

Discrimination and decoherence of Schrödinger cat states in lossy quantum channels

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We investigate environment induced effects of decoherence in discrimination between the Schrödinger cat states transmitted through noisy quantum channels such as optical fibers. We calculate the fidelity and the statistics of photocounts for both even and odd coherent states. The method that uses the beam splitter-like transformation acting in the enlarged Hilbert space to model the quantum channel is compared with the approach based on the Lindblad dynamics of one-mode bosonic systems.

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1. Introduction

Quantum states known as the even (odd) Schrödinger cat states, $|\alpha_+\rangle$ ($|\alpha_-\rangle$), are represented by symmetric (antisymmetric) superpositions of the coherent states:

$$|\alpha_{\pm}\rangle = \frac{1}{\sqrt{M_{\pm}(\alpha)}}(|\alpha\rangle \pm |-\alpha\rangle), \quad (1)$$

where

$$M_{\pm}(\alpha) = 2(1 \pm \langle \alpha | -\alpha \rangle) = 2(1 \pm \exp(-2|\alpha|^2)). \quad (2)$$

These states have a long history dating back to the original paper by Dodonov, Malkin and Man'ko [1]. and, apart from the fundamental interest, are shown to be technologically useful.

In metrology, even coherent states in combination with parity detection method are used to realize super-resolving angular rotation measurements [2]. Optical Schrödinger cat states can also serve as a basis for quantum computations [3], teleportation and cryptography [4–7]. These states have been produced experimentally using the optical Kerr effect in 1986 by Yurke and Stoler [8]. Among more recent experimental procedures suggested for generation of optical cat states are the method based on photon subtraction of an optical squeezed state [9–12] and the protocol that uses homodyne detection and photon number states [13] (see [14] for a review on generating cat states).

In this paper, our goal is to assess the fragility of the cat states propagating in noisy quantum channels. More specifically, we consider the statistics of photocounts underlying the method to discriminate between the parity of the cat states and examine its sensitivity to environment induced decoherence effects. We shall also calculate the fidelity of the states depending on the amplitude and the transmission of the fiber channel. In addition to the model of the quantum channel based on the beam splitter-like transformation acting in the enlarged Hilbert space, we employ Lindblad dynamics of an one-mode bosonic system to model a lossy optical fiber.

2. Results

Following [6,7,15], we begin with the simplest model of the noisy channel representing the fiber. In this model, the lossy mode is combined with the beam splitter of the transmission, τ , along with the input vacuum state in additional environmental (ancillary) mode E :

$$\begin{aligned}
 |\alpha_{\pm}\rangle &\rightarrow |\Psi_{\pm}\rangle = \hat{T}_{BS}(\tau)|\alpha_{\pm}\rangle \otimes |0\rangle_E \\
 &= \frac{1}{\sqrt{M_{\pm}(\alpha)}} \left\{ |\sqrt{\tau}\alpha\rangle \otimes |\sqrt{1-\tau}\alpha\rangle_E \pm |-\sqrt{\tau}\alpha\rangle \otimes |-\sqrt{1-\tau}\alpha\rangle_E \right\},
 \end{aligned}
 \tag{3}$$

where $\hat{T}_{BS}(\tau)$ is the unitary describing the beam splitter with the transmittance τ . Clearly, $|\Psi_{\pm}\rangle$ is the entangled two-mode state generated by the beam splitter. Note that, according to Ref. [16], the phase modulator can be regarded as a multiport beam splitter that transforms a one-mode Schrödinger cat state into entangled multimode state.

The density operators can now be readily derived by tracing out the ancillary mode. The result:

$$\hat{\rho}_{\pm} = \text{Tr}_E |\Psi_{\pm}\rangle\langle\Psi_{\pm}| = P_{\pm}|\gamma_{\pm}\rangle\langle\gamma_{\pm}| + Q_{\pm}|\gamma_{\mp}\rangle\langle\gamma_{\mp}|,
 \tag{4}$$

where $\gamma \equiv \sqrt{\tau}\alpha$, shows that the density matrices are mixtures of even and odd cat states with the coefficients given by:

$$\begin{aligned}
 P_+ &= \cosh((1-\tau)|\alpha|^2) \frac{\cosh(\tau|\alpha|^2)}{\cosh(|\alpha|^2)}, & Q_+ &= 1 - P_+ = \sinh((1-\tau)|\alpha|^2) \frac{\sinh(\tau|\alpha|^2)}{\cosh(|\alpha|^2)}, \\
 P_- &= \cosh((1-\tau)|\alpha|^2) \frac{\sinh(\tau|\alpha|^2)}{\sinh(|\alpha|^2)}, & Q_- &= 1 - P_- = \sinh((1-\tau)|\alpha|^2) \frac{\cosh(\tau|\alpha|^2)}{\sinh(|\alpha|^2)}.
 \end{aligned}
 \tag{5}$$

Since $\langle\gamma_{\mu}|\gamma_{\nu}\rangle = \delta_{\mu\nu}$, the above coefficients P_{ν} and Q_{ν} are the eigenvalues of the density operator $\hat{\rho}_{\nu}$.

The effect of decoherence for even and odd cat states is characterized by the probabilities Q_+ and Q_- . It is not difficult to see that, Q_{\pm} tends to 1/2 as $|\alpha|$ becomes infinitely large, and thus the operators $\hat{\rho}_+$ and $\hat{\rho}_-$ are both approaching the limit of the completely chaotic (equiprobable) mixture as $|\alpha|$ increases. For small $|\alpha|$ with $\alpha \approx 0$, $\hat{\rho}_+$ and $\hat{\rho}_-$ are close to the vacuum state $|0\rangle\langle 0|$ and the mixed state $(1-\tau)|0\rangle\langle 0| + \tau|1\rangle\langle 1|$, respectively.

Given the density operator (4), we can now evaluate the fidelity:

$$F_{\pm} = \langle\alpha_{\pm}|\hat{\rho}_{\pm}|\alpha_{\pm}\rangle
 \tag{6}$$

characterizing the difference between the mixed and pure states: $\hat{\rho}_{\pm}$ and $|\alpha_{\pm}\rangle\langle\alpha_{\pm}|$. For even and odd cat states, we have:

$$\begin{aligned}
 F_+ &= \frac{\cosh((1-\tau)|\alpha|^2)}{\cosh^2(|\alpha|^2)} \cosh^2(\sqrt{\tau}|\alpha|^2), \\
 F_- &= \frac{\cosh((1-\tau)|\alpha|^2)}{\sinh^2(|\alpha|^2)} \sinh^2(\sqrt{\tau}|\alpha|^2).
 \end{aligned}
 \tag{7}$$

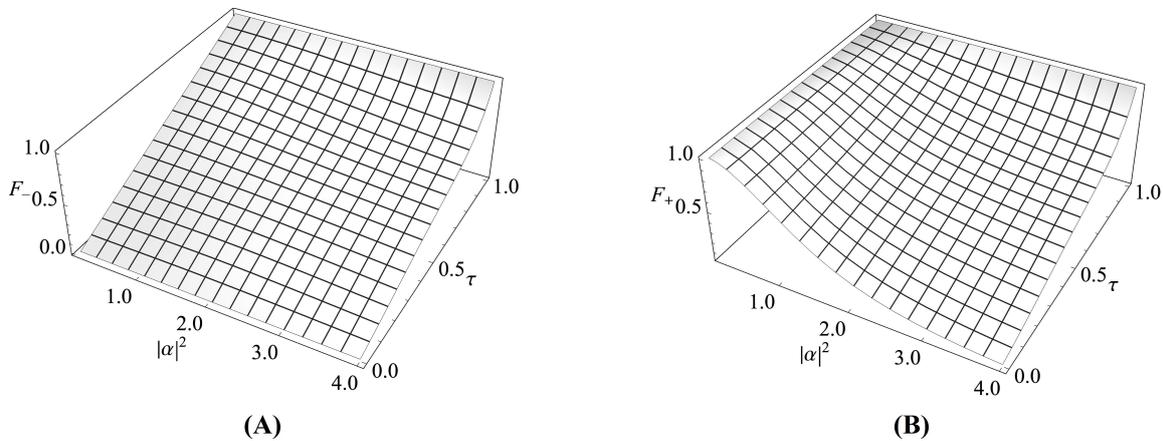


FIG. 1. Fidelity as a function of the transmittance coefficient τ and the mean photon number $|\alpha|^2$ for (A) odd and (B) even cat states

