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A NEW MODEL FOR QUANTUM DOT LIGHT EMITTING-ABSORBING DEVICES: PROOFS AND SUPPLEMENTS

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Motivated by the Jaynes-Cummings (JC) model, we consider here a quantum dot coupled simultaneously to a reservoir of photons and to two electric leads (free-fermion reservoirs). This new Jaynes-Cummings-Leads (JCL) model permits a fermion current through the dot to create a photon flux, which describes a light-emitting device. The same model is also used to describe the transformation of a photon flux into a current of fermions, i.e. a quantum dot light-absorbing device. The key tool to obtain these results is the abstract Landauer-Büttiker formula.

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1. Introduction

In the following the fermion current going through a quantum dot is analyzed as a function of

(1) the electro-chemical potentials on leads and of
(2) the contact with the external photon reservoir.

Although the latter is the canonical JC-interaction, the coupling of the JC model with leads needs certain precautions, if we wish to remain in the one-particle quantum mechanical Hamiltonian approach and scattering theory framework. To this end, we proposed a new Jaynes-Cummings-Leads (JCL-) model [19]. This model makes possible a photon flux into the resonator, created by a fermion current through the dot; i.e. it describes a *light-emitting* device, as well as to transform the external photon flux into a current of fermions, which corresponds to a quantum dot *light-absorbing* device.

The paper is an extended version of [19], which means that the JCL-model, as well as all theorems, corollaries of this article one can already find in [19], however, without any proofs. In the following article, we are going to close this gap and give full proofs of all statements. In doing so, we have added some statements absent in [19].

The paper is organized as follows. The JCL-model is introduced and discussed in Sections 2.1-2.7. For simplicity, we choose for the lead Hamiltonians the one-particle discrete Schrödinger operators, with constant one-site (electric) potentials on each of leads. In Section 2.5, we show that the our model fits into the framework of *trace-class scattering*. In Section 2.7, we verify

the important property that the coupled Hamiltonian has no singular continuous spectrum. Our main tool for analysis of different currents is an abstract Landauer-Büttiker-type formula applied in Sections 3.1 and 3.2 to the case of the JCL-model. This allows us to calculate the outgoing flux of photons induced by electric current via leads. This corresponds to a *light-emitting* device. We also found that pumping the JCL quantum dot with a photon flux from a resonator may induce a current of fermions into the leads. This reversing imitates a quantum *light-absorbing* cell device. These are the main properties of our model and the main application of the Landauer-Büttiker-type formula of Sections 3.1 and 3.2. They are presented in Sections 4 and 5, where we distinguish *contact-induced* and *photon-induced* fermion currents.

To describe the results of Sections 4 and 5, note that in our setup, the sample Hamiltonian is a *two-level* quantum dot *decoupled* from the one-mode photon resonator. Then, the unperturbed Hamiltonian H_0 describes a collection of *four* totally decoupled sub-systems: a sample, a resonator and two leads. The perturbed Hamiltonian H is a fully coupled system, and the feature of our model is that it is *totally* (i.e. including the leads) *embedded* in the external electromagnetic field of the resonator. Hence, each electron can be interpreted as a fermion with *internal* harmonic degrees of freedom, or a Fermi-particle caring its individual *photon cloud*.

Similar to the "Black Box" system-leads (SL-) model $\{H_{SL}, H_0\}$, [1], [4], it turns out that the JCL-model also fits into the framework of the abstract Landauer-Büttiker formula, and in particular, is a trace-class scattering system $\{H_{JCL} = H, H_{SL}\}$. The current in the SL-model is called the *contact-induced* current J^c_{el}. This current was the subject of numerous papers, see e.g. [1,5], or [4]. Note that the current J_{el} is due to the difference of the electro-chemical potentials between two leads, but it may be *zero* even if this difference is not null [12, 13].

The fermion current in the JCL-model, takes into account the effect of the electron-photon interaction under the assumption that the leads are already coupled. This is called the *photon-induced* component J^{ph}_{el} of the total current. To the best of our knowledge, the present paper is the first which rigorously studies this phenomenon. We show that the total free-fermion current J_{el} in the JCL-model decomposes into a sum of the contact- and the photon-induced currents: $J_{el} := J^c_{el} + J^{ph}_{el}$. An extreme case is when the contact-induced current is zero, but the photon-induced component is not, c.f. Section 5.1. In this case, the flux of photons J_{ph} out of the quantum dot (sample) is also non-zero, i.e. the dot serves as a light emitting device, c.f. Section 5.2. In general the $J_{ph} \neq 0$ only when the photon-induced component $J^{ph}_{el} \neq 0$.

By choosing the parameters of the model in an suitable manner, one can get either a photon emitting or a photon absorbing system. Hence, the JCL-model can be used either as a light emission device or as a solar cell. Proofs of explicit formulas for fermion and photon currents $J^{ph}_{l,el}$, J_{ph} are contained in Sections 4 and 5.

Note that the JCL-model is called *mirror symmetric* if (roughly speaking) one can interchange left and right leads and the JCL-model remains unchanged. In Section 5 we discuss a surprising example of a mirror symmetric JCL-model such that the free-fermion current is *zero* but the model is photon emitting. This peculiarity is due to a specific choice of the *photon-electron* interaction, which produces fermions with internal harmonic degrees of freedom.

2. Jaynes-Cummings quantum dot coupled to leads

2.1. Jaynes-Cummings model

The starting point for the construction of our JCL-model is the quantum optics Jaynes-Cummings Hamiltonian H^{JC}. Its simplest version is a *two-level* system (quantum dot) with the energy spacing ε, defined by Hamiltonian h_S on the Hilbert space $\mathfrak{h}_S = \mathbb{C}^2$, see e.g. [16]. It is assumed that this system is "open" and interacts with the one-mode ω photon resonator with Hamiltonian h^{ph}.

Since mathematically h^{ph} coincides with a quantum harmonic oscillator, the Hilbert space of the resonator is the boson Fock space $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C})$ over $\mathbb{C}$ and

$$h^{ph} = \omega\, b^* b\,. \tag{2.1}$$

Here, b^* and b are verifying the Canonical Commutation Relations (CCR) creation and annihilation operators with domains in $\mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$. Operator (2.1) is self-adjoint on its domain:

$$\mathrm{dom}(h^{ph}) = \left\{ (k_0, k_1, k_2, \ldots) \in \ell^2(\mathbb{N}_0) : \sum_{n\in\mathbb{N}_0} n^2 |k_n|^2 < \infty \right\}.$$

Note that the canonical basis $\{\phi_n := (0, 0, \ldots, k_n = 1, 0, \ldots)\}_{n\in\mathbb{N}_0}$ in $\ell^2(\mathbb{N}_0)$ consists of eigenvectors of operator (2.1): $h^{ph}\phi_n = n\omega\,\phi_n$.

To model the *two-level* system with the energy spacing ε, one fixes in $\mathbb{C}^2$ two ortho-normal vectors $\{e_0^S, e_1^S\}$, for example $e_0^S := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $e_1^S := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which are eigenvectors of Hamiltonian h_S with eigenvalues $\{\lambda_0^S = 0,\ \lambda_1^S = \varepsilon\}$. To this end, we set:

$$h_S := \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ , \tag{2.2}$$

and we introduce two *ladder* operators:

$$\sigma^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Then, one obtains $h_S = \varepsilon\ \sigma^+\sigma^-$ as well as $e_1^S = \sigma^+ e_0^S$, $e_0^S = \sigma^- e_1^S$ and $\sigma^- e_0^S = 0$. So, e_0^S is the ground state of Hamiltonian h_S. Note that the *non-interacting* Jaynes-Cummings Hamiltonian H_0^{JC} resides in the space $\mathfrak{H}^{JC} = \mathfrak{h}_S \otimes \mathfrak{h}^{ph} = \mathbb{C}^2 \otimes \mathfrak{F}_+(\mathbb{C})$ and it is defined as the *matrix* operator:

$$H_0^{JC} := h_S \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S} \otimes h^{ph}\ . \tag{2.4}$$

Here, $I_{\mathfrak{h}^{ph}}$ denotes the identity operator in the Fock space $\mathfrak{h}^{ph}$, whereas $I_{\mathfrak{h}_S}$ stays for the identity matrix in the space $\mathfrak{h}_S$.

With operators (2.3), the interaction V_{Sb} between quantum dot and photons (bosons) in the resonator is defined (in the rotating-wave approximation [16]) by the operator:

$$V_{Sb} := g_{Sb}\ (\sigma^+ \otimes b + \sigma^- \otimes b^*)\ . \tag{2.5}$$

Operators (2.4) and (2.5) define the Jaynes-Cummings model Hamiltonian:

$$H_{JC} := H_0^{JC} + V_{Sb}\ , \tag{2.6}$$

which is a self-adjoint operator on the common domain $\mathrm{dom}(H_0^{JC}) \cap \mathrm{dom}(V_{Sb})$. The standard interpretation of H_{JC} is that (2.6) describes an "open" two-level system interacting with an external one-mode electromagnetic field [16].

Since the one-mode resonator is able to absorb *infinitely* many bosons, this interpretation sounds reasonable, but one can see that the spectrum $\sigma(H^{JC})$ of the Jaynes-Cummings model is *discrete*. To this end, note that the so-called number operator $\mathfrak{N}_{JC} := \sigma^+\sigma^- \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}_S} \otimes b^* b$ commutes with H_{JC}. Then, since for any $n \geqslant 0$:

$$\mathfrak{H}_{n>0}^{JC} := \{\zeta_0 e_0^S \otimes \phi_n + \zeta_1 e_1^S \otimes \phi_{n-1}\}_{\zeta_{0,1}\in\mathbb{C}}\ ,\ \mathfrak{H}_{n=0}^{JC} := \{\zeta_0 e_0^S \otimes \phi_0\}_{\zeta_0\in\mathbb{C}}\ ,$$

are eigenspaces of operator $\mathfrak{N}_{JC}$, they reduce H_{JC}, i.e. $H_{JC} : \mathfrak{H}_n^{JC} \to \mathfrak{H}_n^{JC}$. Note that $\mathfrak{H}^{JC} = \bigoplus_{n\geqslant 0} \mathfrak{H}_n^{JC}$, where each $\mathfrak{H}_n^{JC}$ is invariant subspace of operator (2.6). Therefore, it has

the representation:

$$H_{JC} = \bigoplus_{n\in\mathbb{N}_0} H_{JC}^{(n)}\ ,\ \ n>1\ ,\ H_{JC}^{(0)} = 0\,. \tag{2.7}$$

Here operators $H_{JC}^{(n)}$ are the restrictions of H_{JC}, which act in each $\mathfrak{H}_n^{JC}$ as follows:

$$\begin{aligned} &H_{JC}^{(n)}(\zeta_0\, e_0^S\otimes\phi_n + \zeta_1\, e_1^S\otimes\phi_{n-1}) \\ &\quad = [\zeta_0 n\omega + \zeta_1 g_{Sb}\sqrt{n}]\, e_0^S\otimes\phi_n + [\zeta_1(\varepsilon+(n-1)\omega) + \zeta_0 g_{Sb}\sqrt{n}]\, e_1^S\otimes\phi_{n-1}\ . \end{aligned} \tag{2.8}$$

Hence, the spectrum $\sigma(H_{JC}) = \bigcup_{n\geqslant 0}\sigma(H_{JC}^{(n)})$. By virtue of (2.8), the spectrum $\sigma(H_{JC}^{(n)})$ is defined for $n\geqslant 1$ by eigenvalues $E(n)$ of two-by-two matrix $\widehat{H}_{JC}^{(n)}$ acting on the coefficient space $\{\zeta_0,\zeta_1\}$:

$$\widehat{H}_{JC}^{(n)}\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix} = \begin{pmatrix}\varepsilon+(n-1)\,\omega & g_{Sb}\sqrt{n}\\ g_{Sb}\sqrt{n} & n\omega\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix} = E(n)\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix}\ . \tag{2.9}$$

Then, (2.7) and (2.9) imply that the spectrum of the Jaynes-Cummings model Hamiltonian H_{JC} is *pure point*:

$$\sigma(H_{JC}) = \sigma_{p.p.}(H_{JC}) = \{0\}\cup\bigcup_{n\in\mathbb{N}}\left\{n\omega + \frac{1}{2}(\varepsilon-\omega) \pm \sqrt{(\varepsilon-\omega)^2/4 + g_{Sb}^2 n}\right\}.$$

This property evidently persists for any system Hamiltonian h_S with discrete spectrum and linear interaction (2.5) with a finite mode photon resonator [16].

We resume the above observations concerning the Jaynes-Cummings model, which is our starting point, by following remarks:

(a) The standard Hamiltonian (2.6) describes instead of *flux* only oscillations of photons between resonator and quantum dot, i.e. the system h_S is not "open" enough.

(b) Since one our aim is to model a *light-emitting* device, the system h_S needs an *external* source of energy to pump it into the dot, which will be transformed by interaction (2.5) into the outgoing *photon current* by pumping the resonator.

(c) To reach this aim we extend the standard Jaynes-Cummings model to our JCL-model by attaching to the quantum dot h_S (2.2) two *leads*, which are (infinite) reservoirs of *free* fermions. Manipulating with *electro-chemical* potentials of fermions in these reservoirs we can force one of them to inject fermions in the quantum dot, whereas another one to absorb the fermions out the quantum dot with the same rate. This current of fermions throughout the dot will pump the dot and induce a photon current according scenario (b).

(d) The most subtle point is to invent a *leads-dot* interaction V_{lS}, which ensures the above mechanism and which is simple enough that one would still be able to treat this JCL-model using our extension of the Landauer-Büttiker formalism.

2.2. The JCL-model

First, let us make some general remarks and formulate certain indispensable conditions when one follows the modeling (d).

(1) Note that since the Landauer-Büttiker formalism [13] is essentially a scattering theory on a contact between two subsystems, it is developed only on a "one-particle" level. This allows one to study with this formalism only ideal (*non-interacting*) many-body systems. We impose this condition on many-body fermion systems (electrons) in two leads. Thus, only direct interaction between different components of the system: dot-photons V_{Sb} and electron-dot V_{lS} are allowed.

(2) It is well-known that fermion reservoirs are technically simpler to treat than boson ones [13]. Moreover, in the framework of our model, it is also very natural since we study electric current, although produced by "non-interacting electrons". So, below we use fermions/electrons synonymously.
(3) In spite of the precautions formulated above, the first difficulty to consider in an ideal many-body system interacting with quantized electromagnetic field (photons) is induced *indirect* interaction. If electrons can emit and absorb photons, it is possible for one electron to emit a photon that another electron absorbs, thus creating an indirect photon-mediated electron-electron interaction. This interaction makes it impossible to develop the Landauer-Büttiker formula, which requires a non-interacting framework.

Assumption 2.1. To solve this difficulty, we forbid in our model the photon-mediated interaction. To this end, we assume that every electron (in leads and in dot) interacts with its *own* distinct copy of the electromagnetic field. So, considering electrons together with their photon fields as non-interacting "composed particles", allows us to apply the Landauer-Büttiker approach. Formally, it corresponds to the "one-electron" Hilbert space $\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$, where $\mathfrak{h}^{ph}$ is the Hilbert space of the individual photon field. The fermion description of composed-particles $\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$ corresponds to the antisymmetric Fock space $\mathfrak{F}_-(\mathfrak{h}^{el} \otimes \mathfrak{h}^{ph})$.

The composed-particle assumption 2.1 allows us to use the Landauer-Büttiker formalism developed for ideal many-body fermion systems. Now, we come closer to the formal description of our JCL-model with two (infinite) leads and a one-mode quantum resonator.

Recall that the Hilbert space of the Jaynes-Cummings Hamiltonian with two energy levels is $\mathfrak{H}^{JC} = \mathbb{C}^2 \otimes \mathfrak{F}_+(\mathbb{C})$. The boson Fock space is constructed from a one-dimensional Hilbert space, since we consider only photons of a single fixed frequency. We model the electrons in the leads as free fermions residing on discrete semi-infinite lattices. Thus:

$$\mathfrak{h}^{el} = \ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \oplus \ell^2(\mathbb{N}) = \mathfrak{h}^{el}_l \oplus \mathfrak{h}_S \oplus \mathfrak{h}^{el}_r, \tag{2.10}$$

is the one-particle Hilbert space for the electrons and for the dot. Here, $\mathfrak{h}^{el}_\alpha$, $\alpha \in \{l, r\}$, are the respective Hilbert spaces of the *left* and *right* lead, while $\mathfrak{h}_S = \mathbb{C}^2$ is the Hilbert space of the quantum dot. We denote by:

$$\{\delta^\alpha_n\}_{n\in\mathbb{N}}, \qquad \{\delta^S_n\}^1_{j=0},$$

the canonical basis consisting of individual lattice sites of $\mathfrak{h}^{el}_\alpha$, $\alpha \in \{l, r\}$, and of $\mathfrak{h}_S$, respectively. With the Hilbert space for photons, $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, we define the Hilbert space of the *full* system, i.e. quantum dot with leads and with the photon field, as follows:

$$\mathfrak{H} = \mathfrak{h}^{el} \otimes \mathfrak{h}^{ph} = \Big(\ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \oplus \ell^2(\mathbb{N})\Big) \otimes \ell^2(\mathbb{N}_0). \tag{2.11}$$

Remark 2.2. Note that the structure of full space (2.11) takes into account the condition 2.1 and produces composed fermions via the last tensor product. It also manifests that electrons *in the dot* as well as those *in the leads* are composed with photons. This is different than the picture imposed by the the Jaynes-Cummings model, when *only the dot* is composed with photons:

$$\mathfrak{H} = \ell^2(\mathbb{N}) \oplus \mathbb{C}^2 \otimes \ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}) \quad , \quad \mathfrak{H}^{JC} = \mathbb{C}^2 \otimes \ell^2(\mathbb{N}_0)\,, \tag{2.12}$$

see (2.4), (2.5) and (2.6), where $\mathfrak{H}^{JC} = \mathfrak{h}_S \otimes \mathfrak{h}^{ph}$. The next step is a choice of interactions between subsystems: dot-resonator-leads.

