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Fast-forward of standard dynamics with use of electromagnetic field

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We introduce Khujakulov and Nakamura's scheme for the exact fast-forwarding of standard quantum dynamics for a charged particle. The idea allows the acceleration of both amplitude and phase of the wave function throughout the fast-forwarding time range. Firstly we shall apply the proposed method to 1-D free wave packet dynamics and obtain the electromagnetic field to ensure its rapid propagation and diffusion. Then we proceed to study 1-D quantum tunneling phenomenon, namely a rapid penetration of wave function through a delta-function type barrier. We elucidate the distribution of the tunneling current density to show the remarkable enhancement of the tunneling rate (tunneling power) due the fast-forwarding. We introduce two types of time-magnification factors and confirm the stability of fast-forward against the variation of such factors.

Keywords: acceleration of standard dynamics, free wave packets, quantum tunneling.

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1. Introduction

Masuda and Nakamura [1–3] investigated a method of acceleration quantum dynamics with use of a characteristic driving potential determined by the additional phase of the wave function. One can accelerate a given quantum dynamics to obtain a target state in any desired short period. This kind of acceleration is called the fast-forward of quantum dynamics, which constitutes one of the more promising ways of attaining a shortcut to adiabaticity [4–9]. The relationship between the fast forward and the shortcut to adiabaticity is currently clear [10, 11]. Before embarking upon the main part of the text, we briefly summarize the theory of the fast-forward of quantum dynamics updated by Khujakulov and Nakamura [12]. The Schrödinger equation on standard time scale is represented as:

$$\imath\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_0 + V_0(\mathbf{x}, t)\psi_0, \tag{1}$$

$\psi_0 \equiv \psi_0(\mathbf{x}, t)$ is a known function of space $\mathbf{x}$ and time t and is called a standard state. For any long time T called a standard final time, we choose $\psi_0(x, t = T)$ as a target state that we are going to generate.

Let $\tilde{\psi}_0(\mathbf{x}, t)$ be a fast-forwarded state of $\psi_0(\mathbf{x}, t)$ as defined by

$$\tilde{\psi}_0(\mathbf{x}, t) \equiv \psi_0(\mathbf{x}, \Lambda(t)) \equiv \Psi_{FF}(\mathbf{x}, t) \tag{2}$$

with

$$\Lambda(t) = \int_0^t \alpha(\mathrm{t}')\,\mathrm{d}t'. \tag{3}$$

$\alpha(t)$ is a magnification scale factor defined by

$$\begin{aligned} \alpha(0) &= 1, \\ \alpha(t) &> 1 \qquad (0 < t < T_{FF}), \\ \alpha(t) &= 1 \qquad (t \geq T_{FF}). \end{aligned} \tag{4}$$

T_{FF} is the final fast-forward time defined by

$$T = \int_0^{T_{FF}} \alpha(t)\mathrm{d}t. \tag{5}$$

At the final time of the fast-forward (T_{FF}) and we can obtain the exact target state

$$\psi_{FF}(T_{FF}) = \psi_0(T). \tag{6}$$

The explicit expression for $\alpha(t)$ in the fast-forward range ($0 \le t \le T_{FF}$) is proposed by Masuda and Nakamura [1,3] as:

$$\alpha(t) = \bar{\alpha} - (\bar{\alpha} - 1)\cos\left(\frac{2\pi}{T/\bar{\alpha}}t\right), \tag{7}$$

where $\bar{\alpha}$ is the mean value of $\alpha(t)$ and is given by $\bar{\alpha} = T/T_{FF}$. Besides the time-dependent scaling factor in Eq. (7) in the fast-forward range, we can also have recourse to the uniform scaling factor:

$$\alpha(t) = \bar{\alpha} \qquad (0 \le t \le T_{FF}), \tag{8}$$

which may be useful in the quantitative analysis of fast forward. Khujakulov and Nakamura [12] tried to realize ψ_{FF} by applying the electromagnetic field, $\mathbf{E}_{FF}$ and $\mathbf{B}_{FF}$.

