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Original article

The Cauchy problem for a high-order wave equation with a loaded convolution type

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ABSTRACT The present paper is devoted to the problem for one of the loaded wave integro-differential equations, which is equivalent to the nonlocal problem for a higher-order wave equation. The study aims at nonlocal problems and constructs a representation of the solution to the problem for an equation of hyperbolic type. Also, the paper provides examples of some cases where it will be possible to construct solutions to the problem explicitly and in the graphs.

KEYWORDS Integro-dierential equation, Cauchy problem, loaded equation, nonlocal problem.

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1. Introduction

In recent years, differential equations with an integral term and high-order integro-differential equations have been of great interest from the point of view of mathematical engineering, mathematical physics, and chemical reaction-diffusion models. For example, various problems of mathematical engineering, chemical reaction-diffusion models, fundamental and applied physics such as fluid dynamics, beam theory, gas dynamics [1–4], nanoscience [5–7], and various problems of the theory of elasticity, plates, and shells are reduced to such equations (see [8–15] and the references therein).

Equations of the convolution type with the integro-differential operators arise in mathematical models of physical and technical systems where it is necessary to take into account the history of the processes. Constitutive relations in a linear processes of inhomogeneous diffusion and propagation of waves with memory contain a time- and space-dependent memory kernel. Problems of memory kernels [16, 17] identification in parabolic and hyperbolic integro-differential equations have been intensively studied.

In many cases, the equations describing the propagation of electrodynamic and elastic waves are reduced to hyperbolic equations with integral convolution [18–23]. Determination of time- and space-dependent kernels in parabolic integro-differential equations were investigated [24] (see, for example, [25] and references therein).

Various initial, boundary, and non-classical problems for loaded partial differential equations [26] have been studied in many works. We note recent research works [27–35], where there were studies of actual problems for the loaded partial differential and integro-differential equations of classical and mixed types. But we must note that problems for loaded differential and partial differential equations involving convolution types have not been investigated yet.

Proceeding from this, we study an analog of the Cauchy problem for a high-order loaded-wave equation with convolution-type operators in the multidimensional domain. The study is targeted on construction of optimal representations for the solution of the hyperbolic type equation and investigation of the existence and uniqueness of the solution to the Cauchy problem for the loaded differential equation. Representation of a higher-order partial differential operator in the form of a convolution type operator makes it possible, in particular, to reduce the problem of the propagation of electrodynamic and elastic waves and apply the methods of the theory of loaded differential equations.

2. Statement of the problem and its nonlocality

We consider the following loaded integro-differential equation of high order

$$L^m(u) \equiv \left(\frac{\partial^2}{\partial t^2} - A\right)^m u(x,t) = f(x,t) + \mu \int\limits_0^t k(x,\tau)u(0,t-\tau)d\tau,\ (x,t) \in \Omega, \tag{2.1}$$

where $k(x,t),\ f(x,t)$ are given real-valued sufficiently smooth functions, A is a linear differential operator acting on variables $x(x_1, x_2, ...x_n), L^m = L^1(L^{m-1}),\ m \in N,\ \mu$ is given real parameter.

Ω is the domain of the solutions of problem, depending on the form of the operator A.

$$\Omega = \{(x,t) : x \in R^n,\ 0 < t < +\infty\}.$$

In the domain Ω, we study the following problem.

Cauchy problem. Find a solution $u(x,t)$ of equation (2.1) from the class of functions:

$$W = \left\{u(x,t) : u(x,t) \in C^{2m-1}\left(\overline{\Omega}\right) \cap C^{2m}(\Omega)\right\},$$

that satisfies the initial conditions

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = 0,\ \ k = \overline{0, 2m-1}. \tag{2.2}$$

The inhomogeneous problem (2.1) - (2.2) for $\mu = 0$ was studied in [11], [36] but has not yet been studied for $\mu \neq 0$. The following theorem is true.