According to (2.10), the decoupled leads-dot Hamiltonian is the matrix operator:

$$h^{el}_0 = \begin{pmatrix} h^{el}_l & 0 & 0 \\ 0 & h_S & 0 \\ 0 & 0 & h^{el}_r \end{pmatrix} \text{ on } u = \begin{pmatrix} u_l \\ u_S \\ u_r \end{pmatrix} \,,\ \{u_\alpha \in \ell^2(\mathbb{N})\}_{\alpha\in\{l,r\}}\,,\ u_S \in \mathbb{C}^2\,,$$

where $h^{el}_{\alpha} = -\Delta^D + v_\alpha$ with a constant potential bias $v_\alpha \in \mathbb{R}$, $\alpha \in \{l, r\}$, and h_S can be any self-adjoint two-by-two matrix with eigenvalues $\{\lambda^S_0, \lambda^S_1 := \lambda^S_0 + \varepsilon\}$, $\varepsilon > 0$, and eigenvectors $\{e^S_0, e^S_1\}$, cf (2.2). Here, Δ^D denotes the discrete Laplacian on $\ell^2(\mathbb{N})$ with homogeneous Dirichlet boundary conditions given by:

$$\begin{aligned}(\Delta^D f)(x) &:= f(x+1) - 2f(x) + f(x-1), \quad x \in \mathbb{N},\\ \mathrm{dom}(\Delta^D) &:= \{f \in \ell^2(\mathbb{N}_0) : f(0) := 0\},\end{aligned}$$

which is obviously a bounded self-adjoint operator. Notice that $\sigma(\Delta^D) = [0, 4]$.

We define the *lead-dot interaction* for coupling $g_{el} \in \mathbb{R}$ by the matrix operator acting in (2.10) as follows:

$$v_{el} = g_{el} \begin{pmatrix} 0 & \langle \cdot, \delta^S_0\rangle \delta^l_1 & 0 \\ \langle \cdot, \delta^l_1\rangle \delta^S_0 & 0 & \langle \cdot, \delta^r_1\rangle \delta^S_1 \\ 0 & \langle \cdot, \delta^S_1\rangle \delta^r_1 & 0 \end{pmatrix} , \tag{2.13}$$

where non-trivial off-diagonal entries are *projection* operators in the Hilbert space (2.10) with the scalar product $u, v \mapsto \langle u, v\rangle$ for $u, v \in \mathfrak{h}^{el}$. Here, $\{\delta^S_0, \delta^S_1\}$ is ortho-normal basis in $\mathfrak{h}^{el}_S$, which in general may be different from $\{e^S_0, e^S_1\}$. Hence, interaction (2.13) describes quantum *tunneling* between leads and the dot via contact sites of the leads, which are supports of δ^l_1 and δ^r_1.

Then we define the Hamiltonian for the system of interacting leads and dot as $h^{el} := h^{el}_0 + v_{el}$. Here, both h^{el}_0 and h^{el} are bounded self-adjoint operators on $\mathfrak{h}^{el}$.

Recall that photon Hamiltonian in the one-mode resonator is defined by operator $h^{ph} = \omega b^* b$ with domain in the Fock space $\mathfrak{F}_+(\mathbb{C}) \simeq \ell^2(\mathbb{N}_0)$, (2.1). We denote the canonical basis in $\ell^2(\mathbb{N}_0)$ by $\{\Upsilon_n\}_{n\in\mathbb{N}_0}$. Then for the spectrum of h^{ph} one obviously gets:

$$\sigma(h^{ph}) = \sigma_{pp}(h^{ph}) = \bigcup_{n\in\mathbb{N}_0} \{n\omega\}. \tag{2.14}$$

We introduce the following decoupled Hamiltonian H_0, which describes the system when the leads are decoupled from the quantum dot and the electron does not interact with the photon field:

$$H_0 := H^{el}_0 + H^{ph}, \tag{2.15}$$

where

$$H^{el}_0 := h^{el}_0 \otimes I_{\mathfrak{h}^{ph}} \quad \text{and} \quad H^{ph} := I_{\mathfrak{h}^{el}} \otimes h^{ph}.$$

The operator H_0 is self-adjoint on $\mathrm{dom}(H_0) = \mathrm{dom}(I_{\mathfrak{h}^{el}} \otimes h^{ph})$. Recall that h^{el}_0 and h^{ph} are bounded self-adjoint operators. Hence, H^{el}_0 and H^{el} are semi-bounded from below, which yields that H_0 is semi-bounded from below.

The interaction of the photons and the electrons in the quantum dot is given by the coupling of the dipole moment of the electrons to the electromagnetic field in the rotating wave approximation. Namely:

$$V_{ph} = g_{ph} \left((\cdot, e^S_0) e^S_1 \otimes b + (\cdot, e^S_1) e^S_0 \otimes b^* \right), \tag{2.16}$$

for some coupling constant $g_{ph} \in \mathbb{R}$. The total Hamiltonian is given by:

$$H := H^{el} + H^{ph} + V_{ph} = H_0 + V_{el} + V_{ph}, \tag{2.17}$$

where $H^{el} := h^{el} \otimes I_{\mathfrak{h}^{ph}}$ and $V_{el} := v_{el} \otimes I_{\mathfrak{h}^{ph}}$.

In the following, we call $\mathcal{S} = \{H, H_0\}$ the Jaynes-Cummings-leads system, in short JCL-model, which we are going to analyze. In particular, we are interested in the electron and photon currents for that system. The analysis will be based on the abstract Landauer-Büttiker formula, cf. [1, 13].

Lemma 2.3. *H is bounded from below self-adjoint such that* $\mathrm{dom}(H) = \mathrm{dom}(H_0)$.

Proof. Let $c \geqslant 2$. Then,

$$\|b\Upsilon_n\|^2 \leqslant \|b^*\Upsilon_n\|^2 = n+1 \leqslant c^{-1}n^2 + c, \quad n \in \mathbb{N}_0.$$

Consider elements $f \in \mathfrak{h}_S \otimes \mathfrak{h}^{ph} \cap \mathrm{dom}(I_{\mathfrak{h}^{el}} \otimes h^{ph})$ with the following:

$$f = \sum_{j,l} \beta_{jl} e_j \otimes \Upsilon_l, \quad j \in \{0,1\}, \quad l \in \mathbb{N}_0,$$

which are dense in $\mathfrak{H}^{JC} := \mathfrak{h}^{el}_S \otimes h^{ph}$. Then, $\|f\|^2 = \sum_{j,l}|\beta_{jl}|^2$ and $\|(I_{\mathfrak{h}^{el}} \otimes b^*b)f\|^2 = \sum_{j,l=1}|\beta_{jl}|^2 l^2$. We obtain the following:

$$\|((\cdot, e_1^S)e_0^S \otimes b)f\|^2 \leqslant \sum_{j,l}|\beta_{jl}|^2 \|b\Upsilon_l\|^2 \leqslant$$

$$\sum_{j,l}|\beta_{jl}|^2(c^{-1}l^2 + c) = c^{-1}\|(I_{\mathfrak{h}^{el}} \otimes b^*b)f\|^2 + c\|f\|^2$$

Similarly,

$$\|((\cdot, e_1^S)e_0^S \otimes b^*)f\|^2 \leqslant c^{-1}\|(I_{\mathfrak{h}^{el}} \otimes b^*b)f\|^2 + c\,\|f\|^2.$$

If $c \geqslant 2$ is large enough, then we obtain that V_{ph} is dominated by H^{ph} with relative bound less than one. Hence, H is self-adjoint and $\mathrm{dom}(H_0) = \mathrm{dom}(H)$. Since H_0^{el} and V_{el} are bounded and H^{ph} is self-adjoint and bounded from below, it follows that $H = H_0^{el} + H^{ph} + V_{el} + V_{ph}$ is bounded from below [17, Thm. V.4.1]. □

2.3. Time reversible symmetric systems

A system described by the Hamiltonian H is called time reversible symmetric if there is a conjugation Γ defined on $\mathfrak{H}$ such that $\Gamma H = H\Gamma$. Recall that Γ is a conjugation if the conditions $\Gamma^2 = I$ and $(\Gamma f, \Gamma g) = \overline{(f,g)}$, $f, g \in \mathfrak{H}$.

Let $\mathfrak{h}^{ph}_n$, $n \in \mathbb{N}_0$, the subspace spanned by the eigenvector Υ_n in $\mathfrak{h}^{ph}$. We set:

$$\mathfrak{H}_{n_\alpha} := \mathfrak{h}^{el}_\alpha \otimes \mathfrak{h}^{ph}_n, \quad n \in \mathbb{N}_0, \quad \alpha \in \{l, r\}. \tag{2.18}$$

Notice that

$$\mathfrak{H} = \bigoplus_{n \in \mathbb{N}_0, \alpha \in \{l,r\}} \mathfrak{H}_{n_\alpha}$$

Definition 2.4. The JCL-model is called time reversible symmetric if there is a conjugation Γ acting on $\mathfrak{H}$ such that H and H_0 are time reversible symmetric and the subspaces $\mathfrak{H}_{n_\alpha}$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, reduces Γ.

Example 2.5. Let γ^{el}_α and γ^{el}_S be conjugations defined by the following:

$$\gamma^{el}_\alpha f_\alpha := \overline{f_\alpha} := \{\overline{f_\alpha(k)}\}_{k \in \mathbb{N}}, \quad f_\alpha \in \mathfrak{h}^{el}_\alpha, \quad \alpha \in \{l, r\},$$

and

$$\gamma^{el}_S f_S = \gamma^{el}_S \begin{pmatrix} f_S(0) \\ f_S(1) \end{pmatrix} := \begin{pmatrix} \overline{f_S(0)} \\ \overline{f_S(1)} \end{pmatrix}$$

We set $\gamma^{el} := \gamma^{el}_l \oplus \gamma^{el}_S \oplus \gamma^{el}_r$. Further, we set:

$$\gamma^{ph}\psi := \overline{\psi} = \{\overline{\psi(n)}\}_{n \in \mathbb{N}_0}, \quad \psi \in \mathfrak{h}^{ph}.$$

Let $\Gamma := \gamma^{el} \otimes \gamma^{ph}$. One easily checks that Γ is a conjugation on $\mathfrak{H} = \mathfrak{h}^{el} \otimes \mathfrak{h}^{ph}$.

Lemma 2.6. *Let γ^{el}_α, $\alpha \in \{S, l, r\}$, and γ^{ph} be given by Example 2.5.*

(i) *If the conditions $\gamma^{el}_S e^S_0 = e^S_0$ and $\gamma^{el}_S e^S_1 = e^S_1$ are satisfied, then H_0 is time reversible symmetric with respect to Γ and, moreover, the subspaces $\mathfrak{H}_{n_\alpha}$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$, reduces Γ.*

(ii) *If in addition the conditions $\gamma_S^{el}\delta_0^S = \delta_0^S$ and $\gamma_S^{el}\delta_1^S = \delta_1^S$ are satisfied, then JCL-model is time reversible symmetric.*

Proof. (i) Obviously we have

$$\gamma_\alpha^{el} h_\alpha^{el} = h_\alpha^{el}\gamma_\alpha^{el}, \quad \alpha \in \{l, r\}, \quad \text{and} \quad \gamma^{ph} h^{ph} = h^{ph}\gamma^{ph}.$$

If $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$ are satisfied, then $\gamma_S^{el} h_S^{el} = h_S^{el}\gamma_S^{el}$ which yields $\gamma^{el} h_0^{el} = h_0^{el}\gamma^{el}$ and, hence, $\Gamma H_0 = \Gamma H_0$. Since $\gamma^{el}\mathfrak{h}_\alpha^{el} = \mathfrak{h}_\alpha^{el}$ and $\gamma^{ph}\mathfrak{h}^{ph} = \mathfrak{h}^{ph}$ one gets $\Gamma\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_\alpha}$ which shows that $\mathfrak{H}_{n_\alpha}$ reduces Γ.

(ii) Notice that $\gamma_\alpha^{el}\delta_1^\alpha = \delta_1^\alpha$, $\alpha \in \{l, r\}$. If in addition the conditions $\gamma_S^{el}\delta_0^S = \delta_0^S$ and $\gamma_S^{el}\delta_1^S = \delta_1^S$ are satisfied, then $\gamma^{el} v_{el} = v_{el}\gamma^{el}$ is valid, which yields $\gamma^{el} h^{el} = h^{el}\gamma^{el}$. Hence, $\Gamma H = H\Gamma$. Together with (i), this proves that the JCL-model is time reversible symmetric. □

Choosing the following:

$$e_0^S := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1^S := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \delta_0^S := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \delta_1^S := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.19}$$

one satisfies the condition $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$ as well as $\gamma_S^{el} e_0^S = e_0^S$ and $\gamma_S^{el} e_1^S = e_1^S$.

2.4. Mirror symmetric systems

A unitary operator U acting on $\mathfrak{H}$ is called a mirror symmetry if the following conditions are met:

$$U\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{\alpha'}}, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'$$

are satisfied. In particular, this yields $U\mathfrak{H}^{JC} = \mathfrak{H}^{JC}$, $\mathfrak{H}^{JC} := \mathfrak{h}_S^{el} \otimes \mathfrak{h}^{ph}$.

Definition 2.7. The JCL-model is called mirror symmetric if there is a mirror symmetry commuting with H_0 and H.

One can easily verify that if H_0 is mirror symmetric, then

$$H_{n_{\alpha'}} U = U H_{n_\alpha}, \qquad n \in \mathbb{N}_0, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha',$$

where

$$H_{n_\alpha} := h_\alpha^{el} \otimes I_{\mathfrak{h}_n^{ph}} + I_{\mathfrak{h}_\alpha^{el}} \otimes h_n^{ph} = h_\alpha^{el} + n\omega, \quad n \in \mathbb{N}_0, \quad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'.$$

In particular, this yields that $v_\alpha = v_{\alpha'}$. Moreover, one gets $UH_S = H_S U$ where $H_S := h_S^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph}$.

Notice that if H and H_0 commute with the same mirror symmetry U, then also the operator $H_c := h^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph}$ also commutes with U, i.e, is mirror symmetric.

Example 2.8. Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Let $v_l = v_r$ and let e_0^S and e_1^S as well as δ_0^S and δ_1^S be given by (2.19). We set:

$$u_S^{el} e_0^S := e_0^S \quad \text{and} \quad u_S^{el} e_1^S = -e_1^S, \tag{2.20}$$

as well as

$$u^{ph}\Upsilon_n = e^{-in\pi}\Upsilon_n, \quad n \in \mathbb{N}_0. \tag{2.21}$$

Obviously, $U_S := u_S^{el} \otimes u^{ph}$ defines a unitary operator on $\mathfrak{H}^{JC}$. Straightforward computation shows that:

$$U_S H_S = H_S U_S \quad \text{and} \quad U_S V_{ph} = V_{ph} U_S. \tag{2.22}$$

Furthermore, we set:

$$u_{rl}^{el}\delta_n^l := \delta_n^r, \quad \text{and} \quad u_{lr}^{el}\delta_n^r = \delta_n^l, \quad n \in \mathbb{N}, \tag{2.23}$$

and

$$u^{el} := \begin{pmatrix} 0 & 0 & u^{el}_{lr} \\ 0 & u^{el}_S & 0 \\ u^{el}_{lr} & 0 & 0 \end{pmatrix}.$$

We thus have the following:

$$v_{el}\, u^{el} \begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} < f_S, (u^{el}_S)^* \delta^S_0 > \delta^l_1 \\ < f_r, (u^{el}_{lr})^* \delta^l_1 > \delta^S_0 + < f_l, (u^{el}_{rl})^* \delta^r_1 > \delta^S_1 \\ < f_S, (u^{el}_S)^* \delta^S_1 > \delta^r_1 \end{pmatrix} \tag{2.24}$$

Since $\delta^S_0 := \frac{1}{\sqrt{2}}(e^S_0 + e^S_1)$ and $\delta^S_1 := \frac{1}{\sqrt{2}}(e^S_0 - e^S_1)$ we get from (2.20)

$$(u^{el}_S)^* \delta^S_0 = \delta^S_1 \quad \text{and} \quad (u^{el}_S)^* \delta^S_1 = \delta^S_0. \tag{2.25}$$

Obviously, we then have

$$(u^{el}_{lr})^* \delta^l_1 = \delta^r_1 \qquad (u^{el}_{rl})^* \delta^r_1 = \delta^l_1. \tag{2.26}$$

Inserting (2.25) and (2.26) into (2.24), we find

$$v_{el}\, u^{el} \begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} < f_S, \delta^S_1 > \delta^l_1 \\ < f_r, \delta^r_1 > \delta^S_0 + < f_l, \delta^l_1 > \delta^S_1 \\ < f_S, \delta^S_0 > \delta^r_1 \end{pmatrix} \tag{2.27}$$

Further, we have:

$$u^{el} v_{el} \begin{pmatrix} f_l \\ f_S \\ f_r \end{pmatrix} = \begin{pmatrix} < f_S, \delta^S_1 > \delta^l_1 \\ < f_l, \delta^l_1 > \delta^S_1 + < f_r, \delta^r_1 > \delta^S_0 \\ < f_s, \delta^S_0 > \delta^r_1 \end{pmatrix}. \tag{2.28}$$

Comparing (2.27) and (2.28), we get $u^{el} v_{el} = v_{el} u^{el}$. Setting $U := u^{el} \otimes u^{ph}$ one immediately proves that $UH_0 = H_0 U$ and $UH = HU$. Since $U\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{\alpha'}}$, it is satisfied that $\mathcal{S}$ is mirror symmetric.

We note that Example 2.8 $\mathcal{S}$ is also time-reversible symmetric.

2.5. Spectral properties of H: first part

In the following, our goal is to apply the Landauer-Büttiker formula to the JCL-model. By $\mathfrak{L}_p(\mathfrak{H})$, $1 \leqslant p \leqslant \infty$, we denote in the following the Schatten-v.Neumann ideals.