Fig. 1 shows the fidelities F_- and F_+ as an function of the transmittance τ the mean photon number $|\alpha|^2$. It is seen that, in the zero amplitude limit where the symmetric cat state is close the vacuum state, the fidelity F_+ approaches unity. The corresponding value for the fidelity of the antisymmetric cats equals τ . At large $|\alpha|$, the fidelities are exponentially decaying functions of the mean photon number: $F_{\pm} \approx \exp(-(1 - \sqrt{\tau})^2 |\alpha|^2)$.

Our next step is to study how decoherence that has a detrimental effect on the fidelity will affect the distinguishability of the cat states. For this purpose, we shall discuss performing photocounting measurements as the means to discriminate between the even and odd cat states based on the parity of the photon number registered by a photon-number resolving photodetector.

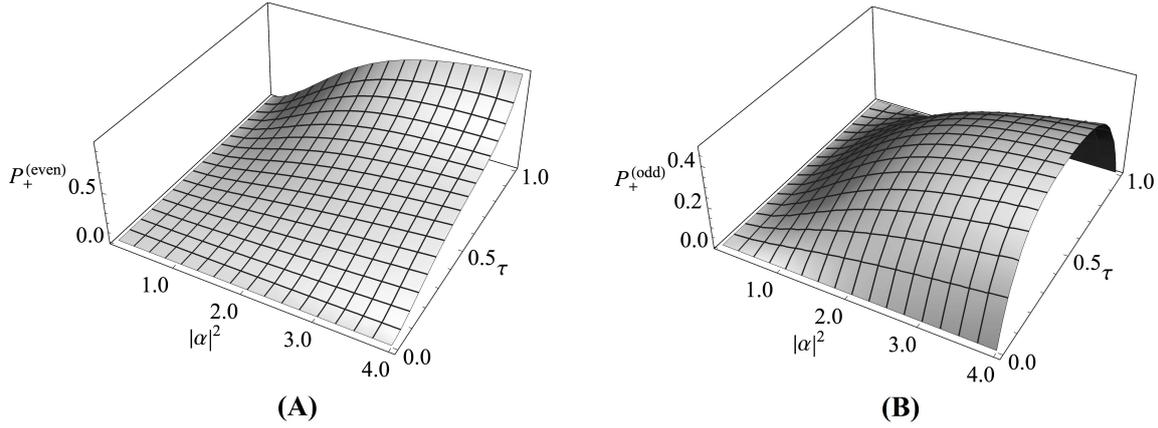


FIG. 2. Probabilities to detect (A) even and (B) odd number of photocounts as a function of the transmittance coefficient τ and the mean photon number $|\alpha|^2$ for the symmetric (even) Schrödinger cat

In order to evaluate the statistics of photocounts for the quantum cat states, $\hat{\rho}_+$ and $\hat{\rho}_-$, we shall use the well-known Kelley-Kleiner formula [17] and deduce the following expression for the probability to detect k photons:

$$\text{Pr}_{\nu}(k) = \frac{1}{k!} \text{Tr} [: (\eta_d \hat{a}^\dagger \hat{a})^k e^{-\eta_d \hat{a}^\dagger \hat{a}} : \rho_{\nu}] = P_{\nu} C_{\nu}(k) + Q_{\nu} C_{-\nu}(k), \tag{8}$$

where

$$\begin{aligned} C_{\nu}(k) &\equiv \frac{1}{k!} \langle \gamma_{\nu} | : (\eta_d \hat{a}^\dagger \hat{a})^k e^{-\eta_d \hat{a}^\dagger \hat{a}} : | \gamma_{\nu} \rangle \\ &= \frac{(\eta_d \tau |\alpha|^2)^k \exp((1 - \eta_d) \tau |\alpha|^2) + (-1)^k \nu \exp(-(1 - \eta_d) \tau |\alpha|^2)}{k! \exp(\tau |\alpha|^2) + \nu \exp(-\tau |\alpha|^2)}, \end{aligned} \tag{9}$$

η_d is the efficiency of the photodetector.

