equations and the number of variables are matched. Here we assume that all imaginary parts of W are zero, that is, $v_{ij} = 0$. So,

$$u_{11} = \frac{1}{2v_F-1}\left(-\frac{a\partial_R a + b\partial_R b}{ad-bc}\frac{c}{a} - \frac{\partial_R b}{a}\right), \quad (25)$$

$$u_{12} = \frac{1}{2v_F-1}\frac{a\partial_R a + b\partial_R b}{ad-bc}, \quad (26)$$

$$u_{21} = \frac{1}{2v_F-1}\left(-\frac{a\partial_R c + b\partial_R d}{ad-bc}\frac{c}{a} - \frac{\partial_R d}{a}\right), \quad (27)$$

$$u_{12} = \frac{1}{2v_F-1}\frac{a\partial_R c + b\partial_R d}{ad-bc}. \quad (28)$$

4. Fast-forward of adiabatic Dirac dynamics

We define the fast-forwarded spinor as

$$\Psi_{FF}(x,y,t) = \Psi^{reg}(x,y,R(\Lambda(t))) = \begin{pmatrix} \phi_1^{reg}(x,y,R(\Lambda(t))) \\ \phi_2^{reg}(x,y,R(\Lambda(t))) \end{pmatrix} e^{i\delta_n(\Lambda(t))} \quad (29)$$

while in this time $\delta_n(\Lambda(t)) = -\int_0^t \mathrm{d}s E_n(R(\Lambda(s)))$ in natural units. Here $\Lambda(t)$ is defined by

$$\Lambda(t) = \int_0^t \alpha(t')dt', \quad (30)$$

with the standard time t. $\alpha(t)$ is an arbitrary magnification time scale factor that satisfies $\alpha(0) = 1$, $\alpha(t) > 1 (0 < t < T_{FF})$ and $\alpha(t) = 1 (t \geq T_{FF})$. For a long final time T in the original adiabatic dynamics, we can consider the fast forward dynamics with a new time variable which reproduces the target state $\Psi_0^{(n)}(R(T))$ in a shorter final time T_{FF} defined by

$$T = \int_0^{T_{FF}} \alpha(t)dt. \quad (31)$$

The simplest expression for $\alpha(t)$ in the fast-forward range ($0 \leq t \leq T_{FF}$) is given by [9] as :

$$\alpha(t) = \bar{\alpha} - (\bar{\alpha} - 1)\cos(\frac{2\pi}{T_{FF}}t), \quad (32)$$

where $\bar{\alpha}$ is the mean value of $\alpha(t)$ given by $\bar{\alpha} = T/T_{FF}$, while the limit $\bar{\alpha} \to \infty$ and $\epsilon \to 0$ under the constraint of $\bar{\alpha}\epsilon \equiv \bar{v}$. Here $R(\Lambda(t))$ is given by [9]

$$R(\Lambda(t)) = R_0 + 2\bar{v}\left(\frac{t}{2} - \frac{T_{FF}\sin\left(\frac{2\pi t}{T_{FF}}\right)}{4\pi}\right). \quad (33)$$

We postulate that the fast-forwarded spinor satisfies the Dirac equation as follows

$$i\frac{\partial \Psi_{FF}}{\partial t} = \begin{pmatrix} m + V_{FF} + V_0^{reg} & \pi_-^{FF} \\ \pi_+^{FF} & -m + V_{FF} + V_0^{reg} \end{pmatrix} \Psi_{FF}, \tag{34}$$

where

$$V_0^{reg} = V_0 + \epsilon\tilde{W}, \tag{35}$$

$$\pi_+^{FF} = -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + A_x^0 + iA_y^0 + A_x^{FF} + iA_y^{FF}, \tag{36}$$

$$\pi_-^{FF} = -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + A_x^0 - iA_y^0 + A_x^{FF} - iA_y^{FF}. \tag{37}$$

By substituting Ψ_{FF} in Eq. (29) to Eq. (34), we see

$$i\frac{\partial \Psi_{reg}}{\partial t} = \begin{pmatrix} m + V_{FF} + V_0^{reg} & \pi_-^{FF} \\ \pi_+^{FF} & -m + V_{FF} + V_0^{reg} \end{pmatrix} \begin{pmatrix} \phi_1^{reg} \\ \phi_2^{reg} \end{pmatrix} \tag{38}$$

We apply the time rescaled t $\to \Lambda(t)$ as

$$i\frac{\partial \Psi_{reg}}{\partial \Lambda(t)} = \begin{pmatrix} m + V_0^{reg} & \pi_-^0 \\ \pi_+^0 & -m + V_0^{reg} \end{pmatrix} \Psi_{reg}, \tag{39}$$

where

$$\pi_\pm^0 = -i\frac{\partial}{\partial x} \pm \frac{\partial}{\partial y} + A_x^0 \pm iA_y^0 \tag{40}$$