Proposition 2.9. *If $\mathcal{S} = \{H, H_0\}$ is the JCL-model, then $(H+i)^{-1} - (H_0+i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$. In particular, the absolutely continuous parts H^{ac} and H^{ac}_0 are unitarily equivalent.*

Proof. We have

$$\begin{aligned}(H+i)^{-1} - (H_0+i)^{-1} &= (H_0+i)^{-1} V (H+i)^{-1} = \\ &(H_0+i)^{-1} V (H_0+i)^{-1} - (H_0+i)^{-1} V (H_0+i)^{-1} V (H+i)^{-1},\end{aligned}$$

where $V = H - H_0 = V_{el} + V_{ph}$. Taking into account Lemma 2.3, it suffices to prove that $(H_0+i)^{-1} V (H_0+i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$. Using the spectral decomposition of h^{ph} with respect to $\mathfrak{h}^{ph} = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}^{ph}_n$, where $\mathfrak{h}^{ph}_n$ are the subspaces spanned by Υ_n, we obtain the following:

$$(H_0+i)^{-1} = \bigoplus_{n\in\mathbb{N}_0} (h^{el}_0 + n\omega + i)^{-1} \otimes I_{\mathfrak{h}^{ph}_n}. \tag{2.29}$$

We have $(H_0+i)^{-1}V(H_0+i)^{-1} = (H_0+i)^{-1}(V_{el}+V_{ph})(H_0+i)^{-1}$. Since v_{el} is a finite rank operator, we have $\|v_{el}\|_{\mathfrak{L}_1} < \infty$. Furthermore, $\mathfrak{h}^{ph}_n$ is obviously one-dimensional for any $n \in \mathbb{N}_0$. Hence, $\|I_{\mathfrak{h}^{ph}_n}\|_{\mathfrak{L}_1} = 1$. From (2.29) and $V_{el} = v_{el} \otimes I_{\mathfrak{h}^{ph}}$, we obtain the following:

$$\begin{aligned}\|(H_0+i)^{-1}V_{el}(H_0+i)^{-1}\|_{\mathfrak{L}_1} &= \sum_{n\in\mathbb{N}_0} \|(h_0^{el}+n\omega+i)^{-1}v_{el}(h_0^{el}+n\omega+i)^{-1}\|_{\mathfrak{L}_1} \\ &\leqslant \sum_{n\in\mathbb{N}_0} \|(h_0^{el}+n\omega+i)^{-2}\|\ \|v_{el}\|_{\mathfrak{L}_1}\end{aligned}$$

Since h_0^{el} is bounded, we get:

$$\|(h_0^{el}+n\omega+i)^{-1}\| = \sup_{\lambda\in\sigma(h_0^{el})} \Big(\sqrt{(\lambda+n\omega)^2+1}\Big)^{-1} \leqslant c(n+1)^{-1}, \tag{2.30}$$

for some $c > 0$. This immediately implies that $\|(H_0+i)^{-1}V_{el}(H_0+i)^{-1}\|_{\mathfrak{L}_1} < \infty$.

We are going to handle $(H_0+i)^{-1}V_{ph}(H_0+i)^{-1}$. Let p_n^{ph} be the projection from $\mathfrak{h}^{ph}$ onto $\mathfrak{h}^{ph}_n$. We have the following:

$$\begin{aligned}&(H_0+i)^{-1}\,(\cdot,e_0^S)e_1^S\otimes b\,(H_0+i)^{-1} \\ &= \sum_{m,n\in\mathbb{N}_0} (h_0^{el}+m\omega+i)^{-1}(\cdot,e_0^S)e_1^S\,(h_0^{el}+n\omega+i)^{-1}\otimes p_m^{ph}bp_n^{ph} \\ &= \sum_{n\in\mathbb{N}} (h_0^{el}+(n-1)\omega+i)^{-1}\,(\cdot,e_0^S)e_1^S\,(h_0^{el}+n\omega+i)^{-1}\otimes\sqrt{n}\Upsilon_{n-1}\langle\cdot,\Upsilon_n\rangle\end{aligned}$$

From (2.30), we get the following:

$$\Big\|(h_0^{el}+(n-1)\omega+i)^{-1}\,(\cdot,e_0^S)e_1^S\,(h_0^{el}+n\omega+i)^{-1}\Big)\otimes\sqrt{n}\Upsilon_n\langle\cdot,\Upsilon_n\rangle\Big\|_{\mathfrak{L}_1} \leqslant c^2\frac{\sqrt{n}}{n(n+1)},$$

$n\in\mathbb{N}$, which yields:

$$\|(H_0+i)^{-1}\,(\cdot,e_0^S)e_1^S\otimes b\,(H_0+i)^{-1}\|_{\mathfrak{L}_1} \leqslant c^2\sum_{n\in\mathbb{N}}^{\infty}\frac{\sqrt{n}}{n(n+1)} < \infty.$$

Since

$$\|(H_0+i)^{-1}\,(\cdot,e_1^S)e_0^S\otimes b^*\,(H_0+i)^{-1}\|_{\mathfrak{L}_1} = \|(H_0+i)^{-1}\,(\cdot,e_0^S)e_1^S\otimes b\,(H_0+i)^{-1}\|_{\mathfrak{L}_1},$$

one gets $(H_0+i)^{-1}V_{ph}(H_0+i)^{-1}\in\mathfrak{L}_1(\mathfrak{H})$, which completes the proof. □

Thus, the JCL-model $\mathcal{S} = \{H, H_0\}$ is a $\mathfrak{L}_1$-scattering system. Let us recall that $h_\alpha^{el} = -\Delta^D + v_\alpha$, $\alpha\in\{l,r\}$, on $\mathfrak{h}_l^{el} = \mathfrak{h}_r^{el} = \ell^2(\mathbb{N})$.

Lemma 2.10. *Let $\alpha\in\{l,r\}$. We have the following:*

$$\sigma(h_\alpha^{el}) = \sigma_{ac}(h_\alpha^{el}) = [v_\alpha, 4+v_\alpha].$$

The normalized generalized eigenfunctions of h_α^{el} are given by:

$$g_\alpha(x,\lambda) = \pi^{-\frac{1}{2}}(1-(-\lambda+2+v_\alpha)^2/4)^{-\frac{1}{4}}\sin\Big(\arccos((-\lambda+2+v_\alpha)/2)x\Big)$$

for $x\in\mathbb{N}$, $\lambda\in(v_\alpha,4+v_\alpha)$.

Proof. We prove the absolute continuity of the spectrum by showing that:

$$\{g_\alpha(x,\lambda)\,|\,\lambda\in(-2,2)\}$$

is a complete set of generalized eigenfunctions. Note that it suffices to prove the lemma for

$$((\Delta^D+2)f)(x) = f(x+1)+f(x-1), \qquad f(0)=0.$$

The lemma then follows by replacing λ with $-\lambda + 2 + v_\alpha$. Let $\lambda \in (-2, 2)$ and

$$g_{\Delta^D}(x, \lambda) = \pi^{-\frac{1}{2}}(1 - \lambda^2/4)^{-\frac{1}{4}} \sin\Big(\arccos(\lambda/2)x\Big).$$

Note that $g_{\Delta^D}(0, \lambda) = 0$, when the boundary condition is satisfied. We substitute $\mu = \arccos(\lambda/2) \in (0, \pi)$, i.e. $\lambda = 2\cos(\mu)$ and obtain the following:

$$\sin(\mu(x + 1)) + \sin(\mu(x - 1)) = 2\sin(\mu x)\cos(\mu),$$

when $g_{\Delta^D}(x, \lambda)$ satisfies the eigenvalue equation. It is obvious that $g_{\Delta^D}(\cdot, \lambda) \notin \ell^2(\mathbb{N}_0)$ for $\lambda \in (-2, 2)$. To complete the proof of the lemma, it remains to show the ortho-normality and the completeness. For the ortho-normality, we have to show that

$$\sum_{x\in\mathbb{N}} g_{\Delta^D}(x, \lambda)g_{\Delta^D}(x, \nu) = \delta(\lambda - \nu).$$

Let $\psi \in C_0^\infty\Big((-2, 2)\Big)$. We use the substitution $\mu = \arccos(\nu/2)$ and the relation

$$\sin(\arccos(y)) = (1 - y^2)^{-\frac{1}{2}}$$

to obtain the following:

$$\begin{aligned}
&\int_{-2}^{2} \mathrm{d}\nu \sum_{x\in\mathbb{N}} g_{\Delta^D}(x, \lambda)g_{\Delta^D}(x, \nu)\psi(\nu) \\
&= 2\pi^{-1} \int_0^\pi \mathrm{d}\mu \sum_{x\in\mathbb{N}} \frac{\sin(\mu)\sin\Big(\arccos(\lambda/2)x\Big)\sin(\mu x)}{(\sin(\mu))^{\frac{1}{2}}(\sin(\arccos(\lambda/2)))^{\frac{1}{2}}}\psi(2\cos(\mu)) \\
&= (2\pi)^{-1} \int_0^\pi \mathrm{d}\mu \sum_{x\in\mathbb{N}} \frac{(\sin(\mu))^{\frac{1}{2}}}{(\sin(\arccos(\lambda/2)))^{\frac{1}{2}}}\Big(e^{i(\arccos(\lambda/2)-\mu)x} + \\
&e^{-i(\arccos(\lambda/2)-\mu)x} - e^{i(\arccos(\lambda/2)+\mu)x} - e^{-i(\arccos(\lambda/2)+\mu)x}\Big)\psi(2\cos(\mu)).
\end{aligned}$$

Observe that for the Dirichlet kernel:

$$\sum_{x\in\mathbb{N}_0} (e^{ixy} + e^{-ixy}) - 1 = 2\pi\,\delta(y),$$

when

$$\begin{aligned}
&\int_{-2}^{2} \mathrm{d}\nu \sum_{x\in\mathbb{N}} g_{\Delta^D}(x, \lambda)g_{\Delta^D}(x, \nu)\psi(\nu) = \\
&\int_0^\pi \mathrm{d}\mu \frac{(\sin(\mu))^{\frac{1}{2}}}{(\sin(\arccos(\lambda/2)))^{\frac{1}{2}}} \left(\delta(\arccos(\lambda/2) - \mu) + \delta(\arccos(\lambda/2) + \mu)\right)\psi(2\cos(\mu)) = \psi(\lambda).
\end{aligned}$$

In the second equality we use that the summand containing $\delta(\arccos(\lambda/2) + \mu)$ is zero since both $\arccos(\lambda/2) > 0$ and $\mu > 0$. Thus, the generalized eigenfunctions are orthonormal. Finally, using once more the substitution $\mu = \arccos(\nu/2)$, we obtain the following:

$$\begin{aligned}
&\int_{-2}^{2} \mathrm{d}\nu\, g_{\Delta^D}(x, \nu)g_{\Delta^D}(y, \nu) = \\
&\int_{-2}^{2} \mathrm{d}\nu \left(1 - (\nu/2)^2\right)^{-\frac{1}{2}} \sin\Big(\arccos(\nu/2)x\Big)\sin\Big(\arccos(\nu/2)y\Big) = \\
&2\pi^{-1} \int_0^\pi \mathrm{d}\mu\, (\sin(\mu))^{-1} \sin(\mu) sin(\mu x)\sin(\mu y) = \delta_{xy},
\end{aligned}$$

for $x, y \in \mathbb{N}$, when the family of generalized eigenfunctions is also complete. □

From these two lemmas, we obtain the following corollary that gives us the spectral properties of H_0.

Proposition 2.11. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Then, $\sigma(H_0) = \sigma_{ac}(H_0) \cup \sigma_{pp}(H_0)$, where*

$$\sigma_{ac}(H_0) = \bigcup_{n\in\mathbb{N}_0} [v_l + n\omega, v_l + 4 + n\omega] \cup [v_r + n\omega, v_r + 4 + n\omega]$$

and

$$\sigma_{pp}(H_0) = \bigcup_{n\in\mathbb{N}_0} \{\lambda_j^S + n\omega : j = 0, 1\}.$$

The eigenvectors are given by $\widetilde{g}(m, n) = e_m^S \otimes \Upsilon_n$, $m = 0, 1$, $n \in \mathbb{N}_0$. The generalized eigenfunctions are given by $\widetilde{g}_\alpha(\cdot, \lambda, n) = g_\alpha(\cdot, \lambda - n\omega) \otimes \Upsilon_n$ for $\lambda \in \sigma_{ac}(H_0)$, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$.

Proof. It is well known (see e.g. [15]) that for two self-adjoint operators A and B with $\sigma_{sc}(A) = \sigma_{sc}(B) = \emptyset$, we have $\sigma_{sc}(A \otimes 1 + 1 \otimes B) = \emptyset$,

$$\sigma_{ac}(A \otimes 1 + 1 \otimes B) = \Big(\sigma_{ac}(A) + \sigma(B)\Big) \cup \Big(\sigma(A) + \sigma_{ac}(B)\Big)$$

and

$$\sigma_{pp}(A \otimes 1 + 1 \otimes B) = \sigma_{pp}(A) + \sigma_{pp}(B).$$

Furthermore, if $\psi_A(\lambda_A)$ and $\psi_B(\lambda_B)$ are (generalized) eigenfunctions of A and B, respectively, then $\psi_A(\lambda_A) \otimes \psi_B(\lambda_B)$ is a (generalized) eigenfunction of $A \otimes I + I \otimes B$ for the (generalized) eigenvalue $\lambda_A + \lambda_B$.

The lemma follows now with $A = h_0^{el}$ and $B = h^{ph}$ using Lemmata 2.10 and (2.14) and the fact that h_S has eigenvectors $\{e_0^S, e_1^S\}$ with eigenvalues $\{\lambda_0^S, \lambda_1^S = \lambda_0^S + \varepsilon\}$. □

2.6. Spectral representation

For the convenience of the reader, we define here what we mean under a spectral representation of the absolutely continuous part K_0^{ac} of a self-adjoint operator K_0 on a separable Hilbert space $\mathfrak{K}$. Let $\mathfrak{k}$ be an auxiliary separable Hilbert space. We consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$. By $\mathcal{M}$, we define the multiplication operator induced by the independent variable λ in $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$. Let $\Phi : \mathfrak{K}^{ac}(K_0) \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{k})$ be an isometry acting from $\mathfrak{K}^{ac}(K_0)$ into $L^2(\mathbb{R}, d\lambda, \mathfrak{k})$ such that: $\Phi\mathrm{dom}(K_0^{ac}) \subseteq \mathrm{dom}(\mathcal{M})$ and

$$\mathcal{M}\Phi f = \Phi K_0^{ac} f, \quad f \in \mathrm{dom}(K_0^{ac}).$$

Obviously, the orthogonal projection $P := \Phi\Phi^*$ commutes with $\mathcal{M}$ which yields the existence of a measurable family, $\{P(\lambda)\}_{\lambda\in\mathbb{R}}$, such that:

$$(P\,\widehat{f}\,)(\lambda) = P(\lambda)\,\widehat{f}\,(\lambda), \qquad \widehat{f}\, \in L^2(\mathbb{R}, \lambda, \mathfrak{k}).$$

We set $L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)) := PL^2(\mathbb{R}, \lambda, \mathfrak{k})$, $\mathfrak{k}(\lambda) := P(\lambda)\mathfrak{k}$, and call the triplet

$$\Pi(K_0^{ac}) := \{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$$

a spectral representation of K_0^{ac}. If $\{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ is a spectral representation of K^{ac}, then K^{ac} is unitarily equivalent $\mathcal{M}_0 := \mathcal{M} \restriction L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda))$. Indeed, one has $\Phi K_0^{ac}\Phi^* = \mathcal{M}_0$. The function $\xi_{K_0}^{ac}(\lambda) := \mathrm{dom}(\mathfrak{k}(\lambda))$, $\lambda \in \mathbb{R}$, is called the spectral multiplicity function of K_0^{ac}. Notice that $0 \leqslant \xi_{K_0}^{ac}(\lambda) \leqslant \infty$ for $\lambda \in \mathbb{R}$.

For $\alpha \in \{l, r\}$, the generalized eigenfunctions of h_α^{el} define generalized Fourier transforms by $\phi_\alpha^{el} : \mathfrak{h}_\alpha^{el} = \mathfrak{h}_\alpha^{el,ac}(h_\alpha^{el}) \to L^2([v_\alpha, v_\alpha + 4])$ and

$$(\phi_\alpha^{el} f_\alpha)(\lambda) = \sum_{x\in\mathbb{N}_0} g_\alpha(x, \lambda) f_\alpha(x), \quad f_\alpha \in \mathfrak{h}_\alpha^{el}. \tag{2.31}$$

We then set:

$$\mathfrak{h}^{el}_\alpha(\lambda) := \begin{cases} \mathbb{C} & \lambda \in [v_\alpha, v_\alpha + 4] \\ 0 & \lambda \in \mathbb{R} \setminus [v_\alpha, v_\alpha + 4]. \end{cases} \tag{2.32}$$

One can easily verify that $\Pi(h^{el}_\alpha) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}_\alpha(\lambda)), \mathcal{M}, \phi^{el}_\alpha\}$ is a spectral representation of $h^{el}_\alpha = h^{el,ac}_\alpha$, $\alpha = l, r$, where we always assumed implicitly that $(\phi^{el}_\alpha f_\alpha)(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus [v_\alpha, v_\alpha + 4]$. Setting:

$$\mathfrak{h}^{el}(\lambda) := \begin{matrix} \mathfrak{h}^{el}_l(\lambda) \\ \oplus \\ \mathfrak{h}^{el}_r(\lambda) \end{matrix} \subseteq \mathbb{C}^2, \quad \lambda \in \mathbb{R}, \tag{2.33}$$

and introducing the map:

$$\phi^{el} : \mathfrak{h}^{el,ac}(h^{el}_0) = \begin{matrix} \mathfrak{h}^{el}_l \\ \oplus \\ \mathfrak{h}^{el}_r \end{matrix} \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}(\lambda)), \tag{2.34}$$

defined by:

$$\phi^{el} f := \begin{pmatrix} \phi^{el}_l f_l \\ \phi^{el}_r f_r \end{pmatrix}, \quad \text{where} \quad f := \begin{pmatrix} f_l \\ f_r \end{pmatrix} \tag{2.35}$$

we obtain a spectral representation $\Pi(h^{el,ac}_0) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}^{el}(\lambda)), \mathcal{M}, \phi^{el}\}$ of the absolutely continuous part $h^{el,ac}_0 = h^{el}_l \oplus h^{el}_r$ of h^{el}_0. One easily verifies that $0 \leqslant \xi^{ac}_{h^{el}_0}(\lambda) \leqslant 2$ for $\lambda \in \mathbb{R}$. Introducing:

$$\lambda^{el}_{\min} := \min\{v_l, v_r\} \quad \text{and} \quad \lambda^{el}_{\max} := \max\{v_l + 4, v_r + 4\}, \tag{2.36}$$

one easily verifies that $\xi^{ac}_{h^{el}_0}(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus [\lambda^{el}_{\min}, \lambda^{el}_{\max}]$.