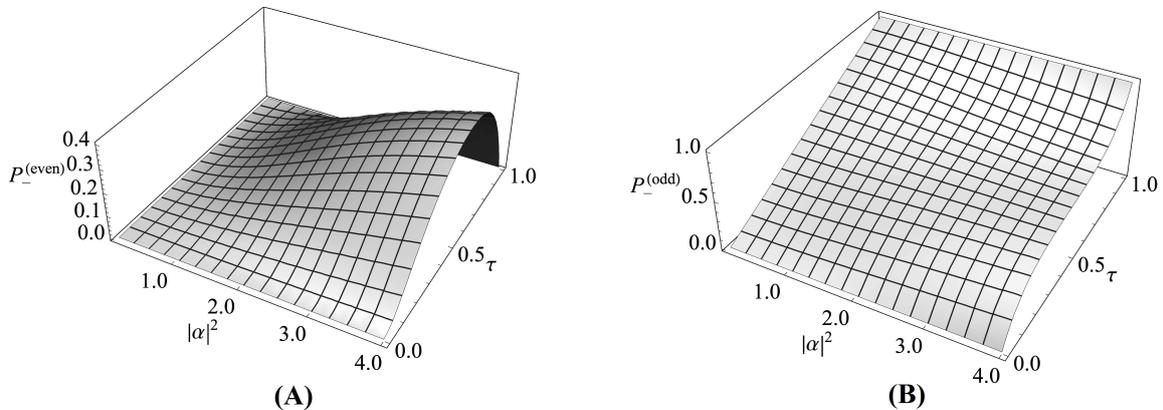


FIG. 3. Probabilities to detect (A) even and (B) odd number of photocounts as a function of the transmittance coefficient τ and the mean photon number $|\alpha|^2$ for the antisymmetric (odd) Schrödinger cat

From Eq. (9) it can be readily seen that, in the case where the efficiency and the transmittance are both equal to unity, $\eta_d = \tau = 1$, the probability of odd (even) number of clicks:

$$P_\nu^{(\text{odd})} = \sum_{k=0}^{\infty} \text{Pr}_\nu(2k+1), \quad P_\nu^{(\text{even})} = \sum_{k=1}^{\infty} \text{Pr}_\nu(2k) \quad (10)$$

vanishes for the even (odd) states, $P_+^{(\text{odd})} = 0$ ($P_-^{(\text{even})} = 0$), and thus the cat states are perfectly distinguishable. Note that, in contrast to the case of pure odd cat state, where the no-click probability is zero, $P_-(0) = 0$, for the pure even cat state, this probability decays with $|\alpha|^2$, $P_+(0) = 1 - P_+^{(\text{even})} = 1/\cosh(|\alpha|^2)$, and represents the non-vanishing contribution of the vacuum state.

As is shown in Figs. 2 and 3, dependence of the decoherence induced probabilities $P_+^{(\text{odd})}$ and $P_-^{(\text{even})}$ on the channel transmittance reveal nonmonotonic behavior. From the above discussion, these probabilities equal zero in the limit of the lossless channel with $\tau = 1$. In the opposite case, with $\tau = 0$, the density operators are projectors on the vacuum state $|0\rangle$ and $P_\pm^{(\text{odd})} = P_\pm^{(\text{even})} = 0$. So, referring to Fig. 4, the no-click probabilities $P_+(0)$ and $P_-(0)$ are both equal to unity in the no-signal limit with $\tau = 0$.