Let's assume ψ_{FF} is the solution of the time-dependent Schrödinger equation for a charged particle in the presence of additional vector $\mathbf{A}_{FF}(\mathbf{x},t)$ and scalar $V_{FF}(\mathbf{x},t)$ potentials:

$$\begin{aligned} \imath\hbar\frac{\partial\psi_{FF}}{\partial t} &= \left(\frac{1}{2m}(\frac{\hbar}{i}\nabla - \mathbf{A}_{FF})^2 + V_{FF} + V_0\right)\psi_{FF} \\ &= -\frac{\hbar^2}{2m}\nabla^2\psi_{FF} + \frac{i\hbar}{2m}(\nabla\cdot\mathbf{A}_{FF})\psi_{FF} \\ &+\frac{i\hbar}{m}\mathbf{A}_{FF}\cdot\nabla\psi_{FF} + \frac{\mathbf{A}_{FF}^2}{2m}\psi_{FF} + (V_{FF} + V_0)\psi_{FF} \end{aligned} \tag{9}$$

where, for simplicity, we employ the prescription of a positive unit charge ($q = 1$) and the unit velocity of light ($c = 1$). The driving electromagnetic field is given by:

$$\mathbf{E}_{FF} = -\frac{\partial\mathbf{A}_{FF}}{\partial t} - \nabla V_{FF}, \quad \mathbf{B}_{FF} = \nabla\times\mathbf{A}_{FF}. \tag{10}$$

Substituting Eqs. (1) and (2) into Eq. (9) and taking its real and imaginary parts, we obtain a pair of equations:

$$\nabla\cdot\mathbf{A}_{FF} + 2\mathrm{Re}\left[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}\right]\mathbf{A}_{FF} + \hbar(\alpha-1)\mathrm{Im}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] = 0 \tag{11}$$

and

$$V_{FF} = -(\alpha-1)\frac{\hbar^2}{2m}\mathrm{Re}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] + \frac{\hbar}{m}\mathbf{A}_{FF}\mathrm{Im}\left[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}\right] - \frac{1}{2m}\mathbf{A}_{FF}^2 + (\alpha-1)V_0. \tag{12}$$

Now, we write $\tilde{\psi}_0$ as:

$$\tilde{\psi}_0 = \rho e^{i\eta} \tag{13}$$

with use of the real amplitude ρ and phase η defined by:

$$\begin{aligned} \rho &\equiv \rho(\mathbf{x},\Lambda(t)), \\ \eta &\equiv \eta(\mathbf{x},\Lambda(t)). \end{aligned} \tag{14}$$

Then, one finds that:

$$\mathbf{A}_{FF} = -\hbar(\alpha-1)\nabla\cdot\eta \tag{15}$$

satisfies Eq. (11), and that

$$V_{FF} = -(\alpha-1)\hbar\frac{\partial\eta}{\partial\Lambda(t)} - \frac{\hbar^2}{2m}(\alpha^2-1)(\nabla\eta)^2. \tag{16}$$

With use of the driving vector $\mathbf{A}_{FF}$ and scalar V_{FF} potentials in Eqs. (15) and (16), we can obtain the fast-forwarded ψ_{FF} in Eq.(2)

Noting $\mathbf{B}_{FF} = \nabla\times\mathbf{A}_{FF} = 0$, only the electric field $\mathbf{E}_{FF}$ is required to accelerate a given dynamics. With use of Eqs. (10), (15) and (16), $\mathbf{E}_{FF}$ is given explicitly by [13]:

$$\mathbf{E}_{FF} = \hbar\dot{\alpha}\nabla\eta + \hbar\frac{\alpha^2-1}{\alpha}\partial_t\nabla\eta + \frac{\hbar^2}{2m}(\alpha^2-1)\nabla(\nabla\eta)^2. \tag{17}$$