Notice, if $v_r + 4 \leqslant v_l$, then

$$\mathfrak{h}^{el}(\lambda) = \begin{cases} \mathbb{C}, & \lambda \in [v_r, v_r + 4] \cup [v_l, v_l + 4], \\ \{0\}, & \text{otherwise} \end{cases}$$

which shows that h^{el}_0 has a simple spectrum. In particular, it holds $\xi^{ac}_{h^{el}_0}(\lambda) = 1$ for $\lambda \in [v_r, v_r + 4] \cup [v_l, v_l + 4]$ and otherwise $\xi^{ac}_{h^{el}_0}(\lambda) = 0$.

Let us introduce the Hilbert space $\mathfrak{h} := l^2(\mathbb{N}_0, \mathbb{C}^2) = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n$, $\mathfrak{h}_n := \mathbb{C}^2$, $n \in \mathbb{N}_0$. Regarding $\mathfrak{h}^{el}(\lambda - n\omega)$ as a subspace of $\mathfrak{h}_n$, one regards:

$$\mathfrak{h}(\lambda) := \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n(\lambda), \quad \mathfrak{h}_n(\lambda) := \mathfrak{h}^{el}(\lambda - n\omega), \quad \lambda \in \mathbb{R}, \tag{2.37}$$

as a measurable family of subspaces in $\mathfrak{h}$. Notice that $0 \leqslant \dim(\mathfrak{h}(\lambda)) < \infty$, $\lambda \in \mathbb{R}$. We consider the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$.

Furthermore, we introduce the isometric map $\Phi : \mathfrak{H}(H^{ac}_0) \longrightarrow L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$ defined by

$$(\Phi f)(\lambda) = \bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} (\phi^{el}_l f_l(n))(\lambda - n\omega) \\ (\phi^{el}_r f_r(n))(\lambda - n\omega) \end{pmatrix}, \quad \lambda \in \mathbb{R}, \tag{2.38}$$

where

$$\bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} f_l(n) \\ f_r(n) \end{pmatrix} \in \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}^{el,ac}(h^{el}_0) \otimes \mathfrak{h}^{ph}_n = \bigoplus_{n\in\mathbb{N}} \begin{pmatrix} \mathfrak{h}^{el}_l \otimes h^{ph}_n \\ \oplus \\ \mathfrak{h}^{el}_r \otimes h^{ph}_n \end{pmatrix}$$

where $\mathfrak{h}_{ph} = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}^{ph}_n$ and $\mathfrak{h}^{ph}_n$ is the subspace spanned by the eigenvectors Υ_n of h^{ph}. One easily verifies that Φ is an isometry acting from $\mathfrak{H}^{ac}(H^{ac}_0)$ onto $L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda))$.

Lemma 2.12. *The triplet* $\{L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda)), \mathcal{M}, \Phi\}$ *forms a spectral representation of* H_0^{ac}*, that is,* $\Pi(H_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}(\lambda)), \mathcal{M}, \Phi\}$ *where there is a constant* $d \in \mathbb{N}_0$ *such that* $0 \leqslant \xi_{H_0}^{ac}(\lambda) \leqslant 2d_{\max}$ *for* $\lambda \in \mathbb{R}$ *where* $d_{\max} := \frac{\lambda_{\max}^{el} - \lambda_{\min}^{el}}{\omega}$ *and* $\lambda_{\max}^{el}$ *and* $\lambda_{\min}^{el}$ *are given by* (2.36)*.*

Proof. It remains to be shown that Φ transforms H_0^{ac} into the multiplication operator $\mathcal{M}$. We have

$$H_0^{ac} f = \bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} (h_l^{el} f_l)(n) + n\omega f_l(n) \\ (h_r^{el} f_r)(n) + n\omega f_r(n) \end{pmatrix}$$

which yields the following:

$$\begin{aligned}(\Phi H_0^{ac} f)(\lambda) &= \bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} (\phi_l^{el}(h_l^{el} f_l)(n))(\lambda - n\omega) + n\omega(\phi_l^{el} f_l(n))(\lambda - n\omega) \\ (\phi_r^{el}(h_r^{el} f_r)(n))(\lambda - n\omega) + n\omega(\phi_r^{el} f_r(n))(\lambda - n\omega) \end{pmatrix} \\ &= \bigoplus_{n\in\mathbb{N}_0} \begin{pmatrix} \lambda(\phi_l^{el} f_l(n))(\lambda - n\omega) \\ \lambda(\phi_r^{el} f_r(n))(\lambda - n\omega) \end{pmatrix} = (\mathcal{M}\Phi f)(\lambda), \quad \lambda \in \mathbb{R}.\end{aligned}$$

which proves the desired property.

One easily checks that $\mathfrak{h}(\lambda)$ might only be only non-trivial if $\lambda - n\omega \in [\lambda_{\min}^{el}, \lambda_{\max}^{el}]$. Hence, we obtain that $\mathfrak{h}(\lambda)$ is non-trivial if the condition:

$$\frac{\lambda - \lambda_{\max}^{el}}{\omega} \leqslant n \leqslant \frac{\lambda - \lambda_{\min}^{el}}{\omega}$$

is satisfied. Hence,

$$0 \leqslant \xi_{H_0}^{ac}(\lambda) \leqslant 2 \text{ card} \left\{ n \in \mathbb{N}_0 : \frac{\lambda - \lambda_{\max}^{el}}{\omega} \leqslant n \leqslant \frac{\lambda - \lambda_{\min}^{el}}{\omega} \right\}, \quad \lambda \in \mathbb{R}.$$

or

$$0 \leqslant \xi_{H_0}^{ac}(\lambda) \leqslant 2\text{card} \left\{ n \in \mathbb{N}_0 : 0 \leqslant n \leqslant \frac{\lambda_{\max}^{el} - \lambda_{\max}^{el}}{\omega} \right\}, \quad \lambda \in \mathbb{R}.$$

Hence $0 \leqslant \xi_{H_0}^{ac}(\lambda) \leqslant d_{\max}$ for $\lambda \in \mathbb{R}$. □

In the following we denote the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_n(\lambda)$ by $P_n(\lambda)$, $\lambda \in \mathbb{R}$, cf (2.37). Since $\mathfrak{h}(\lambda) = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{h}_n(\lambda)$ we have $I_{\mathfrak{h}(\lambda)} = \sum_{n\in\mathbb{N}_0} P_n(\lambda)$, $\lambda \in \mathbb{R}$. Further, we introduce the following subspaces:

$$\mathfrak{h}_{n_\alpha}(\lambda) := \mathfrak{h}_\alpha^{el}(\lambda - n\omega), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

Notice that:

$$\mathfrak{h}_n(\lambda) = \bigoplus_{\alpha\in\{l,r\}} \mathfrak{h}_{n_\alpha}(\lambda), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

By $P_{n_\alpha}(\lambda)$ we denote the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_{n_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$. Clearly, we have $P_n(\lambda) = \sum_{\alpha\in\{l,r\}} P_{n_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$.

Example 2.13. In general, the direct integral $\Pi(H_0^{ac})$ can be very complicated, in particular, the structure of $\mathfrak{h}(\lambda)$ given by (2.37) is difficult to analyze. However, there are interesting simple cases:
(i) Let $v = v_l = v_r$ and $4 \leqslant \omega$. In this case we have $\mathfrak{h}^{el}(\lambda) = \mathbb{C}^2$ for $[v, v+4]$ and

$$\mathfrak{h}(\lambda) = \begin{cases} \mathbb{C}^2, & \lambda \in [v + n\omega, v + n\omega + 4], \quad n \in \mathbb{N}_0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

(ii) Let $v_r = 0$, $v_l = 4$, $\omega_0 = 4$. Then

$$\mathfrak{h}(\lambda) = \begin{cases} \mathfrak{h}_r^{el}(\lambda) = \mathbb{C}, & \lambda \in [0,4), \\ \mathfrak{h}_{lr}^{el}(\lambda) = \mathbb{C}^2, & \lambda \in [4,8), \\ \mathfrak{h}_{rl}^{el}(\lambda) = \mathbb{C}^2, & \lambda \in [8,12), \\ \cdots \end{cases}$$

where

$$\mathfrak{h}_{\alpha\alpha'}^{el}(\lambda) = \begin{matrix} \mathfrak{h}_{\alpha}^{el}(\lambda) \\ \oplus \\ \mathfrak{h}_{\alpha'}^{el}(\lambda) \end{matrix}, \qquad \alpha, \alpha' \in \{l, r\}, \quad \alpha \neq \alpha'.$$

Hence, $\dim(\mathfrak{h}(\lambda)) = 2$ for $\lambda \geqslant 4$. □

Let Z be a bounded operator acting on $\mathfrak{H}^{ac}(H_0)$ and commuting with H_0^{ac}. Since Z commutes with H_0^{ac} there is a measurable family $\{Z(\lambda)\}_{\lambda\in\mathbb{R}}$ of bounded operators acting on $\mathfrak{h}(\lambda)$ such that Z is unitarily equivalent to the multiplication operator induced by $\{Z(\lambda)\}_{\lambda\in\mathbb{R}}$ in $\Pi(H_0^{ac})$. We then set:

$$Z_{m_\alpha n_\varkappa}(\lambda) := P_{m_\alpha}(\lambda) Z(\lambda) \restriction \mathfrak{h}_{n_\varkappa}(\lambda), \quad \lambda \in \mathbb{R}, \qquad m, n \in \mathbb{N}_0, \quad \alpha, \varkappa \in \{l, r\}.$$

Let $Z_{m_\alpha n_\varkappa} := P_{m_\alpha} Z P_{n_\varkappa}$ where P_{m_α} is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_{m_\alpha} \subseteq \mathfrak{H}^{ac}(H_0)$, cf. (2.18). Clearly, the multiplication operator induced $\{Z_{m_\alpha n_\varkappa}(\lambda)\}_{\lambda\in\mathbb{R}}$ in $\Pi(H_0^{ac})$ is unitarily equivalent to $Z_{m_\alpha n_\varkappa}$.

Since, by Lemma 2.12, $\mathfrak{h}(\lambda)$ is a finite dimensional space, the operators $Z(\lambda)$ are finite dimensional ones and we can introduce the following quantity:

$$\sigma_{m_\alpha n_\varkappa}(\lambda) := \mathrm{tr}(Z_{m_\alpha n_\varkappa}(\lambda)^* Z_{m_\alpha n_\varkappa}(\lambda)), \quad \lambda \in \mathbb{R}, \qquad m, n \in \mathbb{N}_0, \quad \alpha, \varkappa \in \{l, r\}.$$

Lemma 2.14. *Let H_0 be the self-adjoint operator defined by* (2.15) *on $\mathfrak{H}$. Furthermore, let Z be a bounded operator on $\mathfrak{H}^{ac}(H_0)$ commuting with H_0^{ac}*
(i) *Let Γ be a conjugation on $\mathfrak{H}$, cf. Section 2.3. If Γ commutes with H_0 and P_{n_α}, $n \in \mathbb{N}_0$, $\alpha \in \{l, r\}$ and $\Gamma Z \Gamma = Z^*$ holds, then $\sigma_{m_\alpha n_\varkappa}(\lambda) = \sigma_{n_\varkappa m_\alpha}(\lambda)$, $\lambda \in \mathbb{R}$.*
(ii) *Let U be a mirror symmetry on $\mathfrak{H}$. If U commutes with H_0 and Z, then $\sigma_{m_\alpha n_\varkappa}(\lambda) = \sigma_{m_{\alpha'} n_{\varkappa'}}(\lambda)$, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}_0$, $\alpha, \alpha', \varkappa, \varkappa' \in \{l, r\}$, $\alpha \neq \alpha'$, $\varkappa \neq \varkappa'$.*

Proof. (i) Since Γ commutes with H_0 the conjugation Γ is reduce by $\mathfrak{H}^{ac}(H_0)$. So without loss of generality, we assume that Γ acts on $\mathfrak{H}^{ac}(H_0)$. We set $\Gamma_{n_\alpha} := \Gamma \restriction \mathfrak{H}_{n_\alpha}$. Notice that:

$$\Gamma = \bigoplus_{n\in\mathbb{N}_0, \alpha\in\{l,r\}} \Gamma_{n_\alpha}.$$

There is a measurable family $\{\Gamma(\lambda)\}_{\lambda\in\mathbb{R}}$ of conjugations such that the multiplication operator induced by $\{\Gamma(\lambda)\}_{\lambda\in\mathbb{R}}$ in $\Pi(H_0^{ac})$ is unitarily equivalent to Γ. Moreover, since Γ commutes with P_{n_α}, we see that the multiplication operator induced by the measurable family:

$$\Gamma_{n_\alpha}(\lambda) := \Gamma(\lambda) \restriction \mathfrak{h}_{n_\alpha}(\lambda), \quad \lambda \in \mathbb{R}, \quad m \in \mathbb{N}_0, \quad \alpha \in \{l, r\},$$

is unitarily equivalent to Γ_{n_α}. Using $\Gamma Z \Gamma = Z^*$ we get $\Gamma_{m_\alpha} Z_{m_\alpha n_\varkappa} \Gamma_{n_\varkappa} = Z_{n_\varkappa m_\alpha}^*$. Hence,

$$\Gamma_{m_\alpha}(\lambda) Z_{m_\alpha n_\varkappa}(\lambda) \Gamma_{n_\varkappa}(\lambda) = Z_{n_\varkappa m_\alpha}(\lambda)^*, \quad \lambda \in \mathbb{R}. \tag{2.39}$$

If X is a trace class operator, then $\mathrm{tr}(\Gamma X \Gamma) = \overline{\mathrm{tr}(X)}$. Using that we find

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \overline{\mathrm{tr}(\Gamma_{n_\varkappa}(\lambda) Z_{m_\alpha n_\varkappa}(\lambda)^* Z_{m_\alpha n_\varkappa}(\lambda) \Gamma_{n_\varkappa}(\lambda))} = \overline{\mathrm{tr}(\Gamma_{n_\varkappa}(\lambda) Z_{m_\alpha n_\varkappa}(\lambda)^* \Gamma_{m_\alpha} \Gamma_{m_\alpha} Z_{m_\alpha n_\varkappa}(\lambda) \Gamma_{n_\varkappa}(\lambda))}$$

From (2.39), we obtain the following:

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \overline{\mathrm{tr}(Z_{n_\varkappa m_\alpha}(\lambda) Z_{n_\varkappa m_\alpha}(\lambda)^*)} = \sigma_{n_\varkappa m_\alpha}(\lambda), \quad \lambda \in \mathbb{R},$$

which proves (i).

(ii) Again, without loss of generality we can assume that U acts only $\mathfrak{H}^{ac}(H_0)$. Since U commutes with H_0, there is a measurable family $\{U(\lambda)\}_{\lambda\in\mathbb{R}}$ of unitary operators acting on $\mathfrak{h}(\lambda)$ such that the multiplication operator induced by $\{U(\lambda)\}_{\lambda\in\mathbb{R}}$ is unitarily equivalent to U. Since $U\mathfrak{H}_{n_\alpha} = \mathfrak{H}_{n_{\alpha'}}$ we have $U(\lambda)\mathfrak{h}_{n_\alpha}(\lambda) = \mathfrak{h}_{n_{\alpha'}}(\lambda)$, $\lambda \in \mathbb{R}$. Hence,

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \mathrm{tr}(U(\lambda) Z_{m_\alpha n_\varkappa}(\lambda)^* Z_{m_\alpha n_\varkappa}(\lambda) U(\lambda)^*) = \mathrm{tr}(U(\lambda) Z_{m_\alpha, n_\varkappa}(\lambda)^* U(\lambda)^* U(\lambda) Z_{m_\alpha, n_\varkappa}(\lambda) U(\lambda)^*).$$

Hence,

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \mathrm{tr}(P_{n_{\varkappa'}} U(\lambda) Z(\lambda)^* U(\lambda)^* P_{m_{\alpha'}}(\lambda) U(\lambda) Z(\lambda) U(\lambda)^* P_{n_{\varkappa'}}(\lambda)).$$

Since U commutes with Z, we find that:

$$\sigma_{m_\alpha n_\varkappa}(\lambda) = \mathrm{tr}(P_{n_{\varkappa'}} Z(\lambda)^* P_{m_{\alpha'}}(\lambda) Z(\lambda) P_{n_{\varkappa'}}(\lambda)) = \sigma_{m_{\alpha'} n_{\varkappa'}}(\lambda), \quad \lambda \in \mathbb{R},$$

which proves (ii). □

2.7. Spectral properties of H: second part

Since we have full information for the spectral properties of H_0, we can use this to show that H has no singular continuous spectrum. Crucial for that is the following lemma: with the help of [6, Cor. IV.15.19], which establishes existence and completeness of wave operators and absence of singular continuous spectrum through a time-falloff method. We cite it as a Lemma for convenience, with slight simplifications that suffice for our purposes.

Lemma 2.15 ([6, Corollary IV.15.19])**.** *Let $\{H_0, H\}$ be a scattering system and let Λ be a closed countable set. Let F_+ and F_- be two self-adjoint operators such that $F_+ + F_- = P^{ac}_{H_0}$ and*

$$s - \lim_{t\to\infty} e^{\mp itH_0} F_\pm e^{\pm itH_0} = 0.$$

If $(H-i)^{-1} - (H_0 - i)^{-1} \in \mathfrak{L}_\infty(\mathfrak{H})$, $(1 - P^{ac}_{H_0})\gamma(H_0) \in \mathfrak{L}_\infty(\mathfrak{H})$, and

$$\Big| \int_0^{\pm\infty} \mathrm{d}t \Big\| \Big((H_0 - i)^{-1} - (H - i)^{-1} \Big) e^{-itH_0} \gamma(H_0) F_\pm \Big\| \Big| < \infty$$

for all $\gamma \in C_0^\infty(\mathbb{R} \setminus \Lambda)$, then $W_\pm(H, H_0)$ exist and are complete and $\sigma_{sc}(H) = \sigma_{sc}(H_0) = \emptyset$. Furthermore, each eigenvalue of H and H_0 in $\mathbb{R} \setminus \Lambda$ is of finite multiplicity and these eigenvalues accumulate the most at points of Λ or at $\pm\infty$.

We already know that the wave operators exist and are complete since the resolvent difference is trace class. Hence, we need Lemma 2.15 only to prove the following proposition.