Theorem 2.1. *Let function $\widetilde{u}(x,t,t_1)$ depending on parameter t_1, be a solution of the equation*

$$L^m(\widetilde{u}) \equiv \left(\frac{\partial^2}{\partial t^2} - A\right)^m \widetilde{u}(x,t,t_1) = 0,\ \ t > t_1, \tag{2.3}$$

satisfying the initial and nonlocal conditions

$$\left.\frac{\partial^k \widetilde{u}}{\partial t^k}\right|_{t=t_1} = 0,\ \left.\frac{\partial^{2m-1}\widetilde{u}}{\partial t^{2m-1}}\right|_{t=t_1} = f(x,t_1) + \mu \int\limits_0^{t_1} ds \int\limits_0^{t_1-s} k(x,z)\widetilde{u}(0,t_1-z,s)dz, \tag{2.4}$$

for $k = \overline{0, 2m-2}$, then function

$$u(x,t) = \int\limits_0^t \widetilde{u}(x,t,t_1)dt_1,\ \ (x,t) \in \Omega,\ \ t > t_1, \tag{2.5}$$

is a solution of the problem (2.1) and (2.2).

Proof of Theorem 2.1. Initially, let us differentiate expressions (2.5) twice in respect to t and, taking into account (2.4), we obtain

$$\frac{\partial^2 u}{\partial t^2} = \widetilde{u}_t(x,t,t_1|_{t_1=t} + \int\limits_0^t \widetilde{u}_{tt}(x,t,t_1)dt_1 = \int\limits_0^t \widetilde{u}_{tt}(x,t,t_1)dt_1, \tag{2.6}$$

On the other hand, applying the operator A to this equality (2.5), we obtain

$$A(u(x,t)) = \int\limits_0^t A(\widetilde{u}(x,t,t_1))dt_1,\ \ (x,t) \in \Omega. \tag{2.7}$$

Therefore, subtracting the resulting equality (2.7) from (2.6), we can easily obtain the first operator function

$$L^1(u) \equiv \left(\frac{\partial^2}{\partial t^2} - A\right) u(x,t) = \int\limits_0^t L^1(\widetilde{u}(x,t,t_1))dt_1. \tag{2.8}$$

Continuing to repeat this process up to $m-1$ and taking into account conditions (2.4), we come to the expression

$$L^{m-1}(u) = \int\limits_0^t L^{m-1}(\widetilde{u}(x,t,t_1))dt_1. \tag{2.9}$$

Hence, differentiating the last expression in respect to t up to the second order and taking (2.2) into account, we have

$$\frac{\partial^2}{\partial t^2}L^{m-1}(u) = \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} - A\right)^{m-1} \left.\widetilde{u}(x,t,t_1)\right|_{t_1=t} + \int\limits_0^t \frac{\partial^2}{\partial t^2}L^{m-1}(\widetilde{u}(x,t,t_1))dt_1 =$$

$$= f(x,t) + \mu \int_0^t ds \int_0^{t-s} k(x,z)\widetilde{u}(0,t-z,s)dz + \int_0^t \frac{\partial^2}{\partial t^2} L^{m-1}(\widetilde{u}(x,t,t_1))dt_1. \tag{2.10}$$

Applying the operator A to the function $L^{m-1}(u)$ and then subtracting the result obtained from (2.10), due to (2.9), (2.3) and $L^m(\widetilde{u}) = 0$, we have

$$\left(\partial^2/\partial t^2 - A\right) L^{m-1}(u) = \mu \int_0^t k(x,t_1)u(0,t-t_1)dt_1 + f(x,t).$$

Also, one can easily make sure that the function $u(x,t)$, defined by equality (2.5), satisfies the initial condition (2.2) which was to be proved. Thus, the theorem 2.1 is proved.