A remarkable issue of the present scheme is the enhancement of the current density $\mathbf{j}_{FF}$. Using a generalized momentum which includes a contribution from the vector potential in Eq. (15), we see:

$$\mathbf{j}_{FF}(\mathbf{x},t) \equiv \psi^*_{FF}(\mathbf{x},t)\frac{1}{m}\left(\frac{\hbar}{i}\nabla - \mathbf{A}_{FF}\right)\psi_{FF}(\mathbf{x},t)$$
$$= \frac{\hbar}{m}\alpha(t)\rho^2(\mathbf{x},\Lambda(t))\nabla\eta(\mathbf{x},\Lambda(t)), \tag{18}$$

where we employ the prescription of a positive unit charge. Noting the current density in the standard dynamics:

$$\mathbf{j}(\mathbf{x},t) \equiv \mathrm{Re}\left[\psi^*_0(\mathbf{x},t)\frac{\hbar}{im}\nabla\psi_0(\mathbf{x},t)\right] = \frac{\hbar}{m}\rho^2(\mathbf{x},t)\nabla\eta(\mathbf{x},t), \tag{19}$$

we find [12]

$$\mathbf{j}_{FF}(\mathbf{x},t) = \alpha(t)\mathbf{j}(\mathbf{x},\Lambda(t)). \tag{20}$$

Thus, the standard current density at each of spatial points becomes both squeezed and magnified by a time-scaling factor $\alpha(t)$ in Eq. (7) or Eq. (8) as a result of the exact fast forwarding which enables acceleration of both amplitude and phase of the wave function throughout the time evolution.

2. Free wave packet dynamics

The time evolution of a free electron wave packet in 1 dimension is described by the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}. \tag{21}$$

Let's consider the initial state of electron as

$$\Psi_0(x,t=0) = \pi^{-1/4}\Delta^{-1/2}\mathrm{exp}\left(-\frac{x^2}{2\Delta} + i\frac{p_0}{\hbar}x\right). \tag{22}$$

Then, the time evolution of Ψ_0 is given by

$$\Psi_0(x,t) = \int_{-\infty}^{\infty} dx' K(x,t;x',0)\Psi_0(x',t=0). \tag{23}$$

Where K is the kernel propagator defined by:

$$K(x,t;x',0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\mathrm{exp}\left[\frac{im(x-x')^2}{2\hbar t}\right] \tag{24}$$

$\Psi_0(x,t)$ in Eq.(23) becomes:

$$\Psi_0(x,t) = \pi^{-1/4}\Delta^{-1/2}\left[1+\frac{i\hbar t}{m\Delta^2}\right]^{-1/2}\times \tag{25}$$
$$\mathrm{exp}\left[-\frac{(x-\frac{p_0}{m}t)^2}{2\Delta^2(1+(\hbar t/m\Delta^2)^2)} + i\left[\frac{p_0}{\hbar}(x-\frac{p_0}{m}t) + \frac{p_0^2}{2\hbar m}t + \frac{\hbar t(x-\frac{p_0}{m}t)^2}{2m\Delta^4(1+(\hbar t/m\Delta^2)^2)}\right]\right].$$

From Eq. (23), the probability amplitude $|\Psi|^2$ and the phase η is given by:

$$|\Psi|^2 = \pi^{-1/4}\Delta^{-1/2}\left[1+(\frac{\hbar t}{m\Delta^2})^2\right]^{-1/4}\mathrm{exp}\left[-\frac{(x-\frac{p_0}{m}t)^2}{2\Delta^2(1+(\hbar t/m\Delta^2)^2)}\right] \tag{26}$$

and

$$\eta = \frac{p_0}{\hbar}\left(x-\frac{p_0}{m}t\right) + \frac{p_0^2}{2\hbar m}t + \frac{\hbar t(x-\frac{p_0}{m}t)^2}{2m\Delta^4(1+(\hbar t/m\Delta^2)^2)} - \frac{1}{2}\mathrm{arctan}\left(\frac{\hbar t}{m\Delta^2}\right), \tag{27}$$

respectively.