Proposition 2.16. *The Hamiltonian H defined by* (2.17) *has no singular continuous spectrum, that is, $\sigma_{sc}(H) = \emptyset$.*

Proof. At first we have to construct the operators $F_\pm$. To this end, let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the usual Fourier transform, i.e

$$(\mathcal{F}f)(\mu) := \widehat{f}\,(\mu) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\mu x} f(x) dx, \quad f \in L^2(\mathbb{R}, dx), \quad \mu \in \mathbb{R}.$$

Further, let $\Pi_\pm$ be the orthogonal projection onto $L^2(\mathbb{R}_\pm)$ in $L^2(\mathbb{R})$. We set:

$$F_\pm = \Phi^* \mathcal{F} \Pi_\pm \mathcal{F}^* \Phi,$$

where Φ is given by (2.38). We immediately obtain $F_- + F_+ = P_{ac}(H_0)$. We still have to show that:

$$s - \lim_{t\to\infty} \|e^{\mp itH_0}\Phi^*\mathcal{F}\Pi_\pm\mathcal{F}^*\Phi e^{\pm itH_0} f\| = 0$$

for $f \in \mathfrak{H}^{ac}(H_0)$. We prove the relation only for F_+ since the proof for F_- is essentially identical. We have the following:

$$\Big(\Pi_+\mathcal{F}^*\Phi e^{itH_0} f\Big)(x) = (2\pi)^{-\frac{1}{2}}\chi_{\mathbb{R}_+}(x)\int_{\mathbb{R}} \mathrm{d}\mu\, e^{i(x+t)\mu}\widehat{f}(\mu) = \chi_{\mathbb{R}_+}(x)\psi(x+t)$$

with $\psi = \mathcal{F}\widehat{f}$. Now,

$$\|e^{-itH_0}\Phi^*\mathcal{F}\Pi_+\mathcal{F}^*\Phi e^{itH_0} f\|^2 =$$
$$\|\Pi_+\mathcal{F}^*\Phi e^{itH_0} f\|^2 = \int_{\mathbb{R}_+} \mathrm{d}x\Big|\psi(x+t)\Big|^2 = \int_t^\infty \mathrm{d}x\Big|\psi(x)\Big|^2 \overset{t\to\infty}{\longrightarrow} 0.$$

Concerning the compactness condition, we already know that $(H-i)^{-1} - (H_0-i)^{-1} \in \mathfrak{L}_1(\mathfrak{H}) \subset \mathfrak{L}_\infty(\mathfrak{H})$ from Proposition 2.9. Let

$$\Lambda := \bigcup_{n\in\mathbb{N}_0} \{v_l + n\omega, v_r + n\omega, v_l + 4 + n\omega, v_r + 4 + n\omega\},$$

which is closed and countable. We know from Corollary 2.11 that H_0 has no singular continuous spectrum and the eigenvalues are of finite multiplicity. It follows then that $(1 - P_{ac}(H_0))\gamma(H_0)$ is compact for every $\gamma \in C_0^\infty(\mathbb{R}\setminus\Lambda)$. The remaining assumption of Lemma 2.15 is as follows:

$$\left|\int_0^{\pm\infty} \mathrm{d}t\, \Big\|\Big((H-i)^{-1} - (H_0-i)^{-1}\Big)\gamma(H_0)e^{-itH_0}F_\pm\Big\|\right| < \infty.$$

If we can prove this, then we immediately obtain that H has no singular continuous spectrum. Now $(H-i)^{-1} - (H_0-i)^{-1} = (H-i)^{-1}(V_{el}+V_{ph})(H_0-i)^{-1}$. But $(H-i)^{-1}$ is bounded,

$$\mathrm{ran}(F_\pm) \subset \mathfrak{H}^{ac}(H_0) = (\mathfrak{h}_l^{el} \oplus \mathfrak{h}_r^{el}) \otimes \mathfrak{h}^{ph},$$

and $V_{ph}P^{ac}(H_0) = 0$. Also, $V_{el} = v_{el} \otimes I_{\mathfrak{h}^{ph}}$ and

$$\ker(v_{el})^\perp \subset \mathbb{C}\delta_1^l \oplus \mathfrak{h}_S \oplus \mathbb{C}\delta_1^r.$$

Hence, it suffices to prove:

$$\left|\int_0^{\pm\infty} \mathrm{d}t\, \Big\|P_1^\alpha(H_0-i)^{-1}\gamma(H_0)e^{-itH_0}F_\pm\Big\|\right| < \infty,$$

$\alpha \in \{l, r\}$, where $P_1^\alpha = p_1^\alpha \otimes I_{\mathfrak{h}^{ph}}$ and p_1^α is the orthogonal projection onto $\mathfrak{h}_\alpha^{el}$. In the following we treat only the case F_+. The calculations for F_- are completely analogous. We use that Φ maps H_0^{ac} into the multiplication operator $\mathcal{M}$ induced by λ. Hence, we get the following:

$$\Big\|P_1^\alpha\widetilde{\gamma}(H_0)e^{-itH_0}\Phi^*\mathcal{F}f\Big\| = \Big\|P_1^\alpha\Phi^*\Phi\widetilde{\gamma}(H_0)e^{-itH_0}\Phi^*\mathcal{F}f\Big\| =$$
$$= (2\pi)^{-\frac{1}{2}}\Big(\sum_{n\in\mathbb{N}_0}\Big|\int_{\delta_{\alpha,n}} \mathrm{d}\lambda\; g_\alpha(1,\lambda-n\omega)\widetilde{\gamma}(\lambda)\int_{\mathbb{R}_+}\mathrm{d}x\, e^{-i\lambda(x+t)}f(x)\Big|^2\Big)^{\frac{1}{2}},$$

where $\mathrm{supp}\,(f) \subseteq \mathbb{R}_+$, $\widetilde{\gamma}(\lambda) := (\lambda-i)^{-1}\gamma(\lambda)$, $\lambda \in \mathbb{R}$, and $\delta_{\alpha,n} := [v_\alpha + n\omega_0, v_\alpha + n\omega + 4]$. Notice that $\widetilde{\gamma}(\lambda) \in C_0^\infty(\mathbb{R}\setminus\Lambda)$. We find:

$$\int_{\delta_{j,n}} \mathrm{d}\lambda\; g_\alpha(1,\lambda-n\omega)\widetilde{\gamma}(\lambda)\int_{\mathbb{R}_+}\mathrm{d}x\, e^{-i\lambda(x+t)}f(x) =$$
$$\int_{v_\alpha}^{v_\alpha+4} \mathrm{d}\lambda\; g_\alpha(1,\lambda)\widetilde{\gamma}(\lambda+n\omega)\int_{\mathbb{R}_+}\mathrm{d}x\, e^{-i(\lambda+n\omega)(x+t)}f(x)$$

which yields:

$$\left\|P_1^\alpha \Phi^* \Phi \widetilde{\gamma}(H_0) e^{-itH_0} \Phi^* \mathcal{F} f\right\| =$$
$$(2\pi)^{-\frac{1}{2}} \Big(\sum_{n\in\mathbb{N}_0} \Big| \int_{v_\alpha}^{v_\alpha+4} \mathrm{d}\lambda \; g_\alpha(1,\lambda) \widetilde{\gamma}(\lambda+n\omega_0) \int_{\mathbb{R}_+} \mathrm{d}x \, e^{-i(\lambda+n\omega_0)(x+t)} f(x) \Big|^2 \Big)^{\frac{1}{2}}.$$

Since the support of $\gamma(\lambda)$ is compact, we see that the sum $\sum_{n\in\mathbb{N}_0}$ is finite. Changing the integrals, we get:

$$\int_{\delta_{\alpha,n}} \mathrm{d}\lambda \; g_\alpha(1,\lambda-n\omega) \widetilde{\gamma}(\lambda) \int_{\mathbb{R}_+} \mathrm{d}x \, e^{-i\lambda(x+t)} f(x) =$$
$$\int_{\mathbb{R}_+} \mathrm{d}x \, f(x) e^{-in\omega_0(x+t)} \int_{v_\alpha}^{v_\alpha+4} \mathrm{d}\lambda \; g_\alpha(1,\lambda) \widetilde{\gamma}(\lambda+n\omega) e^{-i\lambda(x+t)}$$

Integrating by parts m-times, we obtain:

$$\int_{\delta_{\alpha,n}} \mathrm{d}\lambda \; g_\alpha(1,\lambda-n\omega) \widetilde{\gamma}(\lambda) \int_{\mathbb{R}_+} \mathrm{d}x \, e^{-i\lambda(x+t)} f(x) =$$
$$(-i)^m \int_{\mathbb{R}_+} \mathrm{d}x \, f(x) \frac{e^{-in\omega(x+t)}}{(x+t)^m} \int_{v_\alpha}^{v_\alpha+4} \mathrm{d}\lambda \; e^{-i\lambda(x+t)} \frac{d^m}{d\lambda^m} \left(g_\alpha(1,\lambda) \widetilde{\gamma}(\lambda+n\omega) \right)$$

Hence,

$$\Big| \int_{\delta_{\alpha,n}} \mathrm{d}\lambda \; g_\alpha(1,\lambda-n\omega) \widetilde{\gamma}(\lambda) \int_{\mathbb{R}_+} \mathrm{d}x \, e^{-i\lambda(x+t)} f(x) \Big|^2 \leqslant C_n^2 \left(\int_{\mathbb{R}_+} \mathrm{d}x \, |f(x)| \frac{1}{(x+t)^m} \right)^2$$

which yields:

$$\Big| \int_{\delta_{\alpha,n}} \mathrm{d}\lambda \; g_\alpha(1,\lambda-n\omega) \widetilde{\gamma}(\lambda) \int_{\mathbb{R}_+} \mathrm{d}x \, e^{-i\lambda(x+t)} f(x) \Big|^2 \leqslant C_n^2 \frac{1}{t^{(2m-1)}} \|f\|^2$$

for $m \in \mathbb{N}$ where:

$$C_n := \int_{v_\alpha}^{v_\alpha+4} \mathrm{d}\lambda \left| \frac{d^m}{d\lambda^m} \left(g_\alpha(1,\lambda) \widetilde{\gamma}(\lambda+n\omega) \right| \right).$$

Notice that $C_n = 0$ for sufficiently large $n \in \mathbb{N}$. Therefore,

$$\left\|P_1^\alpha \widetilde{\gamma}(H_0) e^{-itH_0} \Phi^* \mathcal{F} f\right\| \leqslant \left(\sum_{n\in\mathbb{N}_0} C_n^2 \right)^{1/2} \frac{1}{t^{m-1/2}} \|f\|, \quad f \in L^2(\mathbb{R}_+, dx),$$

which shows that $\left\|P_1^\alpha \widetilde{\gamma}(H_0) e^{-itH_0} F_+\right\| \in L^1(\mathbb{R}_+, dt)$ for $m \geqslant 2$. □

3. Landauer-Büttiker formula and applications

3.1. Landauer-Büttiker formula

The abstract Landauer-Büttiker formula can be used to calculate currents through devices. Usually one considers a pair $\mathcal{S} = \{K, K_0\}$ be of self-adjoint operators where the unperturbed Hamiltonian K_0 describes a totally decoupled system, that means, the inner system is closed and the leads are decoupled from it, while the perturbed Hamiltonian K describes the system where the leads are coupled to the inner system. An important component is system $\mathcal{S} = \{K, K_0\}$, which represents a complete scattering or even a trace class scattering system.

In [1], an abstract Landauer-Büttiker formula was derived in the framework of a trace class scattering theory for semi-bounded self-adjoint operators which allows one to reproduce the results of [18] and [7] rigorously. In [13], the results of [1] were generalized to non-semi-bounded operators. Following [1], we consider a trace class scattering system $\mathcal{S} = \{K, K_0\}$. We

recall that $\mathcal{S} = \{K, K_0\}$ is called a trace class scattering system if the resolvent difference of K and K_0 belongs to the trace class. If $\mathcal{S} = \{K, K_0\}$ is a trace class scattering system, then the wave operators $W_\pm(K, K_0)$ exist and are complete. The scattering operator is defined by $S(K, K_0) := W_+(K, K_0)^* W_-(K, K_0)$. The main components, besides the trace class scattering system $\mathcal{S} = \{K, K_0\}$, are the density and the charge operators ρ and Q, respectively.

The density operator ρ is a non-negative bounded self-adjoint operator commuting with K_0. The charge Q is a bounded self-adjoint operator commuting also with K_0. If K has no singular continuous spectrum, then the current related to the density operator ρ and the charge Q is defined as follows:

$$J^{\mathcal{S}}_{\rho,Q} = -i \operatorname{tr} \left(W_-(K, K_0) \rho W_-(K, K_0)^* [K, Q] \right), \tag{3.1}$$

where $[K, Q]$ is the commutator of K and Q. In fact, the commutator $[K, Q]$ might be not defined. In this case, the regularized definition:

$$J^{\mathcal{S}}_{\rho,Q} = -i \operatorname{tr} \left(W_-(K, K_0)(I + K_0^2) \rho W_-(K, K_0)^* \frac{1}{K - i} [K, Q] \frac{1}{K + i} \right), \tag{3.2}$$

is used, where it is assumed that $(I + K_0^2)\rho$ is a bounded operator. Since the condition $(H - i)^{-1}[H, Q](H + i)^{-1} \in \mathfrak{L}_1(\mathfrak{H})$ is satisfied, definition (3.2) makes sense. By $\mathfrak{L}_1(\mathfrak{H})$ is the ideal of trace class operators is denoted.

Let K_0 be self-adjoint operator on the separable Hilbert space $\mathfrak{K}$. We call ρ be a density operator for K_0 if ρ is a bounded non-negative self-adjoint operator commuting with K_0. Since ρ commutes with K_0, one sees that ρ leaves invariant the subspace $\mathfrak{K}^{ac}(K_0)$. We then set

$$\rho_{ac} := \rho \restriction \mathfrak{K}^{ac}(K_0).$$

call ρ_{ac} the ac-density part of ρ.

A bounded self-adjoint operator, Q, commuting with K_0, is called a charge. If Q is the charge, then:

$$Q_{ac} := Q \restriction \mathfrak{K}^{ac}(K_0),$$

is called its ac-charge component.

Let $\Pi(K_0^{ac}) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)), \mathcal{M}, \Phi\}$ be a spectral representation of K_0^{ac}. If ρ is a density operator, then there is a measurable family $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ of bounded self-adjoint operators such that the multiplication operator:

$$(\mathcal{M}_{\rho_{ac}} \widehat{f})(\lambda) := \rho_{ac}(\lambda)\, \widehat{f}\,(\lambda), \quad \widehat{f} \in \operatorname{dom}(M_{\rho_{ac}}) := L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)),$$

is unitarily equivalent to the ac-part ρ_{ac}, that is, $\mathcal{M}_{\rho_{ac}} = \Phi \rho_{ac} \Phi^*$. In particular, this yields that: ess-sup $_{\lambda\in\mathbb{R}} \|\rho_{ac}(\lambda)\|_{\mathcal{B}(\mathfrak{k}(\lambda)} = \|\rho_{ac}\|_{\mathcal{B}(\mathfrak{K}^{ac}(K_0))}$. In the following, we call $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ the density matrix of ρ_{ac}.

Similarly, one obtains that if Q is a charge, then there is a measurable family $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ of bounded self-adjoint operators, such that the multiplication operator:

$$\begin{aligned} (\mathcal{M}_{Q_{ac}} \widehat{f})(\lambda) &:= Q_{ac}(\lambda)\, \widehat{f}\,(\lambda), \\ \widehat{f} \in \operatorname{dom}(Q_{ac}) &:= \{f \in L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda)) : Q_{ac}(\lambda)\, \widehat{f}\,(\lambda) \in L^2(\mathbb{R}, d\lambda, \mathfrak{k}(\lambda))\}, \end{aligned}$$

is unitarily equivalent to Q_{ac}, i.e. $\mathcal{M}_{Q_{ac}} = \Phi Q_{ac} \Phi^*$. In particular, one has:

$$\text{ess-sup}_{\lambda\in\mathbb{R}} \|Q_{ac}(\lambda)\|_{\mathcal{B}(\mathfrak{k}(\lambda))} = \|Q_{ac}\|_{\mathcal{B}(\mathfrak{K}^{ac}(K_0))}. \tag{3.3}$$

If Q is a charge, then the family $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the charge matrix of the ac-component of Q.

Let $\mathcal{S} = \{K, K_0\}$ be a trace scattering system. By $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$, we denote the scattering matrix, which corresponds to the scattering operator $S(K, K_0)$ with respect to the spectral representation $\Pi(K_0^{ac})$. The operator $T := S(K, K_0) - P^{ac}(K_0)$ is called the transmission operator.

By $\{T(\lambda)\}_{\lambda\in\mathbb{R}}$, we denote the transmission matrix which is related to the transmission operator. Scattering and transmission matrices are related by $S(\lambda) = T_{\mathfrak{k}(\lambda)} + T(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. Notice that $T(\lambda)$ belongs for to the trace class a.e. $\lambda \in \mathbb{R}$.

Theorem 3.1 ([13, Corollary 2.14]). *Let $\mathcal{S} := \{K, K_0\}$ be a trace class scattering system and let $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ be the scattering matrix of $\mathcal{S}$ with respect to the spectral representation $\Pi(K_0^{ac})$. Furthermore, let ρ and Q be density and charge operators and let $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ be the density and charge matrices of the ac-components ρ_{ac} and charge Q_{ac} with respect to $\Pi(K_0^{ac})$, respectively. If $(I + K_0^2)\rho$ is bounded, then the current $J^{\mathcal{S}}_{\rho,Q}$ defined by* (3.2) *admits the representation:*

$$J^{\mathcal{S}}_{\rho,Q} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}\Big(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S^*(\lambda)Q_{ac}(\lambda)S(\lambda))\Big)\mathrm{d}\lambda, \tag{3.4}$$

where the integrand on the right side and the current $J^{\mathcal{S}}_{\rho,Q}$ satisfy the estimate:

$$\begin{aligned}|\mathrm{tr}\,(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S^*(\lambda)Q_{ac}(\lambda)S(\lambda)))| \leqslant \\ 4\|\rho(\lambda)\|_{\mathfrak{L}(\mathfrak{k}(\lambda))}\|T(\lambda)\|_{\mathfrak{L}_1(\mathfrak{k}(\lambda))}\|Q(\lambda)\|_{\mathfrak{L}(\mathfrak{k}(\lambda))},\end{aligned} \tag{3.5}$$

for a.e. $\lambda \in \mathbb{R}$ and

$$|J^{\mathcal{S}}_{\rho,Q}| \leqslant C_0\|(H + i)^{-1} - (H_0 + i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})}, \tag{3.6}$$

where $C_0 := \frac{2}{\pi}\|(1 + H_0^2)\rho\|_{\mathfrak{L}(\mathfrak{K})}$.