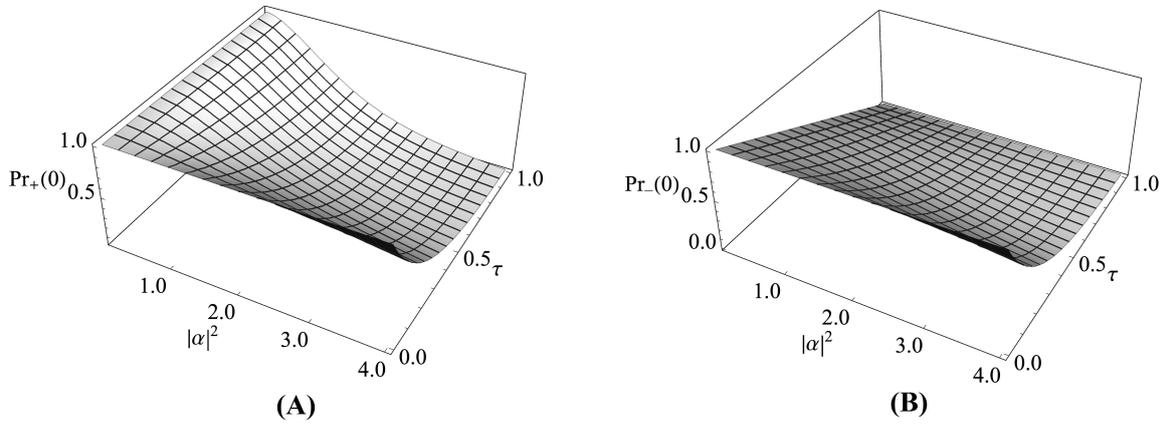


FIG. 4. No-click probability as a function of the transmittance coefficient τ and the mean photon number $|\alpha|^2$ for (A) symmetric (even) and (B) antisymmetric (odd) Schrödinger cats

The results indicate that the statistics of photocounts for the odd cat state at small $|\alpha|^2$ are less sensitive to the decoherence effects as compared to the even states. Interestingly, there is the optimal value of $|\alpha|$ that gives the greatest possible difference $P_+^{(\text{even})} - P_+^{(\text{odd})}$, whereas $P_-^{(\text{odd})} - P_-^{(\text{even})}$ monotonically decreases with $|\alpha|$.

An alternative approach to modelling of the quantum channel representing a noisy fiber is based on the master equation method [16, 18, 19]. In our case we can start from the one-mode Markovian thermal bath version of the Lindblad equation for the density matrix given by:

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} &= \mathcal{L}\hat{\rho} = -i\Omega[\hat{n}, \hat{\rho}] + \kappa(\mathcal{D}_{\hat{a}\hat{a}^\dagger}\hat{\rho} + e^{-z_T}\mathcal{D}_{\hat{a}^\dagger\hat{a}}\hat{\rho}), \\ \hat{n} &= \hat{a}^\dagger\hat{a}, \quad \mathcal{D}_{\hat{A}\hat{B}}\hat{\rho} \equiv 2\hat{A}\hat{\rho}\hat{B} - \hat{B}\hat{A}\hat{\rho} - \hat{\rho}\hat{B}\hat{A}, \end{aligned} \quad (11)$$

where the dagger denotes Hermitian conjugation, $\hat{\rho}$ is the density matrix representing the quantum state; \hat{a}^\dagger (\hat{a}) is the creation (annihilation) operator; $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ stands for the commutator; Ω (κ) is the element of the frequency (relaxation) constant, $z_T = \frac{\hbar\Omega_0}{k_B T}$ is the dimensionless inverse temperature parameter, where Ω_0 is the bare frequency, \hbar is the reduced Planck constant, k_B is the Boltzmann constant and T is the temperature of the environment.

By using the method of characteristics, it can be shown that temporal evolution of the normally ordered characteristic function is described by the relation [20]:

$$\chi_N(\beta, t) = \text{Tr}[e^{\beta\hat{a}^\dagger} e^{-\beta^*\hat{a}} \hat{\rho}(t)] = e^{-\langle n \rangle |\beta|^2} \chi_{\text{ini}}(e^{(i\Omega - \Gamma)t}\beta), \quad \langle n \rangle = (1 - e^{-2\Gamma t})n_T \quad (12)$$

where $\Gamma = (1 - e^{-zT})\kappa$ is the relaxation rate; $n_T = (e^{zT} - 1)^{-1}$ is the mean number of thermal photons; and $\chi_{\text{ini}}(\beta)$ is determined by the initial condition $\hat{\rho}(0) = |\alpha_\nu\rangle\langle\alpha_\nu|$ as follows:

$$\begin{aligned}\chi_{\text{ini}}(\beta) &= \chi_\nu(\beta) = \langle\alpha_\nu|e^{\beta\hat{a}^\dagger}e^{-\beta^*\hat{a}}|\alpha_\nu\rangle \\ &= \frac{\cosh(\beta\alpha^* - \beta^*\alpha) + \nu e^{-2|\alpha|^2} \cosh(\beta\alpha^* + \beta^*\alpha)}{1 + \nu e^{-2|\alpha|^2}}.\end{aligned}\quad (13)$$