By taking the time derivation of Eq. (39) and noting $d\Lambda(t) = \alpha(t)dt$, with the limit $\epsilon \to 0$ and $\alpha \to \infty$, $\alpha\epsilon \to v(t)$, we see (the detailed derivation given at Appendix)

$$\begin{pmatrix} V_{FF} & A_x^{FF} - iA_y^{FF} \\ A_x^{FF} + iA_y^{FF} & V_{FF} \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = v(t) \begin{pmatrix} W & W - iW \\ W + iW & W \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} \tag{41}$$

The electric and magnetic fields are related to the driving vector potential V_{FF} and the driving vector potential A_{FF} as $E_{FF} = -\frac{dA_{FF}}{dt} - \nabla V_{FF}$ and $B_{FF} = \nabla \times A_{FF}$, respectively.

5. Examples

We apply our calculation to a system that has a constant magnetic field perpendicular to the graphene plane in the positive direction of the z axis. This is the very well-known case of Landau levels [41]. We use their solution, for $n > 0$, written as:

$$\begin{pmatrix} \phi_1^n \\ \phi_2^n \end{pmatrix} = e^{iky} \begin{pmatrix} \varphi_{n-1}(x) \\ i\varphi_n(x) \end{pmatrix}, \tag{42}$$

where the eigenfunctions are given in terms of the Hermite polynomials, which correspond to harmonic oscillator potential:

$$\varphi_n(x) = c_n e^{-z^2/2} H_n(z(x)), \quad z(x) = \sqrt{\frac{\omega}{2}}\left(x - \frac{2k}{\omega}\right). \tag{43}$$

We define the adiabatic parameter as transforming the spatial variable of x as

$$x \to x - R(t). \tag{44}$$

After calculation from Eqs. 25-28, we found the solution for W,

$$W = \sqrt{\frac{\omega}{2}} \frac{1}{2v_F - 1} \begin{pmatrix} 0 & z\frac{\varphi_{n-1}}{\varphi_n} - 2(n-1)\frac{c_{n-1}}{c_{n-2}}\frac{\varphi_{n-2}}{\varphi_n} \\ z\frac{\varphi_n}{\varphi_{n-1}} - 2n\frac{c_n}{c_{n-1}} & 0 \end{pmatrix}. \tag{45}$$

For a constant magnetic field $B(x) = 1/2$ and $\omega = k = 1$, the density of the wave function ρ (solid line) and the current density j (dotted line) with $n = 1$ are depicted in Fig. 1. Using the regularization term W, we obtained the same wave function from the initial to the final time of evolution, and the time to obtain the target state is shorter. The regularization term preserves the wave function, and the final state is obtained in a shorter time.

The regularization term for $(2 + 1)$ Dirac systems appears only in terms of the real part. There are four freedoms of choice to solve the regularization term. Here, we suppress the imaginary part of W in order to become more practical in realization. We see that the regularization term W fulfills the Dirac equation and guarantees the adiabatic wave function.

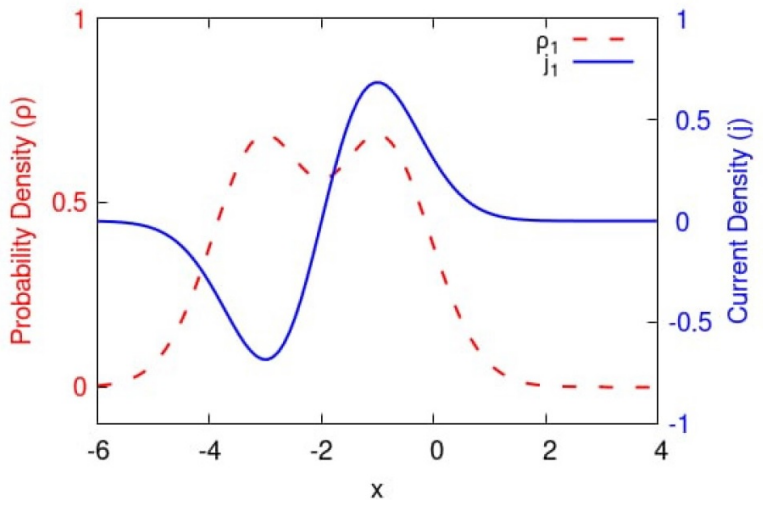

FIG. 1. Probability and current density of wave function with constant magnetic field and $n = 1$

Once the driving scalar and vector potential have been obtained, the targeted wave function will be obtained in a shorter time.