By introducing a new variable, $\tau = t - t_1$, the equations (2.3) - (2.4) can be reduced to the problem:

$$L^m(\widetilde{u}) \equiv \left(\frac{\partial^2}{\partial t^2} - A\right)^m \widetilde{u} = 0, \ \ \tau > 0, \tag{2.11}$$

$$\left.\frac{\partial^k \widetilde{u}}{\partial \tau^k}\right|_{\tau=0} = 0, \ \left.\frac{\partial^{2m-1}\widetilde{u}}{\partial \tau^{2m-1}}\right|_{\tau=0} = f(x,t_1) + \mu \int_0^{t_1} ds \int_s^{t_1} k(x,t_1 - z)\widetilde{u}(0,z,s)dz, \tag{2.12}$$

for $k = \overline{0, 2m-2}$, $\tau > 0$. Consequently, by solving the problem (2.11) - (2.12) and again passing to the variables introduced at the beginning of the section, we can find the required function $\widetilde{u}(x,t,t_1)$. Thus, from formula (2.5), we restore the solution of the problem (2.1) - (2.2).

In studying equation (2.11) concerning nonlocal terms, it becomes necessary to use the spherical means wave equation. The problem under consideration refers to a loaded wave equation involving convolution-type operators, for which the spherical mean method has not yet been studied.

3. Solutions of the problem on R^1

Let $n = 1$, $A = \partial^2/\partial x^2$. Find a solution $u(x,t)$ of equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^m u(x,t) = f(x,t) + \mu \int_0^t k(x,\tau)u(0,t-\tau)d\tau, \ (x,t) \in \Omega, \tag{3.1}$$

from the class of functions W that satisfies the initial conditions

$$u(x,t)|_{t=0} = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{t=0} = \ldots = \left.\frac{\partial^{2m-1}u(x,t)}{\partial t^{2m-1}}\right|_{t=0} = 0, \tag{3.2}$$

where $k(x,t)$, $f(x,t)$ are given real-valued functions.

We introduce the following notations

$$\upsilon_0(x,t,\tau) = \widetilde{u}(x,t,\tau), \ \upsilon_1(x,t,\tau) = L_1\upsilon_0(x,t,\tau), \ldots \upsilon_{m-1}(x,t,\tau) = L_{m-1}\upsilon_0(x,t,\tau). \tag{3.3}$$

Thus, considering the loaded part analogously to [28], one reduces problem (2.11) - (2.12) to the following system for the functions $\upsilon_k(x,t,t_1)$

$$\begin{cases} \dfrac{\partial^2 \upsilon_0}{\partial t^2} - \dfrac{\partial^2 \upsilon_0}{\partial x^2} = \upsilon_1, \\ \dfrac{\partial^2 \upsilon_1}{\partial t^2} - \dfrac{\partial^2 \upsilon_1}{\partial x^2} = \upsilon_2, \\ \ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial^2 \upsilon_{m-2}}{\partial t^2} - \dfrac{\partial^2 \upsilon_{m-2}}{\partial x^2} = \upsilon_{m-1}, \\ \dfrac{\partial^2 \upsilon_{m-1}}{\partial t^2} - \dfrac{\partial^2 \upsilon_{m-1}}{\partial x^2} = 0, \end{cases} \tag{3.4}$$

and initial conditions

$$\upsilon(x,t,t_1)|_{t=0} = \left.\frac{\partial \upsilon_k(x,t,t_1)}{\partial t}\right|_{t=0} = 0, \ \ldots.., \ \left.\frac{\partial \upsilon_{m-1}}{\partial t}\right|_{t=0} = g_{m-1}(x,t_1), \tag{3.5}$$

where

$$g_{m-1} = \sum_{j=0}^{m-1} (-1)^j c_k^j (f^{(j)}(x,t_1) + \mu \int_0^{t_1} ds \int_s^{t_1} k(x,t_1 - z)\upsilon_0^{(j)}(0,z,s)dz), \quad c_k^j = k![j!(k-j)!]$$

is the binomial coefficient. Hence, by applying the classical method of spherical means [37] analogously to work [11], we can find the solution to the problem (2.1)-(2.2).

Consequently, after determining the solution to the problem (3.4) - (3.5) by (3.11) from [36] and using the introduced notation (3.3), the solution to the problem (2.11) - (2.12) can be written as:

$$\widetilde{u}(x,t,t_1) = 2^{-2m+1} \int\limits_{x-(t-t_1)}^{x+(t-t_1)} k_1(x,s,t-t_1) f(s,t_1) ds +$$

$$+2^{-2m+1}\mu \int\limits_{x-(t-t_1)}^{x+(t-t_1)} k_1(x,s,t-t_1) \int\limits_0^{t_1} dz \int\limits_z^{\tau} k(x,t_1-\eta)\widetilde{u}(0,\eta,z) d\eta \, ds, \tag{3.6}$$

where $k_1(x,s,t) = \left[(t^2-(s-x)^2\right]^{m-1}/[(m-1)!]^2$.