An important point is that the probability of clicks (8) is expressed in terms of the moments $\langle:\hat{n}^m:\rangle$:

$$P_{\Gamma\nu}(k) = \sum_{m=0}^{\infty} \frac{(-1)^m \eta_d^{m+k}}{k!m!} \langle:\hat{n}^{m+k}:\rangle. \quad (14)$$

In particular, for the cat states, generating function of the moments:

$$\begin{aligned}\tilde{\chi}_\nu(|\beta|) &= \frac{1}{2\pi} \int_0^{2\pi} \chi_\nu(|\beta|e^{i\phi})d\phi = \sum_{m=0}^{\infty} \frac{(-1)^m |\beta|^{2m}}{(m!)^2} \langle:\hat{n}^m:\rangle_\nu, \\ \langle:\hat{n}^m:\rangle_\nu &= \frac{1 + (-1)^m \nu e^{-2|\alpha|^2}}{1 + \nu e^{-2|\alpha|^2}} |\alpha|^{2m}\end{aligned}\quad (15)$$

is obtained by averaging $\chi_\nu(\beta)$ over the phase of β and leads to the expression for the probability of photocounts given by Eq. (9). Then the phase averaged characteristic function (12):

$$\tilde{\chi}_N(|\beta|, t) = \frac{1}{2\pi} \int_0^{2\pi} \chi_N(|\beta|e^{i\phi}, t)d\phi = e^{-\langle n \rangle |\beta|^2} \tilde{\chi}_\nu(\sqrt{T}|\beta|) = \sum_{m=0}^{\infty} \frac{(-1)^m |\beta|^{2m}}{(m!)^2} \langle:\hat{n}^m:\rangle \quad (16)$$

gives the expression for the normally ordered moments of the photon number operator

$$\langle:\hat{n}^m:\rangle = \sum_{l=0}^m \left[\frac{m!}{(m-l)!} \right]^2 \frac{\langle n \rangle^l}{l!} T^{m-l} \langle:\hat{n}^{m-l}:\rangle_\nu, \quad (17)$$

where $T = e^{-2\Gamma t}$ is the channel transmittance. Formula (17) can now be combined with Eq. (14) to compute the probability of photocounts (10). It is not difficult to see that, at $\eta_d = 1$, the probabilities $P_+^{(\text{odd})}$ and $P_-^{(\text{even})}$ vanish in the zero-temperature limit with $n_T = 0$. Thus we arrive at the conclusion that within the framework of Lindblad dynamics distinguishability of the Schrödinger cat states is solely governed by the mean number of thermal photons $\langle n \rangle \propto n_T$ rather than the channel transmittance.

3. Conclusion and discussion

We have studied the fidelity and the statistics of photocounts for the even and odd Schrödinger cat states subjected to the effects of decoherence in the noisy quantum channel representing a lossy optical fiber. For the beam splitter model of the quantum channel characterized by the transmittance τ , the probabilities that determine the photocounting statistics are computed in relation to τ and the mean photon number $|\alpha|^2$. By contrast to this model, exact results describing Lindblad dynamics of photon states show that the key parameter governing decoherence induced effects on photocounting is the mean number of thermal photons n_T .

In our analysis, the criterion used to discriminate between the even and odd states is based on the parity of photocounts registered by a photodetector. Since the vacuum state contribution to the even Schrödinger cat states is found to reduce their distinguishability, the odd coherent states present the most robust case.

In general, from our results, it might be concluded that, despite daunting nature of the task to utilize Schrödinger cat states for quantum communications, these states can be used in entanglement distribution (swapping) protocols at sufficiently short distances.

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