Figure 1 shows $|\Psi|^2$ of the standard wave packet as a function of x and t.

Now, we shall proceed to analyze the fast-forward the above dynamics. The fast-forward state is given by:

$$\Psi_{\mathrm{FF}} = \Psi_0(x,(\Lambda(t))), \tag{28}$$

Figure 2 shows $|\Psi_{\mathrm{FF}}|^2$ where the mean time-magnification factor $\bar{\alpha}$ = 5 is used.

To realize the fast-forward state, the electric field is given by Eq. (17). Using Eqs. (17) and (27), we can evaluate $\mathbf{E}_{FF}$, which is depicted in Fig. 3.

Now, we shall apply the present scheme to tunneling phenomena in quantum mechanics.

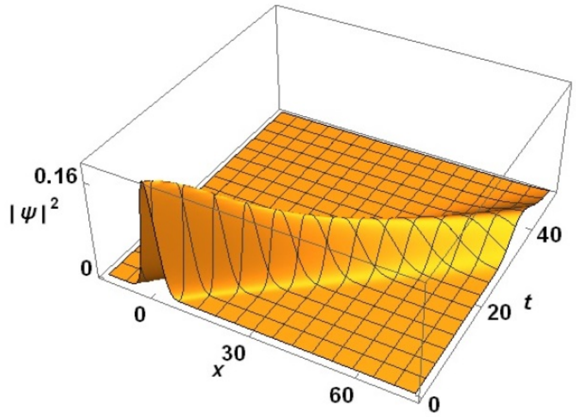

FIG. 1. D plot of $|\Psi|^2$ for a standard wave packet as a function of x and t. The final time $T = 50$

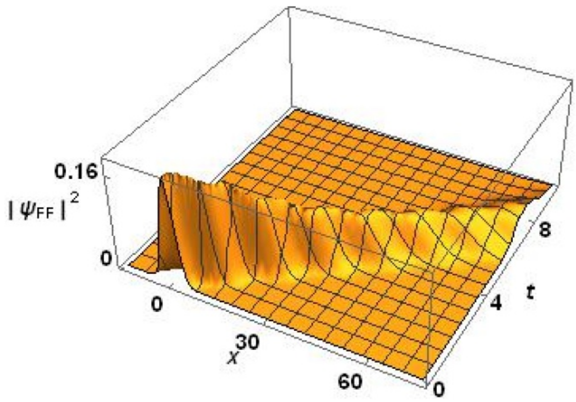

FIG. 2. 3D plot of $|\Psi_{\rm FF}|^2$ fast-forward wave packet with $\bar{\alpha} = 5$ as a function of x and t for the $\bar{\alpha} = 5$, $T_{FF} = 10$

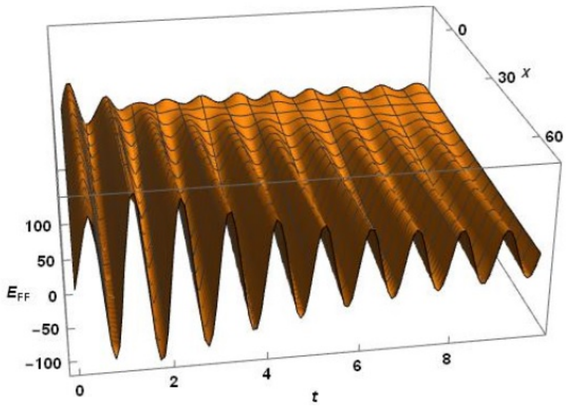

FIG. 3. 3D plot of E_{FF} as a funtion of x and t, $\bar{\alpha} = 5$

3. Fast-forward of tunneling of wave packet dynamics

Confining ourselves again to 1-D motion, we now investigate the time evolution of a localized wave packet when it runs through a delta-function barrier. The initial wave packet centered at $x = -x_0$ and having a momentum k is expressed as:

$$\psi^{(0)}(x,0) = \sqrt{\beta} e^{-\beta|x+x_0|} e^{ik(x+x_0)}. \tag{29}$$

$\psi^{(0)}(x,0)$ satisfies the normalization condition $\int_{-\infty}^{\infty} |\psi^{(0)}(x,0)|^2 dx = 1$. Therefore, $\langle x \rangle = -x_0$ and $\langle p \rangle = k$ at $t = 0$.

The time-dependent Schrödinger equation with a δ function barrier at $x = 0$ is given by

$$\left[i\hbar\partial_t + (\hbar^2/2m)\partial_x^2\right] \psi_0(x,t) = V(x)\psi_0(x,t), \tag{30}$$

with $V(x) = V_0\delta(x)$. In order to simplify notation, we shall use "natural unit" ($\hbar = m = 1$).

The time evolution of ψ_0 follows for $t > 0$ from:

$$\psi_0(x,t) = \psi^{(0)}(x,t) - V_0 \int_{-\infty}^{\infty} \mathrm{d}x' \times M(|x| + |x'|; -iV_0; t)\psi_0(x',0). \tag{31}$$

Here $M(x;k;t)$ is "Moshinsky" function defined in terms of the complementary error function by

$$M(x;k;t) = \frac{1}{2} e^{i(kx - k^2 t/2)} \mathrm{erfc}\left(\frac{x - kt}{\sqrt{2it}}\right), \tag{32}$$

which is interpreted as the wave function of a monochromatic particle that is confined to the left half-space $x \leq 0$ at $t = 0$. On the other hand, $\psi^{(0)}(x,t)$ is the free-particle wave function:

$$\psi^{(0)}(x,t) = \int_{-\infty}^{\infty} dx' K(x,t|x',0)\psi^{(0)}(x',0), \tag{33}$$

with K the free-particle propagator given in Eq. (24).

The explicit solution for $t > 0$ was given by Elberfeld and Kleber [13] as:

$$\begin{aligned} \psi_0(x,t) = \sqrt{\beta}[M(x + x_0; k - i\beta; t) + M(-x - x_0; -k - i\beta; t)] \\ +V_0\sqrt{\beta}[S(x_0, \lambda^*; t) - S(x_0, -\lambda; t) + e^{-\lambda x_0}[S(0, -\lambda; t) + S(0, \lambda; t)]], \end{aligned} \tag{34}$$

where $\lambda = \beta - ik$ and $S(\xi, \lambda; t)$ is defined by:

$$S(\xi, \lambda; t) = [1/(V_0 - \lambda)][M(|x| + \xi; -iV_0; t) - M(|x| + \xi; -i\lambda; t)]. \tag{35}$$

The first bracket on r.h.s. of Eq. (34) describes the time evolution of the free $(V_0 = 0)$ wave packet, and the second bracket denotes the sum of reflected and transmitted waves.

The tunneling current density is:

$$j(x,t) = \mathrm{Im}[\psi_0^*(x,t)\partial_x\psi_0(x,t)]. \tag{36}$$

Now, we analyze the fast forward of tunneling of wave packets, and find the corresponding current density. Here we shall present the results not investigated by Khujakulov and Nakamura [13]. By extracting the space-time dependent phase η of the wave function in Eq. (34), one can obtain both vector and scalar potentials in Eqs. (15) and (16). Under these driving potentials, one can generate the fast-forward state of a tunneling wave packet through the barrier as:

$$\psi_{FF}(x,t) \equiv \psi_0(x, \Lambda(t)), \tag{37}$$

which accelerates both amplitude and phase of Eq. (34) exactly. From Eq. (20), the tunneling current density for