6. Conclusion

We have presented the fast-forward method of adiabatic quantum dynamics for a Dirac particle. This scheme is proposed to accelerate the spinor component of the wave function without disturbing the eigenstate of the system. During the evolution, we will obtain exactly the same eigenstate from the initial time to the final time. Using this scheme, we can reach the target state at any desired time. The strategy used in this scheme is the combination of two opposite ideas: infinitely large time-magnification factor ($\bar{\alpha}$) and infinitely small growth rate (ϵ) of the adiabatic parameter. To accelerate the adiabatic wave function we have to obtain the regularization term of Hamiltonian to guarantee the adiabatic motion. Then, by calculating the driving scalar potential and the driving vector potential, the time to obtain the target state will be shorter. As a typical example, we considered the adiabatic dynamics of the Dirac particle. We obtain the driving scalar potential V_{FF} and the driving vector potential A_{FF}. By tuning the electric field, we can accelerate the adiabatic electron dynamics in the ground state trapped in the plane xy, an electric field in the x direction, and a constant magnetic field in the z direction. This acceleration will preserve the ground state of the wave function from the initial time to the final time. The fast-forward method gives one a useful protocol for understanding and controlling graphene as a two-dimensional honeycomb lattice, which is important for their applications in nanoscale electronic devices and superconductor.

Appendix : Derivation for the driving potential in $(2+1)$ Dirac equation

We see on the left hand side of Eq. (39)

$$i\hbar\frac{\partial\Psi_{reg}}{\alpha(t)\partial t} = i\hbar\Big[-\frac{i}{\hbar}\alpha E\Psi_{reg} + \frac{\partial\Psi_{reg}}{\partial t}\Big]e^{i\delta_n(\Lambda(t))}, \tag{A.46}$$

and on the right hand side

$$\alpha\begin{pmatrix} m+V_0^{reg} & \pi_-^0 \\ \pi_+^0 & -m+V_0^{reg}\end{pmatrix}\Psi_{reg}, \tag{A.47}$$

then we have

$$\alpha E\begin{pmatrix}\phi_1^{reg}\\ \phi_2^{reg}\end{pmatrix} + i\hbar\frac{\partial}{\partial t}\Psi_{reg} = \alpha\begin{pmatrix} m+V_0^{reg} & \pi_-^0 \\ \pi_+^0 & -m+V_0^{reg}\end{pmatrix}\Psi_{reg}. \tag{A.48}$$

By deleting $\dfrac{\partial\Psi_{reg}}{\partial t}$, and from Eq. (38) and Eq. (A.48), we see

$$\begin{pmatrix} m+V_{FF}+V_0^{reg} & \pi_-^{FF} \\ \pi_+^{FF} & -m+V_{FF}+V_0^{reg}\end{pmatrix}\begin{pmatrix}\phi_1^{reg}\\ \phi_2^{reg}\end{pmatrix} = \alpha\begin{pmatrix} m+V_0^{reg} & \pi_-^0 \\ \pi_+^0 & -m+V_0^{reg}\end{pmatrix}\Psi_{reg} - (\alpha-1)E\begin{pmatrix}\phi_1^{reg}\\ \phi_2^{reg}\end{pmatrix} \tag{A.49}$$

Now we apply

$$\Psi_{reg}(x,y,R(t) = \Psi^{reg}(x,y,R(t) = \begin{pmatrix} \bar{\phi}_1(x,y,R(\Lambda(t))) \\ \bar{\phi}_2(x,y,R(\Lambda(t))) \end{pmatrix} \tag{A.50}$$

Substituting above spinor state to Eq. (A.49), we see

$$\begin{pmatrix} m+V_{FF}+V_0^{reg} & \pi_-^{FF} \\ \pi_+^{FF} & -m+V_{FF}+V_0^{reg} \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = \alpha \begin{pmatrix} m+V_0^{reg} & \pi_-^0 \\ \pi_+^0 & -m+V_0^{reg} \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} - (\alpha-1)E \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} \tag{A.51}$$

From Eq. (A.51), we see the eigen value problem as

$$-(\alpha-1) \begin{pmatrix} m+V_0 & \pi_-^0 \\ \pi_+^0 & -m+V_0 \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = -(\alpha-1)E \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} \tag{A.52}$$

Eq. (A.51) can be rewritten as

$$\begin{pmatrix} V_{FF} & A_x^{FF} - iA_y^{FF} \\ A_x^{FF} + iA_y^{FF} & V_{FF} \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = \epsilon(\alpha-1) \begin{pmatrix} W & W-iW \\ W+iW & W \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} \tag{A.53}$$

Noting $d\Lambda(t) = \alpha(t)dt$ and taking the limit $\epsilon \to 0$ and $\alpha \to \infty$, $\alpha\epsilon \to v(t)$, we will obtain Eq. (41)

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