In applications, not every charge Q is a bounded operator. We say the self-adjoint operator Q commuting with K_0 is a p-tempered charge if $Q(H_0 - i)^{-p}$ is a bounded operator for $p \in \mathbb{N}_0$. As above, we can introduce $Q_{ac} := Q \restriction \mathrm{dom}(Q) \cap \mathfrak{K}^{ac}(K_0)$. It follows that $QE_{K_0}(\Delta)$ is a bounded operator for any bounded Borel set Δ. This yields that the corresponding charge matrix, $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$, is a measurable family of bounded self-adjoint operators such that:

$$\text{ess-sup}_{\,\lambda\in\mathbb{R}}(1 + \lambda^2)^{p/2}\|Q_{ac}(\lambda)\|_{\mathfrak{L}(\mathfrak{k}(\lambda))} < \infty.$$

To generalize the current $J^{\mathcal{S}}_{\rho,Q}$ to tempered charges Q, one uses the fact that $Q(\Delta) := QE_{K_0}(\Delta)$ is a charge for any bounded Borel set Δ. Hence, the current $J^{\mathcal{S}}_{\rho,Q(\Delta)}$ is well-defined by (3.2) for any bounded Borel set Δ. Using Theorem 3.1 one gets that for p-tempered charges, the limit

$$J^{\mathcal{S}}_{\rho,Q} := \lim_{\Delta\to\mathbb{R}} J^{\mathcal{S}}_{\rho,Q(\Delta)} \tag{3.7}$$

exists, provided $(H_0 - i)^{p+2}\rho$ is a bounded operator. This gives rise to the following corollary:

Corollary 3.2. *Let the assumptions of the Theorem 3.1 be satisfied. If for some $p \in \mathbb{N}_0$ the operator $(H_0 - i)^{p+2}\rho$ is bounded and Q is a p-tempered charge for K_0, then the current defined by* (3.7) *admits the representation* (3.4)*, where the right hand side of* (3.4) *satisfies the estimate* (3.5)*. Moreover, the current $J^{\mathcal{S}}_{\rho,Q}$ can be estimated in the following manner:*

$$|J^{\mathcal{S}}_{\rho,Q}| \leqslant C_p\|(H + i)^{-1} - (H_0 + i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})}, \tag{3.8}$$

where $C_p := \frac{2}{\pi}\|(1 + H_0^2)^{p+2/2}\rho\|_{\mathfrak{L}(\mathfrak{K})}\|Q(I + H_0^2)^{-p/2}\|_{\mathfrak{L}(\mathfrak{K})}$.

At first glance, the formula (3.4) is not very similar to the original Landauer-Büttiker formula of [7, 18]. To make the formula more convenient, we recall that a standard application example for the Landauer-Büttiker formula is the so-called black-box model, cf. [1]. In this case, the Hilbert space $\mathfrak{K}$ is given by:

$$\mathfrak{K} = \mathfrak{K}_S \oplus \bigoplus_{j=1}^{N} \mathfrak{K}_j, \quad 2 \leqslant N < \infty. \tag{3.9}$$

and K_0 by:

$$K_0 = K_S \oplus \bigoplus_{j=1}^{N} K_j, \quad 2 \leqslant N < \infty. \tag{3.10}$$

The Hilbert space $\mathfrak{K}_S$ is called the sample or dot and K_S is the sample or dot Hamiltonian. The Hilbert spaces $\mathfrak{K}_j$ are called reservoirs or leads and K_j are the reservoir or lead Hamiltonians. For simplicity, we assume that the reservoir Hamiltonians K_j are absolutely continuous and the sample Hamiltonian K_S has a point spectrum. A typical choice for the density operator is:

$$\rho = f_S(K_S) \oplus \bigoplus_{j=1}^{N} f_j(K_j), \tag{3.11}$$

where $f_S(\cdot)$ and $f_j(\cdot)$ are non-negative bounded Borel functions, and for the charge:

$$Q = g_S(H_s) \oplus \bigoplus_{j=1}^{N} g_j(H_j), \tag{3.12}$$

where $g_S(\cdot)$ and $g_j(\cdot)$ a bounded Borel functions. Making this choice the Landauer-Büttiker formula (3.4) takes the form:

$$J^{\mathcal{S}}_{\rho,Q} = \frac{1}{2\pi} \sum_{j,k=1}^{N} \int_{\mathbb{R}} (f_j(\lambda) - f_k(\lambda)) g_j(\lambda) \sigma_{jk}(\lambda) d\lambda, \tag{3.13}$$

where

$$\sigma_{jk}(\lambda) := \mathrm{tr}(T_{jk}(\lambda)^* T_{jk}(\lambda)), \quad j,k = 1, \ldots, N, \quad \lambda \in \mathbb{R}, \tag{3.14}$$

are called the *total transmission probability* from reservoir k to reservoir j, cf. [1]. We call it the *cross-section* of the scattering process going from channel k to channel j at energy $\lambda \in \mathbb{R}$. $\{T_{jk}(\lambda)\}_{\lambda \in \mathbb{R}}$ is called the transmission matrix from channel k to channel j at energy $\lambda \in \mathbb{R}$ with respect to the spectral representation $\Pi(K_0^{ac})$. We note that $\{T_{jk}(\lambda)\}_{\lambda \in \mathbb{R}}$ corresponds to the transmission operator:

$$T_{jk} := P_j T(K, K_0) P_k, \quad T(K, K_0) := S(K, K_0) - P^{ac}(K_0), \tag{3.15}$$

acting from the reservoir k to reservoir j where $T(K, K_0)$ is called the transmission operator. Let $\{T(\lambda)\}_{\lambda \in \mathbb{R}}$ be the transmission matrix. Following [1], the current $J^{\mathcal{S}}_{\rho,Q}$ given by (3.13) is directed from the reservoirs into the sample.

The quantity $\|T(\lambda)\|_{\mathfrak{L}_2} = \mathrm{tr}(T(\lambda)^* T(\lambda))$ is well-defined and is called the cross-section of the scattering system $\mathcal{S}$ at energy $\lambda \in \mathbb{R}$. Notice that:

$$\sigma(\lambda) = \|T(\lambda)\|_{\mathfrak{L}_2} = \mathrm{tr}(T(\lambda)^* T(\lambda)) = \sum_{j,k=1}^{N} \sigma_{jk}(\lambda). \quad \lambda \in \mathbb{R},$$

We point out that the channel cross-sections $\sigma_{jk}(\lambda)$ admit the property:

$$\sum_{j=1}^{N} \sigma_{jk}(\lambda) = \sum_{j=1}^{N} \sigma_{kj}(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.16}$$

which is a consequence of the unitarity of the scattering matrix. Moreover, if there is a conjugation J, such that $KJ = JK$ and $K_0 J = J K_0$ holds, that is, if the scattering system $\mathcal{S}$ is time reversible symmetric, then we have even more, namely, it holds that:

$$\sigma_{jk}(\lambda) = \sigma_{kj}(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.17}$$

Usually, the Landauer-Büttiker formula (3.13) is used to calculated the electron current entering the reservoir j from the sample. In this case one has to choose $Q := Q_j^{el} := -\mathfrak{e} P_j$ where

P_j is the orthogonal projection form $\mathfrak{K}$ onto $\mathfrak{K}_j$ and $\mathfrak{e} > 0$ is the magnitude of the elementary charge. This is equivalent to choosing $g_j(\lambda) = -\mathfrak{e}$ and $g_k(\lambda) = 0$ for $k \neq j$, $\lambda \in \mathbb{R}$. In doing so, we get the Landauer-Büttiker formula simplifying to:

$$J^{\mathcal{S}}_{\rho,Q^{el}_j} = -\frac{\mathfrak{e}}{2\pi} \sum_{k=1}^{N} \int_{\mathbb{R}} (f_j(\lambda) - f_k(\lambda)) \sigma_{jk}(\lambda) d\lambda. \tag{3.18}$$

To restore the original Landauer-Büttiker formula, one sets:

$$f_j(\lambda) = f(\lambda - \mu_j), \quad \lambda \in \mathbb{R}, \tag{3.19}$$

where μ_j is the chemical potential of the reservoir $\mathfrak{K}_j$ and $f(\cdot)$ is a bounded non-negative Borel function called the distribution function. This gives rise to the following formula:

$$J^{\mathcal{S}}_{\rho,Q^{el}_j} = -\frac{\mathfrak{e}}{2\pi} \sum_{k=1}^{N} \int_{\mathbb{R}} (f(\lambda - \mu_j) - f(\lambda - \mu_k)) \sigma_{jk}(\lambda) d\lambda. \tag{3.20}$$

In particular, if we choose one:

$$f(\lambda) := f_{FD}(\lambda) := \frac{1}{1 + e^{\beta\lambda}}, \quad \beta > 0, \quad \lambda \in \mathbb{R}, \tag{3.21}$$

where $f_{FD}(\cdot)$ is the Fermi-Dirac distribution function, and inserting (3.21) into (3.20) we obtain:

$$J^{\mathcal{S}}_{\rho,Q^{el}_j} = -\frac{\mathfrak{e}}{2\pi} \sum_{k=1}^{N} \int_{\mathbb{R}} (f_{FD}(\lambda - \mu_j) - f_{FD}(\lambda - \mu_k)) \sigma_{jk}(\lambda) d\lambda. \tag{3.22}$$

If we have only two reservoirs, then they are usually denoted by l (left) and r (right). Let $j = l$ and $k = r$. Then,

$$J^{\mathcal{S}}_{\rho,Q^{el}_l} = -\frac{\mathfrak{e}}{2\pi} \int_{\mathbb{R}} (f_{FD}(\lambda - \mu_l) - f_{FD}(\lambda - \mu_r)) \sigma_{lr}(\lambda) d\lambda. \tag{3.23}$$

One easily checks that $J^{\mathcal{S}}_{\rho,Q_l} \leqslant 0$ if $\mu_l \geqslant \mu_r$. That means, the current is leaving the left reservoir and is entering the right one which is in accordance with physical expectations.

Example 3.3. Notice that $s_c := \{h^{el}, h^{el}_0\}$ is a $\mathfrak{L}_1$ scattering system. The Hamiltonian h^{el} takes into account the effect of coupling of reservoirs or leads $\mathfrak{h}_l := l^2(\mathbb{N})$ and $\mathfrak{h}_r := l^2(\mathbb{N})$ to the sample $\mathfrak{h}_S = \mathbb{C}^2$ which is also called the quantum dot. The Hamiltonians for the leads are given by: $h^{el}_\alpha = -\Delta^D + v_\alpha$, $\alpha = l, r$. The sample or quantum dot Hamiltonian is given by h^{el}_S. The wave operators are given by:

$$w_\pm(h^{el}, h^{el}_0) := s\text{-}\lim_{t\to\infty} e^{ith^{el}} e^{-ith^{el}_0} P^{ac}(h^{el}_0). \tag{3.24}$$

The scattering operator is given by $s_c := w_+(h^{el}, h^{el}_0)^* w_-(h^{el}, h^{el}_0)$. Let $\Pi(h^{el,ac}_0)$ be the spectral representation of $h^{el,ac}_0$ introduced in Section 2.6. If ρ^{el} and q^{el} are density and charge operators for h^{el}_0, then the Landauer-Büttiker formula takes the following form:

$$J^{s_c}_{\rho^{el},q^{el}} = \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{tr}\left(\rho^{el}_{ac}(\lambda)\left(q^{el}_{ac} - s_c(\lambda)^* q^{el}_{ac}(\lambda) s_c(\lambda)\right)\right), \tag{3.25}$$

where $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$, $\{q^{el}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{\rho^{el}(\lambda)\}_{\lambda\in\mathbb{R}}$ are the scattering, charge and density matrices with respect to $\Pi(h^{el,ac}_0)$, respectively. The condition that $((h^{el}_0)^2 + I_{\mathfrak{h}^{el}})\rho^{el}$ is a bounded operator is superfluous because h^{el}_0 is a bounded operator. For the same reason we have that every p-tempered charge q^{el} is in fact a charge, that means, q^{el} is a bounded self-adjoint operator.

The scattering system s_c is a black-box model with reservoirs $\mathfrak{h}^{el}_l$ and $\mathfrak{h}^{el}_r$. Choosing

$$\rho^{el} = f_l(h^{el}) \oplus f_S(h^{el}_S) \oplus f_r(h^{el}_r),$$

where $f_\alpha(\cdot)$, $\alpha = l, r$, are bounded Borel functions, and

$$q^{el} = g_l(h_l^{el}) \oplus g_S(h_S^{el}) \oplus g_r(h_r^{el}),$$

where $g_\alpha(\cdot)$, $\alpha \in \{l, r\}$, are locally bounded Borel functions, then from (3.13) it follows that:

$$J^{s_c}_{\rho^{el},q^{el}} = \frac{1}{2\pi} \sum_{\substack{\alpha,\varkappa\in\{l,r\}\\ \alpha\neq\varkappa}} \int_{\mathbb{R}} (f_\alpha(\lambda) - f_\varkappa(\lambda)) g_\alpha(\lambda)\sigma_c(\lambda) d\lambda,$$

where $\{\sigma_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is the channel cross-section from left to right and vice versa. Indeed, let $\{t_c(\lambda)\}_{\lambda\in\mathbb{R}}$ be the transition matrix which corresponds to the transition operator $t_c := s_c - I_{\mathfrak{h}^{el}}$. Clearly, one has $t_c(\lambda) = I_{\mathfrak{h}(\lambda)} - s_c(\lambda)$, $\lambda \in \mathbb{R}$. Let $\{p^{el}_\alpha(\lambda)\}_{\lambda\in\mathbb{R}}$ be the matrix which corresponds to the orthogonal projection p^{el}_α from $\mathfrak{h}^{el}$ onto $\mathfrak{h}^{el}_\alpha$. Further, let $t^c_{rl}(\lambda) := p^{el}_r(\lambda) t_c(\lambda) p^{el}_l$ and $t^c_{lr} := p^{el}_l(\lambda) t_c(\lambda) p^{el}_r$. Notice that both quantities are in fact scalar functions. Accordingly, the channel cross-sections $\sigma^c_{lr}(\lambda)$ and $\sigma^c_{rl}(\lambda)$ at energy $\lambda \in \mathbb{R}$ are given by $\sigma_c(\lambda) := \sigma^c_{lr}(\lambda) = |t^c_{lr}(\lambda)|^2 = |t^c_{rl}(\lambda)|^2 = \sigma^c_{rl}(\lambda)$, $\lambda \in \mathbb{R}$.

In particular, if $g_l(\lambda) = 1$ and $g_r = 0$, then:

$$J^{s_c}_{\rho^{el},q^{el}_l} = \frac{1}{2\pi} \int_{\mathbb{R}} (f_l(\lambda) - f_r(\lambda)) \sigma_c(\lambda) d\lambda, \tag{3.26}$$

and $q^{el}_l := p^{el}_l$. Following [1], $J^{s_c}_{\rho^{el},q^{el}_l}$ denotes the current entering the quantum dot from the left lead.

3.2. Application to the JCL-model

Let $\mathcal{S} = \{H, H_0\}$ now be the JCL-model. Furthermore, let ρ and Q be the density operator and a charge for H_0, respectively. Under these assumptions, the current $J^{\mathcal{S}}_{\rho,Q}$ is defined by:

$$J^{\mathcal{S}}_{\rho,Q} := -i\mathrm{tr}\left(W_-(H, H_0)(I + H_0^2)\rho W_-(H, H_0)^* \frac{1}{H - i}[H, Q]\frac{1}{H + i}\right), \tag{3.27}$$

and admits representation (3.4). If Q is a p-tempered charge and $(H_0 - i)^{p+2}\rho$ is a bounded operator, then the current $J^{\mathcal{S}}_{\rho,Q}$ is defined in accordance with (3.7) and the Landauer-Büttiker formula (3.4) is also valid.

We introduce the intermediate scattering system $\mathcal{S}_c := \{H, H_c\}$, where:

$$H_c := h^{el} \otimes I_{\mathfrak{h}^{ph}} + I_{\mathfrak{h}^{el}} \otimes h^{ph} = H_0 + V_{el}.$$

The Hamiltonian H_c describes the coupling of the leads to the quantum dot, but under the assumption that the photon interaction is not switched on.

Accordingly, $\mathcal{S}_{ph} := \{H, H_c\}$ and $\mathcal{S}_c := \{H_c, H_0\}$ are $\mathfrak{L}_1$-scattering systems. The corresponding scattering operators are denoted by S_{ph} and S_c, respectively. Let $\Pi(H^{ac}_c) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_c(\lambda)), \mathcal{M}, \Phi_c\}$ of H^{ac}_c be a spectral representation of H_c. The scattering matrix of the scattering system $\{H, H_c\}$ with respect to $\Pi(H^{ac}_c)$ is denoted by $\{S_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$. The scattering matrix of the scattering system $\{H_c, H_0\}$ with respect to $\Pi(H^{ac}_0) = \{L^2(\mathbb{R}, d\lambda, \mathfrak{h}_0(\lambda)), \mathcal{M}, \Phi_0\}$ is denoted by $\{S_c(\lambda)\}_{\lambda\in\mathbb{R}}$.

Since $\mathcal{S}_c$ is a $\mathfrak{L}_1$-scattering system, the wave operators $W_\pm(H_c, H_0)$ exist and are complete and since $\Phi_c W_\pm(H_c, H_0)\Phi_0^*$ commutes with $\mathcal{M}$, there are measurable families $\{W_\pm(\lambda)\}_{\lambda\in\mathbb{R}}$ of isometries acting from $\mathfrak{h}_0(\lambda)$ onto $\mathfrak{h}_c(\lambda)$ for a.e. $\lambda \in \mathbb{R}$ such that:

$$(\Phi_c W_\pm(H_c, H_0)\Phi_0^* \widehat{f}\,)(\lambda) = W_\pm(\lambda)\, \widehat{f}\,(\lambda), \quad \lambda \in \mathbb{R}, \quad \widehat{f} \in L^2(\mathbb{R}, d\lambda, \mathfrak{h}_0(\lambda)).$$

The families $\{W_\pm(\lambda)\}_{\lambda\in\mathbb{R}}$ are called wave matrices.