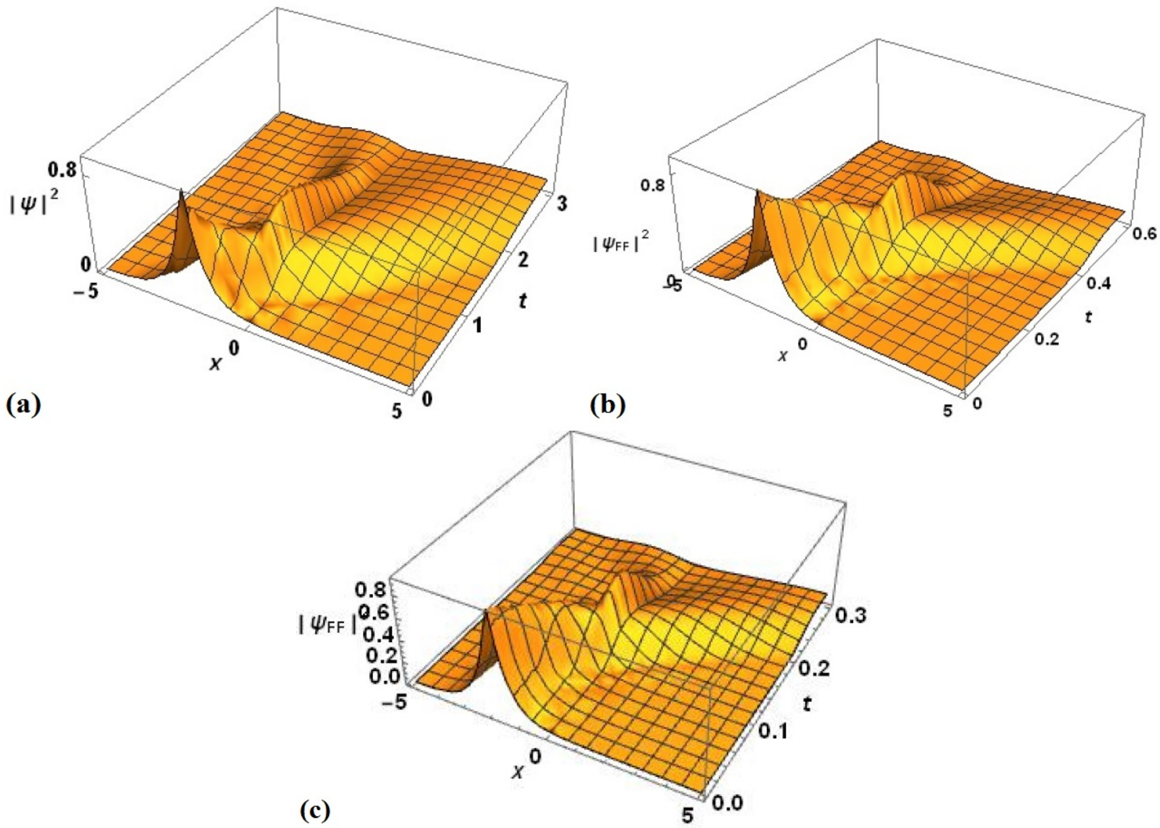

FIG. 4. 3D plot of wave function amplitude (a) $|\Psi|^2$; (b) $|\Psi_{\mathrm{FF}}|^2$ with $\bar{\alpha} = 5$; (c) $|\Psi_{\mathrm{FF}}|^2$ with $\bar{\alpha} = 10$

the fast-forward tunneling phenomenon is:

$$j_{FF}(x,t) = \alpha(t) j(x, \Lambda(t)). \tag{38}$$

Figure 4 shows the probability amplitude as a function of x and t. In our numerical analysis we choose $x_0 = 2, k = 2$ and $\beta = 1$. We use typical space and time scales like $L = 10^{-2}\times$ *the linear dimension of a device* and $\tau = 10^{-2}\times$ *the phase coherent time* and put $\frac{\hbar}{m} = 1(\times L^2\tau^{-1})$. Therefore, the above choice means $x_0 = 2(\times L), k = 2(\times L^{-1})$ and $\beta = 1(\times L^{-1})$. We shall show the standard dynamics up to $T = 3(\times\tau)$ and its fast-forward version up to $T_{FF} \equiv \frac{T}{\bar{\alpha}}(\tau)$ with use of the mean time acceleration factor $\bar{\alpha} = 5$ and $\bar{\alpha} = 10$.