Substituting the obtained function into (2.5), we find the solution to the problem (2.1), and (2.2) in the following form:

$$u(x,t) = 2^{-2m+1}\mu \int\limits_0^t dt_1 \int\limits_{x-(t-t_1)}^{x+(t-t_1)} k_1(x,s,t-t_1) \, ds \int\limits_0^{t_1} k(s,t_1-z)u(0,z)dz +$$

$$+2^{-2m+1} \int\limits_0^t dt_1 \int\limits_{x-(t-t_1)}^{x+(t-t_1)} k_1(x,s,t-t_1) f(s,t_1) ds. \tag{3.7}$$

The resulting expressions (3.6) and (3.7) are represented as loaded integral equations concerning the unknown functions $\widetilde{u}(x,t,t_1)$ and $u(x,t)$, respectively. Therefore, to solve the integral equation (3.7), we put $x=0$:

$$u(0,t) - \mu \int\limits_0^t K(0,t,\tau)u(0,\tau) \, d\tau = \widetilde{f}(0,t), \tag{3.8}$$

where

$$K(0,t,\tau) = \frac{\mu}{2^{2m-1}} \int\limits_{\tau}^{t} dz \int\limits_{-(t-z)}^{(t-z)} k(s,z-\tau) k_1(0,s,t-\tau) \, ds,$$

$$\widetilde{f}(x,t) = \frac{1}{2^{2m-1}} \int\limits_0^t dz \int\limits_{x-(t-z)}^{x+(t-z)} k_1(x,s,t-z) f(s,z) ds.$$

Hence, taking into account the notation $k_1(x,s,t-\tau)$ and some changes of variables, it is easy to verify that:

$$\left|\widetilde{f}(x,t)\right|_{x=0} = \left| \frac{1}{2^{2m-1}} \int\limits_0^t dz \int\limits_{-(t-z)}^{(t-z)} k_1(0,s,t-z) f(s,z) ds \right| \le$$

$$\le \left| \frac{C}{2^{2m-1}[(m-1)!]^2} \int\limits_0^t (t-z)^{2m-1} dz \int\limits_{-1}^{1} (1-s^2)^{m-1} ds \right| \le$$

$$\le \left| \frac{C}{(2m-1)!} \int\limits_0^t (t-z)^{2m-1} dz \right| \le \left| C \frac{t^{2m}}{(2m)!} \right|. \tag{3.9}$$

Considering $k(x,t) \in H^{l,l/2}(\Omega)$ analogously, we obtain $|K(0,t,\tau)| \le C \left|(t-\tau)^{2m}\right|$, at $m \ge 1$.

Thus, we can write the solutions of equation (3.8) in the form:

$$u(0,t) = \widetilde{f}(0,t) + \mu \int\limits_0^t R(0,t,\tau)\widetilde{f}(0,\tau) \, d\tau, \tag{3.10}$$

where $R(0,t,\tau)$ is the resolvent of the kernel $K(0,t,\tau)$. The solution (3.8), according to the theory of integral equations, can be easily verified that it is unique in a class of functions that can have a weak singularity.

Thus, from formula (3.7), taking into account (3.10), we can write an explicit form of the solution to the problem (2.1) - (2.2).

4. Examples

This section of the work is devoted to the study of the obtained results, giving several examples confirming the validity of the conclusions, from a numerical perspective. Note that the problems under consideration are illustrated by figures with limited solution parameters, which can be continued further.

Example 4.1. Find the solution $u(x,t)$ of the Cauchy problem in the class of functions

$$W = \left\{u(x,t) : u(x,t) \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)\right\},$$

for the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = t - x + \int\limits_0^t u(0,t-s)ds,\ (x,t) \in \Omega, \tag{4.1}$$

with the initial conditions

$$u(x,t)|_{t=0} = u_t(x,t)|_{t=0} = 0, \tag{4.2}$$

where $\Omega = \{(x,t) : x \in R,\ t > 0\}$.