Straightforward computation shows that $\widehat{S}_{ph} := W_+(H_c, H_0)^* S_{ph} W_+(H_c, H_0)$ commutes with H_0. Hence, with respect to the spectral representation $\Pi(H^{ac}_0)$, the operator $\widehat{S}_{ph}$ is unitarily

equivalent to a multiplication induced by a measurable family $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ of unitary operators in $\mathfrak{h}_0(\lambda)$. Straightforward computation shows that:

$$\widehat{S}_{ph}(\lambda) = W_+(\lambda)^* S_{ph}(\lambda) W_+(\lambda), \tag{3.28}$$

for a.e. $\lambda \in \mathbb{R}$. Roughly speaking, $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is the scattering matrix of S_{ph} with respect to the spectral representation $\Pi(H_0^{ac})$.

Furthermore, let

$$\rho^c := W_-(H_c, H_0)\rho W_-(H_c, H_0)^* \tag{3.29}$$

and

$$Q^c := W_+(H_c, H_0) Q W_+(H_c, H_0)^*. \tag{3.30}$$

The operators ρ^c and Q^c are the density and tempered charge operators for the scattering system $\mathcal{S}_{ph}$. Indeed, one easily verifies that ρ^c and Q^c are commute with H_c. Moreover, ρ^c is non-negative. Furthermore, if Q is a charge, then Q^c is also a charge. This gives rise to the introduction of currents $J^c_{\rho,Q} := J^{\mathcal{S}_c}_{\rho,Q}$,

$$J^c_{\rho,Q} := -i\mathrm{tr}\left(W_-(H_c, H_0)\rho W_-(H_c, H_0)^* \frac{1}{H_c - i}[H_c, Q]\frac{1}{H_c + i}\right), \tag{3.31}$$

and $J^{ph}_{\rho,Q} := J^{\mathcal{S}_{ph}}_{\rho^c,Q^c}$,

$$J^{ph}_{\rho,Q} := -i\mathrm{tr}\left(W_-(H, H_c)\rho^c W_-(H, H_c)^* \frac{1}{H - i}[H, Q^c]\frac{1}{H + i}\right), \tag{3.32}$$

which are well defined. If Q is p-tempered charge and $(H_0 - i)^{p+2}\rho$ is a bounded operator, then one easily checks that Q^c is a p-tempered charge and $(H_c - i)^{p+2}\rho^c$ is a bounded operator. Hence the definition of the currents $J^{\mathcal{S}_c}_{\rho^c,Q^c}$ can be extended to this case and the Landauer-Büttiker formula (3.4) holds.

Finally, we note that the corresponding matrices $\{\rho^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ are related to the matrices $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ by

$$\rho^c_{ac}(\lambda) = W_-(\lambda)\rho_{ac}(\lambda)W_-(\lambda)^* \quad \text{and} \quad Q^c_{ac}(\lambda) = W_+(\lambda)Q_{ac}(\lambda)W_+(\lambda)^* \tag{3.33}$$

for a.e. $\lambda \in \mathbb{R}$.

Proposition 3.4 (Current decomposition)**.** *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Furthermore, let ρ and Q be the density operator and a p-tempered charge, $p \in \mathbb{N}_0$, for H_0, respectively. If $(H_0 - i)^{p+2}\rho$ is a bounded operator, then the decomposition,*

$$J^{\mathcal{S}}_{\rho,Q} = J^c_{\rho,Q} + J^{ph}_{\rho,Q}, \tag{3.34}$$

holds where $J^c_{\rho,Q}$ and $J^{ph}_{\rho,Q}$ are given by (3.31) *and* (3.32)*.*

In particular, let $\{S_c(\lambda)\}_{\lambda\in\mathbb{R}}$, $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ be scattering, density and charge matrices of S_c, ρ and Q with respect to $\Pi(H_0^{ac})$ and let $\{S_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$, $\{\rho^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ and $\{Q^c_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ be the scattering, density and charge matrices of the scattering operator S_{ph}, density operator ρ^c, cf. (3.29)*, and charge operator Q^c, cf.* (3.30)*, with respect to the spectral representation $\Pi(H_c^{ac}\}$. Then, the following representations:*

$$J^c_{\rho,Q} := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda))d\lambda, \tag{3.35}$$

$$J^{ph}_{\rho,Q} := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\rho^c_{ac}(\lambda)(Q^c_{ac}(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda)))d\lambda, \tag{3.36}$$

take place.

Proof. Since $\mathcal{S}_c$ and $\mathcal{S}_{ph}$ are $\mathfrak{L}_1$-scattering systems from Theorem 3.1 the representations (3.35) and (3.36) are easily follow. Taking into account (3.33), we get the following:

$$\begin{aligned}&\mathrm{tr}(\rho^c_{ac}(\lambda)(Q^c_{ac}(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda))) = \\ &\quad \mathrm{tr}(W_-(\lambda)\rho_{ac} W_-(\lambda)^*(W_+(\lambda)Q_{ac}(\lambda)W_+(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda))).\end{aligned}$$

Using $S_c(\lambda) = W_+(\lambda)^* W_-(\lambda)$ we find that:

$$\begin{aligned}&\mathrm{tr}(\rho^c_{ac}(\lambda)(Q^c_{ac}(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda))) = \mathrm{tr}\,(\rho_{ac}(\lambda)\times \\ &(S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda) - W_-(\lambda)^* S_{ph}(\lambda)^* W_+(\lambda) Q_{ac}(\lambda) W_+(\lambda)^* S_{ph}(\lambda) W_-(\lambda)))\,.\end{aligned} \tag{3.37}$$

Since $\{H_c, H_0\}$ and $\{H, H_c\}$ are $\mathfrak{L}_1$-scattering systems, the existence of the wave operators $W_\pm(H, H_c)$ and $W_\pm(H_c, H_0)$ follows. Using the chain rule, we find $W_\pm(H, H_0) = W_\pm(H, H_c)W_\pm(H_c, H_0)$ which yields:

$$\begin{aligned}S &= W_+(H, H_0)^* W_+(H, H_0) \\ &= W_+(H_c, H_0)^* W_+(H, H_c) W_-(H, H_c) W_-(H_c, H_0) = W_+(H_c, H_0)^* S_{ph} W_-(H_c, H_0).\end{aligned}$$

Hence, the scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of $\{H, H_0\}$ admits the representation

$$S(\lambda) = W_+(\lambda)^* S_{ph}(\lambda) W_-(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.38}$$

Inserting (3.38) into (3.37), we get the following:

$$J^{ph}_{\rho,Q} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\rho_{ac}(\lambda)(S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda) - S(\lambda)^* Q_{ac}(\lambda) S(\lambda)))d\lambda \tag{3.39}$$

Using (3.39), we obtain the following:

$$J^c_{\rho,Q} + J^{ph}_{\rho,Q} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\rho_{ac}(\lambda)(Q_{ac}(\lambda) - S(\lambda)^* Q_{ac}(\lambda) S(\lambda)))d\lambda.$$

Finally, taking into account (3.4), we obtain (3.34). □

Remark 3.5.
(i) The current $J^c_{\rho,Q}$ is due to the coupling of the leads to the quantum dot and it is therefore called the *contact induced current*.
(ii) The current $J^{ph}_{\rho,Q}$ is due to the interaction of photons with electrons and it is called the *photon induced current*. Notice the this current is calculated under the assumption that the leads are already in contact with the dot.

Corollary 3.6. *Let the assumptions of Proposition 3.4 be satisfied. With respect to the spectral representation $\Pi(H^{ac}_0)$ of H^{ac}_0 the photon induced current $J^{ph}_{\rho,Q}$ can be represented by:*

$$J^{ph}_{\rho,Q} := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{tr}(\,\widehat{\rho}_{ac}(\lambda)\,(Q_{ac}(\lambda) - \,\widehat{S}_{ph}(\lambda)^*\,Q_{ac}(\lambda)\,\widehat{S}_{ph}\,(\lambda)))d\lambda, \tag{3.40}$$

where the measurable families $\{\,\widehat{S}_{ph}(\lambda)\,\}_{\lambda\in\mathbb{R}}$ and $\{\,\widehat{\rho}_{ac}(\lambda)\,\}_{\lambda\in\mathbb{R}}$ are given by (3.28) *and*

$$\widehat{\rho}_{ac}(\lambda) := S_c(\lambda)\rho_{ac}(\lambda)S_c(\lambda)^* \quad \lambda \in \mathbb{R}, \tag{3.41}$$

respectively.

Proof. Using (3.33) and $S_c(\lambda) = W_+(\lambda)^* W_-(\lambda)$, we find:

$$\begin{aligned}&\mathrm{tr}(\rho^c_{ac}(\lambda)(Q^c_{ac}(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda))) = \\ &\quad \mathrm{tr}\,(S_c(\lambda)\rho_{ac}(\lambda)S_c(\lambda)^*\,(Q_{ac}(\lambda) - W_+(\lambda)^* S_{ph}(\lambda)^* W_+(\lambda) Q_{ac}(\lambda) W_+(\lambda)^* S_{ph}(\lambda) W_+(\lambda)))\,.\end{aligned}$$

Taking into account the representations (3.28) and (3.41), we get the following:

$$\begin{aligned}&\mathrm{tr}(\rho^c_{ac}(\lambda)(Q^c_{ac}(\lambda) - S_{ph}(\lambda)^* Q^c_{ac}(\lambda) S_{ph}(\lambda))) = \\ &\quad \mathrm{tr}(S_c(\lambda)\rho_{ac}(\lambda)S_c(\lambda)^*(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^*\, Q_{ac}(\lambda)\, \widehat{S}_{ph}(\lambda))),\end{aligned}$$

which immediately yields (3.40). □

Remark 3.7. In the following, we call $\{\,\widehat{\rho}_{ac}(\lambda)\,\}_{\lambda\in\mathbb{R}}$, cf. (3.41), the photon modified electron density matrix. Notice that $\{\,\widehat{\rho}_{ac}(\lambda)\,\}_{\lambda\in\mathbb{R}}$ might be non-diagonal, even if the electron density matrix $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is diagonal.

4. Analysis of currents

In the following, we analyze currents $J^c_{\rho,Q}$ and $J^{ph}_{\rho,Q}$ under the assumption that ρ and Q have the tensor product structure:

$$\rho = \rho^{el}\otimes\rho^{ph} \quad \text{and} \quad Q = q^{el}\otimes q^{ph}, \tag{4.1}$$

where ρ^{el} and ρ^{ph} as well as q^{el} and q^{ph} are density operators and (tempered) charges for h^{el}_0 and h^{ph}, respectively. Since ρ^{ph} commutes with h^{ph}, which is discrete, the operator ρ^{ph}has the form:

$$\rho^{ph} = \sum_{n\in\mathbb{N}_0} \rho^{ph}(n)(\cdot,\Upsilon_n)\Upsilon_n, \tag{4.2}$$

where $\rho^{ph}(n)$ are non-negative numbers. Similarly, q^{ph} can be represented as:

$$q^{ph} = \sum_{n\in\mathbb{N}_0} q^{ph}(n)(\cdot,\Upsilon_n)\Upsilon_n, \tag{4.3}$$

where $q^{ph}(n)$ are real numbers.

Lemma 4.1. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Assume that $\rho \neq 0$ and Q have the structure* (4.1) *where ρ^{el} is a density operator and q^{el} is a charge for h^{el}_0.*

(i) *The operator $(H_0 - i)^{p+2}\rho$, $p\in\mathbb{N}_0$, is bounded if and only if the condition:*

$$\sup_{n\in\mathbb{N}_0} \rho^{ph}(n) n^{p+2} < \infty, \tag{4.4}$$

is satisfied.

(ii) *The charge Q is p-tempered if and only if:*

$$\sup_{n\in\mathbb{N}} |q^{ph}(n)| n^{-p} < \infty, \tag{4.5}$$

is valid

Proof. (i) The operator $(H_0 - i)^{p+2}\rho$ admits the representation:

$$(H_0 - i)^{p+2}\rho = \bigoplus_{p\in\mathbb{N}_0} \rho^{ph}(n)(h^{el}_0 + n\omega - i)^{p+2}\rho^{el}.$$

We have:

$$\begin{aligned}\|(H_0 - i)^{p+2}\rho\|_{\mathfrak{L}(\mathfrak{H})} &= \sup_{p\in\mathbb{N}_0} \rho^{ph}(n)\|(h^{el}_0 + n\omega - i)^{p+2}\rho^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})} \\ &= \sup_{p\in\mathbb{N}_0} \rho^{ph}(n) n^{p+2} n^{-(p+2)} \left\|(h^{el}_0 + n\omega - i)^{p+2}\rho^{el}\right\|_{\mathfrak{L}(\mathfrak{h}^{el})}.\end{aligned} \tag{4.6}$$

Since $\lim_{n\to\infty} n^{-(p+2)} \left\|(h^{el}_0 + n\omega - i)^{p+2}\rho^{el}\right\|_{\mathfrak{L}(\mathfrak{h}^{el})} = \omega^{p+2}\|\rho^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})}$, we obtain for sufficiently large $n\in\mathbb{N}_0$ that:

$$\frac{\omega^{p+2}}{2}\|\rho^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})} \leqslant n^{-(p+2)}\|(h^{el}_0 + n\omega - i)^{p+2}\rho^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})}.$$

Using that and (4.6), we immediately obtain (4.4). Conversely, from (4.6) and (4.4), we obtain that $(H_0 - i)^{p+2}\rho$ is a bounded operator.

(ii) As above, we have:

$$Q(H_0 - i)^{-p} = \bigoplus_{n\in\mathbb{N}_0} q^{ph}(n)q^{el}.$$

Hence:

$$\|Q(H_0 - i)^{-p}\|_{\mathfrak{L}(\mathfrak{H})} = \sup_{n\in\mathbb{N}_0} |q^{ph}(n)| \|q^{el}(h_0^{el} + n\omega - i)^{-p}\|_{\mathfrak{L}(\mathfrak{h}^{el})}.$$

Since $\lim_{n\to\infty} n^p \|(h_0^{el} + n\omega - i)^{-p}\|_{\mathfrak{L}(\mathfrak{h}^{el})} = \omega^{-p}\|q^{el}\|_{\mathfrak{L}(\mathfrak{h}^{el})}$, we similarly obtain, as above, that (4.5) holds. The converse is obvious. □

4.1. Contact induced current

Let us recall that $\mathcal{S}_c = \{H_c, H_0\}$ is a $\mathfrak{L}_1$-scattering system. Straightforward computation shows that:

$$W_\pm(H_c, H_0) = w_\pm(h^{el}, h_0^{el}) \otimes I_{\mathfrak{h}^{ph}},$$

where $w_\pm(h^{el}, h_0^{el})$ is given by (3.24). Hence:

$$S_c = s_c \otimes I_{\mathfrak{h}^{ph}}, \quad \text{where} \quad s_c := w_+(h_c^{el}, h_0^{el})^* w_-(h_c^{el}, h_0^{el}).$$

Proposition 4.2. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Assume that ρ and Q are given by* (4.1) *where ρ^{el} and q^{el} are density and charge operators for h_0^{el} and ρ^{ph} and q^{ph} for h^{ph}, respectively. If for some $p \in \mathbb{N}_0$ the conditions* (4.4) *and* (4.5) *are satisfied, then the current $J^c_{\rho,Q}$ is well defined and admits the representation:*

$$J^c_{\rho,Q} = \gamma J^{s_c}_{\rho^{el},q^{el}}, \qquad \gamma := \sum_{n\in\mathbb{N}_0} q^{ph}(n)\rho^{ph}(n), \tag{4.7}$$

where $J^{s_c}_{\rho^{el},q^{el}}$ is defined by (3.2). *In particular, if $\mathrm{tr}(\rho^{ph}) = 1$ and $q^{ph} = I_{\mathfrak{h}^{ph}}$, then $J^c_{\rho,Q} = J^{s_c}_{\rho^{el},q^{el}}$.*

Proof. First, we note that by lemma 4.1 the operator $(H_0 - i)^{p+2}\rho$ is bounded and Q is p-tempered. Hence, the current $J^{\mathcal{S}_c}_{\rho,Q}$ is correctly defined and the Landauer-Büttiker formula (3.4) is valid.

With respect to the spectral representation $\Pi(H_0^{ac})$ of Lemma 2.12, the charge matrix $\{Q_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ of $Q_{ac} = q_{ac}^{el} \otimes q^{ph}$ admits the representation:

$$Q_{ac}(\lambda) = \bigoplus_{n\in\mathbb{N}_0} q_{ac}^{el}(\lambda - n\omega)q^{ph}(n), \quad \lambda \in \mathbb{R}. \tag{4.8}$$

Since $S_c = s_c \otimes I_{\mathfrak{h}^{ph}}$, the scattering matrix $\{S_c(\lambda)\}_{\lambda\in\mathbb{R}}$ admits the representation:

$$S_c(\lambda) = \bigoplus_{n\in\mathbb{N}_0} s_c(\lambda - n\omega), \quad \lambda \in \mathbb{R}.$$

Hence:

$$\begin{aligned} &Q_{ac}(\lambda) - S_c(\lambda)^* Q_{ac}(\lambda) S_c(\lambda) = \\ &\quad \bigoplus_{n\in\mathbb{N}_0} q^{ph}(n) \left(q_{ac}^{el}(\lambda - n\omega) - s_c(\lambda - n\omega)^* q_{ac}^{el}(\lambda - \omega n) s_c(\lambda - n\omega) \right). \end{aligned} \tag{4.9}$$

Moreover, the density matrix $\{\rho_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ admits the representation:

$$\rho_{ac}(\lambda) = \bigoplus_{n\in\mathbb{N}_0} \rho^{ph}(n)\rho_{ac}^{el}(\lambda - n\omega) \tag{4.10}$$

Inserting (4.10) into (4.9) we find the following:

$$\rho^{ac}(\lambda)\left(Q_{ac}(\lambda)-S_c(\lambda)^*Q_{ac}(\lambda)S_c(\lambda)\right)= \bigoplus_{n\in\mathbb{N}_0} q^{ph}(n)\rho^{ph}(n)\rho^{el}_{ac}(\lambda-n\omega)\left(q^{el}_{ac}(\lambda-\omega n)-s_c(\lambda-n\omega)^*q^{el}_{ac}(\lambda-\omega n)s_c(\lambda-n\omega)\right)$$

Since $\gamma=\sum_{n\in\mathbb{N}_0} q^{ph}(n)\rho^{ph}(n)$ is absolutely convergent by (4.4) and (4.5), we obtain that:

$$\mathrm{tr}\left(\rho^{ac}(\lambda)\left(Q_{ac}(\lambda)-S_c(\lambda)^*Q_{ac}(\lambda)S_c(\lambda)\right)\right)= \sum_{n\in\mathbb{N}_0} q^{ph}(n)\rho^{ph}(n)\mathrm{tr}\left(\rho^{el}_{ac}(\lambda-n\omega)\left(q^{el}_{ac}(\lambda-\omega n)-s_c(\lambda-n\omega)^*q^{el}_{ac}(\lambda-\omega n)s_c(\lambda-n\omega)\right)\right) \tag{4.11}$$

Clearly, we have:

$$\left|\mathrm{tr}\left(\rho^{el}_{ac}(\lambda-n\omega)\left(q^{el}_{ac}(\lambda-\omega n)-s_c(\lambda-n\omega)^*q^{el}_{ac}(\lambda-\omega n)s_c(\lambda-n\omega)\right)\right)\right|\leqslant 4\|\rho^{el}_{ac}(\lambda-n\omega)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))}\|q^{el}_{ac}(\lambda-n\omega)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))},\quad \lambda\in\mathbb{R}.$$

We insert (4.11) into the Landauer-Büttiker formula (3.35). Using (4.4) and (4.5) as well as

$$\int_{\mathbb{R}}\|\rho^{el}_{ac}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))}\|q^{el}_{ac}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))}d\lambda<\infty,$$

we see that we can interchange the integral and the sum. By doing so, we get:

$$J^c_{\rho,Q}=\sum_{n\in\mathbb{N}_0} q^{ph}(n)\rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\rho^{el}_{ac}(\lambda-n\omega)\times \left(q^{el}_{ac}(\lambda-\omega n)-s_c(\lambda-n\omega)^*q^{el}_{ac}(\lambda-\omega n)s_c(\lambda-n\omega)\right)\right)d\lambda.$$

Using (3.25) we prove (4.7).