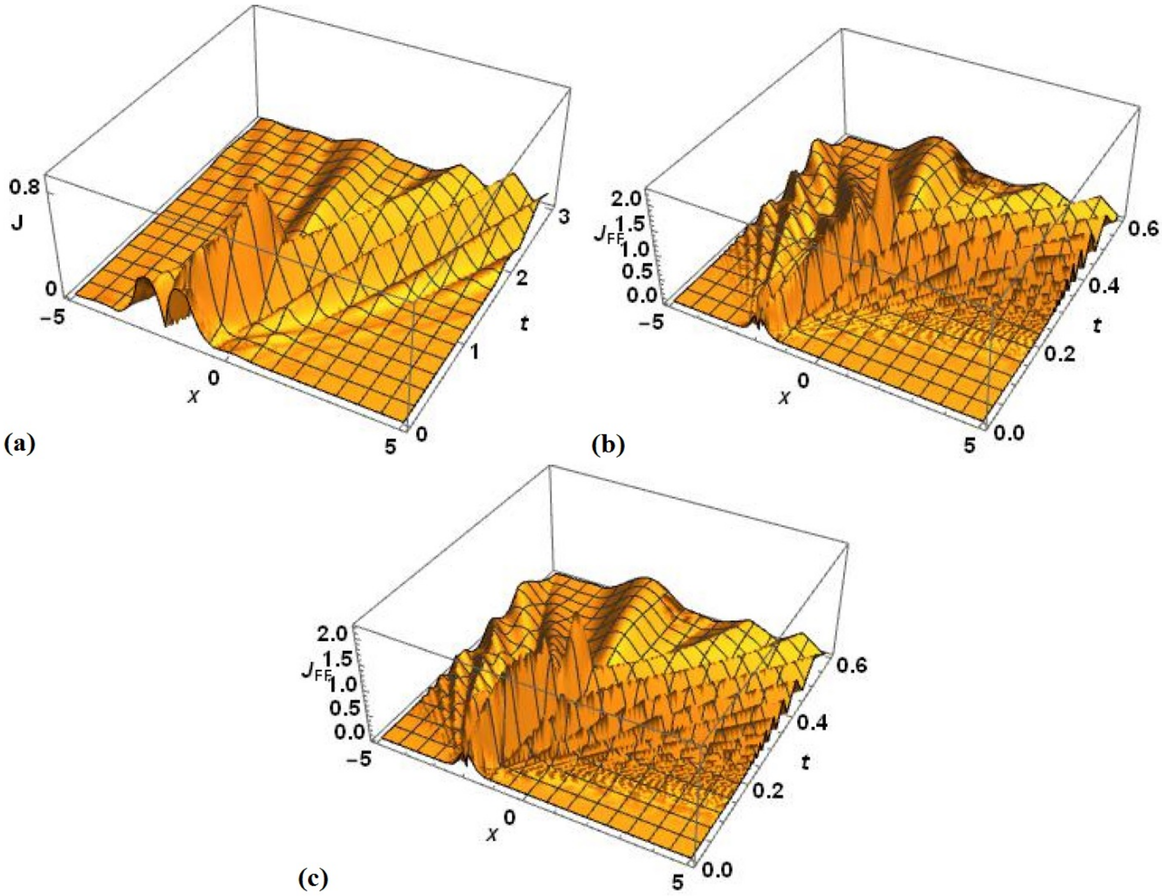

FIG. 5. 3D plots of current density as a function of x and t (a) standard current density $j(x,t)$ in Eq. (36); (b) fast-forward current in Eq. (38) with $\alpha = \bar{\alpha} - (\bar{\alpha}-1)\cos(\frac{2\pi}{T_{FF}}t)$ and $\bar{\alpha} = 5$; (c) fast-forward current in Eq. (38) with $\alpha = 1 + 6(\bar{\alpha}-1)\frac{t}{T_{FF}}(1 - \frac{t}{T_{FF}})$ and $\bar{\alpha} = 5$