In this case, for $k(x,t) \equiv 1$, taking into account the following replacement, $s = x + (t-\tau)s'$, and some properties of special functions, we have

$$K(0,t,\tau) = \frac{\mu}{2^{2m-1}[(m-1)!]^2} \int\limits_\tau^t (t-z)^{2m-1}dz \int\limits_{-1}^1 \left(1-s^2\right)^{m-1} ds =$$

$$= \frac{\mu}{2^{2m-1}[(m-1)!]^2} \int\limits_\tau^t (t-z)^{2m-1}dz \int\limits_0^1 (1-s)^{-\frac{1}{2}} s^{m-1}\, ds =$$

$$= \frac{\mu}{2^{2m-1}[(m-1)!]^2} B(m,1/2) \int\limits_\tau^t (t-z)^{2m-1}dz = \frac{\mu(t-\tau)^{2m}}{2m!}. \tag{4.3}$$

Hence, taking into account the method of successive approximations, we can conclude that for $k(x,t) \equiv 1$ and for any μ there exists a resolvent of the kernel $K(0,t,\tau)$ in the form

$$R(0,t,\tau) = \sum_{i=1}^{\infty} \left[(t-\tau)^{(2m+1)i-1}\right] / \left[((2m+1)i-1)!\right]. \tag{4.4}$$

By the same method, it will be possible to find the explicit form of the resolvent in other cases from the kernel function $k(x,t) \neq 1$.

Thus, from (3.10), taking into account (4.4) at $f(x,t) = -x + t$, we have

$$u(0,t) = \sum_{i=0}^{\infty} \frac{t^{3i+3}}{(3i+3)!}.$$

Therefore, setting $\mu = 1$, we find the final solution of problem (4.1) and (4.2) in the form:

$$u(x,t) = \frac{t^3}{6} - \frac{xt^2}{2} + \sum_{i=0}^{\infty} \frac{t^{3i+6}}{(3i+6)!},$$

which is represented in Fig. 1.

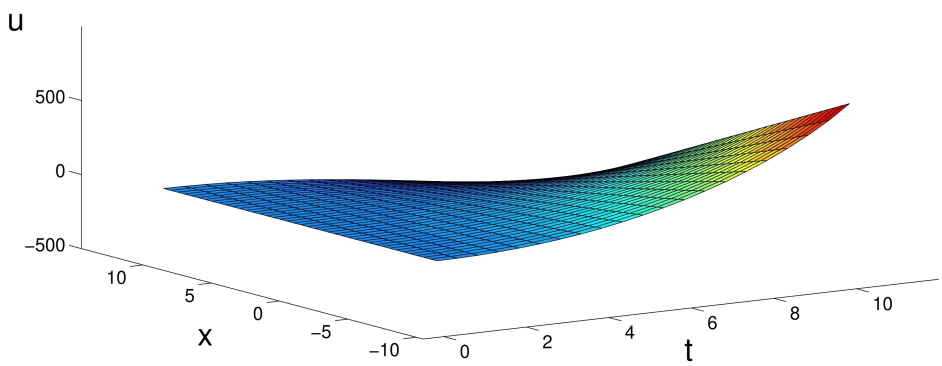

FIG. 1. The solution to the Example 4.1

Example 4.2. Find a solution $u(x,t)$ of the Cauchy problem in the class of functions W for the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = xt + \int_0^t su(0,t-s)ds, \tag{4.5}$$

with initial conditions (4.2).