We see the exponential wave function partially goes through the barrier and is partially reflected back. The dynamics up to T on the standard time scale is reproduced in the fast-forward dynamics up to T_{FF}. The phenomena in the latter time scale is just the squeezing (along the time axis) of those in the former time scale.

Figure 5 shows the standard and fast-forwarded tunneling currents as a function of x and t. Here, we choose $T = 5, T_{FF} = 1$ and $\bar{\alpha} = 5$.

As for fast-forwarding, we have employed two kinds of time-magnification factor: (i) cos-type, $\alpha = \bar{\alpha} - (\bar{\alpha}-1)\cos(\frac{2\pi t}{T_{FF}})$ and (ii) parabola-type, $1 + 6(\bar{\alpha}-1)\frac{t}{T_{FF}}(1 - \frac{t}{T_{FF}})$. We find the temporal behavior of the current density is both squeezed and amplifield, as compared to the standard version of j. We also see this result is not affected by the fuctional form of $\alpha(t)$. Fig. 6 shows E_{FF} for two kinds of time-magnification factors, which also shows that E_{FF} is not sensitive to the functional form of $\alpha(t)$.

4. Conclusion

By using the fast-forward theory which makes possible the exact acceleration of the phase and amplitude of a standard wave function, we investigated fast-forward of wave packet dynamics with and without a potential

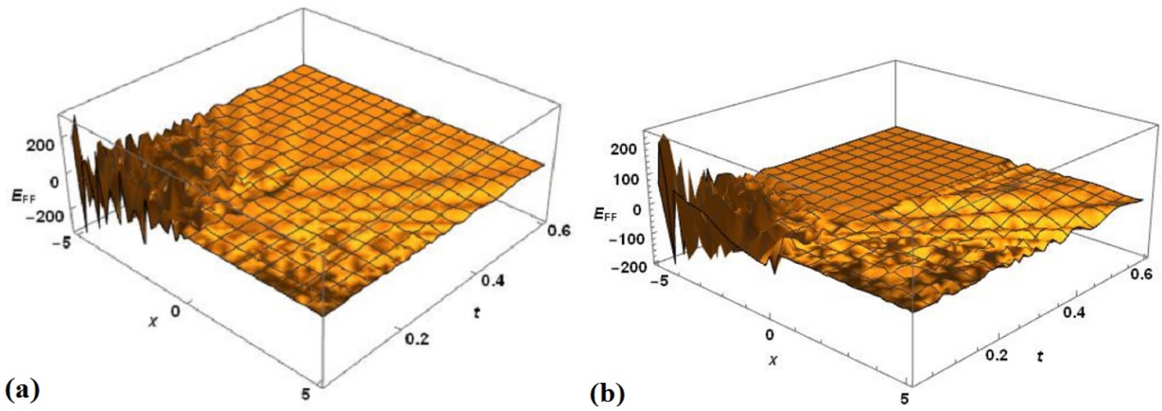

FIG. 6. Electric field E_{FF}: (a) cos-type time-magnification factor; (b) parabola-type time-magnification factor

barrier. We choose two kinds of time-magnification scaling factors $\alpha(t)$ ((i) cos type and (ii) parabolic type). The fast-forwarded current density distribution and the driving electromagnetic field have proved to be unaffected by the details of $\alpha(t)$, which indicate the stability of fast-forwarding mechanism.

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