If we substitute $k(x,t) = t$, first, we should find the solution of the integral equation (3.8). To find the solution, we should determine the kernel $K(0,t,\tau)$ and its resolvent $R(0,t,\tau)$. Thus, the function K from the definition in equation (3.8) has the form

$$K(0,t,\tau) = \frac{\mu}{2^{2m-1}} \int_\tau^t (z-\tau)dz \int_{-(t-z)}^{(t-z)} k_1(0,s,t-\tau)\,ds,$$

Hence, similarly to (4.3), we obtain

$$K(0,t,\tau) = \frac{(t-\tau)^{2m+1}}{[(m+1)!]}, \quad if \;\; k(x,t) = t. \tag{4.6}$$

From here, we can easily find the resolvent of the kernel (4.6) which has the form

$$R(0,t,\tau) = \sum_{i=1}^{\infty} (t-\tau)^{2i(m+1)-1}/(2i(m+1)-1)!. \tag{4.7}$$

Thus, similarly to example 4.1, we can find a solution to the problem (4.5) and (4.2), for $m = 1,2,....$ If m=1, we can illustrate the explicit solution as

$$u(x,t) = \frac{xt^3}{3!} + \sum_{i=0}^{\infty} \frac{t^{4i+8}}{(4i+8)!},$$

which is represented in Fig. 2.

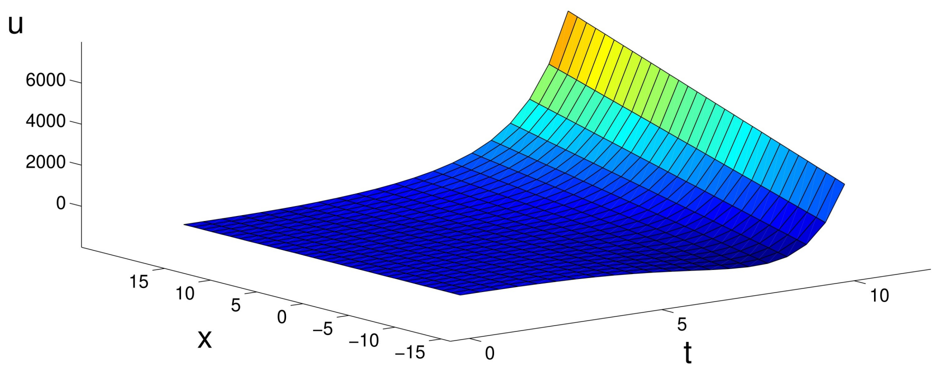

FIG. 2. The solution to the Example 4.2

The solution to the problem is found similarly, respectively, for the cases $m = 2,3....$

Example 4.3. Find a solution $u(x,t) \in C^3\left(\overline{\Omega}\right) \cap C^6(\Omega)$ of the Cauchy problem in $\Omega \in R^2$, for the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^2 u(x,t) = xt + \int_0^t su(0,t-s)ds, \tag{4.8}$$

with initial conditions

$$u(x,t)|_{t=0} = u_t(x,t)|_{t=0} = u_{tt}(x,t)|_{t=0} = u_{ttt}(x,t)|_{t=0} = 0. \tag{4.9}$$

Similarly to examples 4.1 and 4.2, we can find solutions to the problem (4.8) - (4.9), for $m = 1,2.....$ If m=1 (for $m = 2,3...$, appropriately), we can illustrate the solution as

$$u(x,t) = \frac{xt^5}{5!} + \frac{3}{2}\sum_{i=0}^{\infty} \frac{t^{6i+12}}{(6i+12)!},$$

which is represented in Fig. 3.

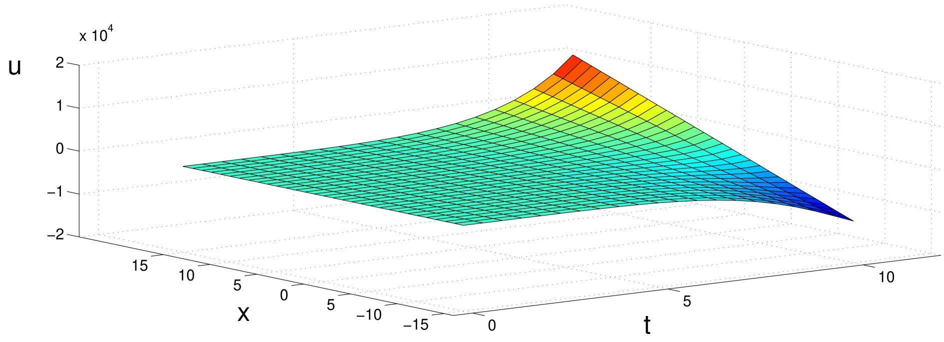

FIG. 3. The solution to the Example 4.3

5. Solutions of the problem in R^3

Let $n = 3,\ A \equiv \sum_{i=1}^{3} \partial^2/\partial x_i^2$. Find a solution $u(x,t) = u(x_1, x_2, x_3, t)$ of equation

$$\left(\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}\right)\right)^m u(x,t) = f(x,t) + \mu \int_0^t k(x,s)u(0,t-s)ds, \tag{5.1}$$

from the class W of functions that satisfies the initial conditions (2.2), where $k(x,t)$, $f(x,t)$ are given smooth functions.

When considering problem (5.1) - (2.2), which equivalently reduces to problem (2.11) - (2.12) as $A \equiv \sum_{i=1}^{3} \partial^2/\partial x_i^2$ [34], using the methods of spherical means [37], we have

$$\widetilde{u}(x,t,t_1) = \frac{1}{2^{2m-1}\pi} \int_{|\xi-x|\le t-t_1} K_1(x,\xi,t-t_1)f(\xi,t_1)d\xi +$$

$$+\frac{\mu}{2^{2m-1}\pi} \int_{|\xi-x|\le t-t_1} K_1(x,\xi,t-t_1)d\xi \int_0^{t_1} ds \int_s^{t_1} k(\xi,t_1-z)\widetilde{u}(0,z,s)dz, \tag{5.2}$$

$K_1(x,\xi,t) = \left[t^2 - |\xi - x|^2\right]^{m-2} / [(m-2)!(m-1)!],\ |\xi - x|^2 = \sum_{k=1}^{3} (\xi_k - x_k)^2$. Therefore, substituting $x = 0$, we obtain the following integral equation

$$\widetilde{u}(0,t,t_1) - \int_0^{t_1} ds \int_s^{t_1} \widetilde{K}(0,t;z,t_1)\widetilde{u}(0,z,s)dz = \widetilde{f}(0,t,t_1), \tag{5.3}$$

where

$$\widetilde{K}(x,t;z,t_1) = \frac{\mu}{2^{2m-1}\pi} \int_{|\xi-x|\le t-t_1} k(\xi,t_1-z)K_1(x,\xi,t-t_1)d\xi, \tag{5.4}$$

$$\widetilde{f}(x,t,t_1) = \frac{1}{2^{2m-1}\pi} \int_{|\xi-x|\le t-t_1} K_1(x,\xi,t-t_1)f(\xi,t_1)d\xi. \tag{5.5}$$

Similarly, taking into account $k(x,t) \in H^{l,l/2}(\Omega)$ at $m > 1$, and solution (5.3) by writing via the resolvent $R(0,t;z,t_1)$ of the kernel equation, we obtain

$$\widetilde{u}(0,t,t_1) = \widetilde{f}(0,t,t_1) + \int_0^{t_1} ds \int_s^{t_1} R(0,t;z,t_1)\widetilde{f}(0,z,s)dz, \tag{5.6}$$

Thus, from formula (2.5), the solution of problem (5.1) - (2.2) has the form:

$$u(x,t) = \frac{1}{2^{2m-1}\pi} \int_0^t \int_{|\xi-x|\le t-\tau} K_1(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau +$$

$$+\frac{1}{2^{2m-1}\pi}\int\limits_{0}^{t}\int\limits_{|\xi-x|\leq t-\tau} K_1(x,\xi,t-\tau)\widetilde{F}(0,\tau)d\xi d\tau, \tag{5.7}$$

here $\widetilde{F}(0,\tau)=\int\limits_{0}^{\tau}ds\int\limits_{s}^{\tau}k(\xi,\tau-z)\widetilde{u}(0,z,s)dz,\ \widetilde{u}(0,t,\tau)$ is a known function according to the formula (5.6).

Note that solutions to problem (2.1)-(2.2) can also be written in other cases n since $n=4,5,....$

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