If $\mathrm{tr}(\rho^{ph})=1$, then $\sum_{\mathbb{N}_0}\rho^{ph}(n)=1$. Furthermore, if $\rho^{ph}=I_{\mathfrak{h}^{ph}}$, then $q^{ph}(n)=1$. Hence, $\gamma=1$. □

4.2. Photon induced current

To calculate the current $J^{ph}_{\rho,Q}$, we use representation (3.40). We then set:

$$\widehat{S}^{ph}_{mn}(\lambda):=P_m(\lambda)\,\widehat{S}_{ph}(\lambda)\upharpoonright\mathfrak{h}_n(\lambda),\quad \lambda\in\mathbb{R},$$

where $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is defined by (3.28) and $P_m(\lambda)$ is the orthogonal projection from $\mathfrak{h}(\lambda)$, cf. (2.37), onto $\mathfrak{h}_m(\lambda):=\mathfrak{h}^{el}(\lambda-m\omega)$, $\lambda\in\mathbb{R}$.

Proposition 4.3. *Let $\mathcal{S}=\{H,H_0\}$ be the JCL-model. Assume that ρ and Q are given by* (4.1) *where ρ^{el} and q^{el} are density and charge operators for h^{el}_0 and ρ^{ph} and q^{ph} for h^{ph}, respectively. If for some $p\in\mathbb{N}_0$ the conditions* (4.4) *and* (4.5) *are satisfied, then the current $J^{ph}_{\rho,Q}$ is well-defined and it admits the following representation:*

$$J^{ph}_{\rho,Q}=\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)\sum_{n\in\mathbb{N}_0}q^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\,\mathrm{tr}\left(\widehat{\rho}^{el}_{ac}(\lambda-m\omega)\times \left(q^{el}_{ac}(\lambda-n\omega)\delta_{mn}-\widehat{S}^{ph}_{nm}(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right), \tag{4.12}$$

where $\{\widehat{\rho}^{el}_{ac}(\lambda)\}_{\lambda\in\mathbb{R}}$ is the photon modified electron density defined, cf. (3.41)*, which takes the following form:*

$$\widehat{\rho}^{el}_{ac}(\lambda)=s_c(\lambda)\rho^{el}(\lambda)s_c(\lambda)^*,\quad \lambda\in\mathbb{R}. \tag{4.13}$$

Proof. By Lemma 4.1 we get that that the charge Q is p-tempered and $(H_0 - i)^p \rho$ is a bounded operator. By Corollary 3.2, the current $J^{ph}_{\rho,Q} := J^{S_{ph}}_{\rho^c,Q^c}$ is well-defined.

Since $\left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda)\right)$ is a trace class operator for $\lambda \in \mathbb{R}$, we get from (3.40) and (4.10) that:

$$\mathrm{tr}\left(\widehat{\rho}_{ac}(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda)\right)\right) = \sum_{m\in\mathbb{N}_0} \rho^{ph}(m) \mathrm{tr}\left(\widehat{\rho}^{el}(\lambda - m\omega) P_m(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda)\right) P_m(\lambda)\right)$$

Furthermore, we have:

$$\begin{aligned} P_m(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda)\right) P_m(\lambda) \\ = q^{ph}(m) \left(q^{el}(\lambda - m\omega) - P_m(\lambda) \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda)\right) P_m(\lambda) \\ = q^{ph}(m) q^{el}(\lambda - m\omega) - \sum_{n\in\mathbb{N}_0} q^{ph}(n) \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}^{ph}_{nm}(\lambda), \end{aligned}$$

for $\lambda \in \mathbb{R}$ where $\widehat{S}^{ph}_{nm}(\lambda)^* := P_n(\lambda) \widehat{S}_{ph}(\lambda) P_m(\lambda)$, $\lambda \in \mathbb{R}$. Notice that $\sum_{n\in\mathbb{N}_0}$ is a sum with a finite number of summands. Hence:

$$\mathrm{tr}\left(\widehat{\rho}_{ac}(\lambda) \left(Q_{ac}(\lambda) - \widehat{S}_{ph}(\lambda)^* Q_{ac}(\lambda) \widehat{S}_{ph}(\lambda)\right)\right) = \sum_{m\in\mathbb{N}_0} \rho^{ph}(m) \sum_{n\in\mathbb{N}_0} q^{ph}(n) \times \mathrm{tr}\left(\widehat{\rho}^{el}(\lambda - m\omega) \left(q^{el}(\lambda - m\omega)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}^{ph}_{nm}(\lambda)\right)\right)$$

We are going to show that

$$\sum_{m\in\mathbb{N}_0} \rho^{ph}(m) \sum_{n\in\mathbb{N}_0} |q^{ph}(n)| \int_{\mathbb{R}} \left|\mathrm{tr}\left(\widehat{\rho}^{el}(\lambda - m\omega) \times \left(q^{el}(\lambda - m\omega)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}^{ph}_{nm}(\lambda)\right)\right)\right| d\lambda < \infty.$$

Clearly, one has the following estimate:

$$\left|\mathrm{tr}\left(\widehat{\rho}^{el}(\lambda - m\omega) \left(q^{el}(\lambda - m\omega)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}(\lambda - n\omega) \widehat{S}^{ph}_{nm}(\lambda)\right)\right)\right| \leqslant 2 \| \widehat{\rho}^{el}(\lambda - m\omega) \|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \left(\|q^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))}\delta_{nm} + \|q^{el}(\lambda - n\omega)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))}\right).$$

Furthermore, we get:

$$\int_{\lambda\in\mathbb{R}} \| \widehat{\rho}^{el}(\lambda - m\omega) \|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda - m\omega)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))}\delta_{nm} \leqslant \int_{\lambda\in\mathbb{R}} \| \widehat{\rho}^{el}(\lambda) \|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} d\lambda$$

and

$$\int_{\mathbb{R}} \| \widehat{\rho}^{el}(\lambda - m\omega) \|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda - n\omega)\|_{\mathfrak{L}(\mathfrak{h}_n(\lambda))} d\lambda \leqslant \|q^{el}_{ac}\|_{\mathfrak{L}(\mathfrak{h}^{el})} \int_{\lambda\in\mathbb{R}} \| \widehat{\rho}^{el}(\lambda - (m-n)\omega) \|_{\mathfrak{L}(\mathfrak{h}_{m-n}(\lambda))} d\lambda$$

If the conditions (4.4) and (4.5) are satisfied, then

$$\sum_{m\in\mathbb{N}_0} \rho^{ph}(m) |q^{ph}(m)| \int_{\mathbb{R}} \| \widehat{\rho}^{el}(\lambda) \|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} \|q^{el}(\lambda)\|_{\mathfrak{L}(\mathfrak{h}_m(\lambda))} d\lambda < \infty.$$

Furthermore, we obtain:

$$\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)\sum_{n\in\mathbb{N}_0}|q^{ph}(n)|\int_{\lambda\in\mathbb{R}}\|\,\widehat{\rho}^{el}(\lambda-(m-n)\omega)\,\|_{\mathfrak{L}(\mathfrak{h}_{m-n}(\lambda))}d\lambda\leqslant$$
$$(v_{\max}-v_{\min}+4)\|\rho^{el}_{ac}\|_{\mathfrak{L}(\mathfrak{h}^{el})}\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)\sum_{|m-n|\leqslant d_{\max}}|q^{ph}(n)|<\infty,$$

where $d_{\max}$ is introduced by Lemma 2.12. To prove the following:

$$\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)\sum_{|m-n|\leqslant d_{\max}}|q^{ph}(n)|<\infty,$$

we again use (4.4) and (4.5). The last step allows us to interchange the integral and the sums, which immediately proves (4.12). □

Corollary 4.4. *Let $\mathcal{S}=\{H,H_0\}$ be the JCL-model. We assume that ρ and Q are given by* (4.1)*, where ρ^{el} and q^{el} are density and charge operators for h^{el}_0 and ρ^{ph} and q^{ph} for h^{ph}, respectively. If ρ^{el} is an equilibrium state, i.e. $\rho^{el}=f^{el}(h^{el}_0)$, then:*

$$J^{ph}_{\rho,Q}=\sum_{m,n\in\mathbb{N}_0}q^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\rho^{ph}(n)f^{el}(\lambda-n\omega)-\rho^{ph}(m)f^{el}(\lambda-m\omega)\Big)\times$$
$$\mathrm{tr}\Big(\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big)\,d\lambda. \tag{4.14}$$

Proof. From (4.12), we obtain the following:

$$J^{ph}_{\rho,Q}=\sum_{n\in\mathbb{N}_0}q^{ph}(n)\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\;f^{el}(\lambda-m\omega)\times$$
$$\mathrm{tr}\Big(q^{el}_{ac}(\lambda-n\omega)\delta_{mn}-\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big).$$

Hence,

$$J^{ph}_{\rho,Q}=\sum_{n\in\mathbb{N}_0}q^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)f^{el}(\lambda-m\omega)\times$$
$$\mathrm{tr}\Big(q^{el}_{ac}(\lambda-n\omega)\delta_{mn}-\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big).$$

This gives the following:

$$J^{ph}_{\rho,Q}=\sum_{n\in\mathbb{N}_0}q^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}d\lambda\,\Big(\rho^{ph}(n)f^{el}(\lambda-n\omega)\mathrm{tr}\Big(q^{el}_{ac}(\lambda-n\omega)\Big)- \tag{4.15}$$
$$\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)f^{el}(\lambda-m\omega)\mathrm{tr}\Big(\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big)\Big).$$

Since

$$\sum_{m\in\mathbb{N}_0}\rho^{ph}(m)f^{el}(\lambda-m\omega)\mathrm{tr}\Big(\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big)=$$
$$\sum_{m\in\mathbb{N}_0}\Big(\rho^{ph}(m)f^{el}(\lambda-m\omega)-\rho^{ph}(n)f^{el}(\lambda-n\omega)\Big)\,\mathrm{tr}\Big(\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big)+$$
$$\rho^{ph}(n)f^{el}(\lambda-n\omega)\sum_{m\in\mathbb{N}_0}\mathrm{tr}\Big(\,\widehat{S}^{ph}_{nm}\,(\lambda)^*q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}\,(\lambda)\Big),$$

then inserting this into (4.15) we obtain (4.14). □

5. Electron and photon currents

5.1. Electron current

To calculate the electron current induced by contacts and photon contact, we make the following choice throughout this section. We set

$$Q^{el}_\alpha := q^{el}_\alpha \otimes q^{ph}, \quad q^{el}_\alpha := -\mathfrak{e}p^{el}_\alpha \quad \text{and} \quad q^{ph} := I_{\mathfrak{h}^{ph}}, \quad \alpha \in \{l, r\}, \tag{5.1}$$

where p^{el}_α denotes the orthogonal projection from $\mathfrak{h}^{el}$ onto $\mathfrak{h}^{el}_\alpha$. By $\mathfrak{e} > 0$, we denote the magnitude of the elementary charge. Since p^{el}_α commutes with h^{el}_α, one can easily verify that Q^{el}_α commutes with H_0, which shows that Q^{el}_α is a charge. Following [1], the flux related to Q^{el}_α gives us the electron current $J^{\mathcal{S}}_{\rho,Q^{el}_\alpha}$ entering the lead α from the sample. Notice $Q^{el}_\alpha = -\mathfrak{e}P_\alpha$ where P_α is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_\alpha := \mathfrak{h}^{el}_\alpha \otimes \mathfrak{h}^{ph}$. Since $q^{ph} = I_{\mathfrak{h}^{ph}}$, condition (4.5) is immediately satisfied for any $p \geqslant 0$.

Let $f(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative bounded measurable function. We set:

$$\rho^{el} = \rho^{el}_l \oplus \rho^{el}_S \oplus \rho^{el}_r, \quad \rho^{el}_\alpha := f(h^{el}_\alpha - \mu_\alpha), \quad \alpha \in \{l, r\}, \tag{5.2}$$

and $\rho = \rho^{el} \otimes \rho^{ph}$. By μ_α the chemical potential of the lead α is denoted. In applications, one sets $f(\lambda) := f_{FD}(\lambda)$, $\lambda \in \mathbb{R}$, where $f_{FD}(\lambda)$ is the so-called Fermi-Dirac distribution given by (3.21). If $\beta = \infty$, then $f_{FD}(\lambda) := \chi_{\mathbb{R}_-}(\lambda)$, $\lambda \in \mathbb{R}$. Notice that $[\rho^{el}, p^{el}] = 0$. For ρ^{ph}, we choose the Gibbs state:

$$\rho^{ph} := \frac{1}{Z} e^{-\beta h^{ph}}, \quad Z = \mathrm{tr}(e^{-\beta h^{ph}}) = \frac{1}{1 - e^{-\beta\omega}}. \tag{5.3}$$

Hence, $\rho^{ph} = (1 - e^{-\beta\omega})e^{-\beta h^{ph}}$. If $\beta = \infty$, then $\rho^{ph} := (\cdot, \Upsilon_0)\Upsilon_0$. Clearly, $\mathrm{tr}(\rho^{ph}) = 1$. We note that $\rho^{ph}(n) = (1 - e^{-\beta\omega})e^{-n\beta\omega}$, $n \in \mathbb{N}_0$, satisfies the condition (4.4) for any $p \geqslant 0$. Accordingly, $\rho_0 = \rho^{el} \otimes \rho^{ph}$ is the density operator for H_0.

Definition 5.1. Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. If $Q := Q^{el}_\alpha$, where Q^{el}_α is given by (5.1), and $\rho := \rho_0 := \rho^{el} \otimes \rho^{ph}$, where ρ^{el} and ρ^{ph} are given by (5.2) and (5.3), then $J^{el}_{\rho_0,Q^{el}_\alpha} := J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha}$ is called the electron current entering the lead α. The currents $J^{c}_{\rho_0,Q^{el}_\alpha}$ and $J^{ph}_{\rho_0,Q^{el}_\alpha}$ are called the contact-induced and photon-induced electron currents.

5.1.1. Contact induced electron current. The following proposition immediately follows from Proposition 4.2.

Proposition 5.2. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Then the contact induced electron current $J^{c}_{\rho_0,Q^{el}_\alpha}$, $\alpha \in \{l, r\}$, is given by $J^{c}_{\rho_0,Q^{el}_\alpha} = J^{s_c}_{\rho^{el},q^{el}_\alpha}$. In particular, one has:*

$$J^{c}_{\rho_0,Q^{el}_\alpha} = -\frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} (f(\lambda - \mu_\alpha) - f(\lambda - \mu_\varkappa)\sigma_c(\lambda)d\lambda, \quad \alpha, \varkappa \in \{l, r\}, \quad \alpha \neq \varkappa, \tag{5.4}$$

where $\{\sigma_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is the channel cross-section from left to the right of the scattering system $s_c = \{h^{el}, h^{el}_0\}$, cf. Example 3.3.

Proof. Since $\mathrm{tr}(\rho^{ph}) = 1$ it follows from Proposition 4.2 that $J^{c}_{\rho_0,Q^{el}_\alpha} = J^{s_c}_{\rho^{el},q^{el}_\alpha}$. From (3.26), cf. Example 3.3, we find (5.4). □

If $\mu_l > \mu_r$ and $f(\cdot)$ is decreasing, then $J^{c}_{\rho_0,Q^{el}_l} < 0$. Hence, the electron contact current is going from the left lead to the right which is in accordance with the physical expectations. In particular, this is valid for the Fermi-Dirac distribution.

Proposition 5.3. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Further, let ρ^{el} and ρ^{ph} be given by* (5.2) *and* (5.3)*, respectively. If the charge Q^{el}_α is given by* (5.1)*, then the following holds:*

(E) If $\mu_l = \mu_r$, then $J^c_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l, r\}$.
(S) If $v_l \geqslant v_r + 4$, then $J^c_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l, r\}$, even if $\mu_l \neq \mu_l$.
(C) If $e^S_0 = \delta^S_0$ and $e^S_1 = \delta^S_1$, then $J^c_{\rho_0,Q^{el}_\alpha} = 0$, $\alpha \in \{l, r\}$, even if $\mu_l \neq \mu_l$.

Proof. (E) If $\mu_l = \mu_r$, then $f(\lambda - \mu_l) = f(\lambda - \mu_r)$. Applying formula (5.4) we obtain $J^c_{\rho_0,Q^{el}_\alpha} = 0$.

(S) If $v_l \geqslant v_r + 4$, then $h^{el,ac}_0$ has simple spectrum. Hence the scattering matrix $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $s_c = \{h^{el}, h^{el}_0\}$ is a scalar function which immediately yields $\sigma_c(\lambda) = 0$, $\lambda \in \mathbb{R}$, which yields $J^c_{\rho_0,Q^{el}_\alpha} = 0$.

(C) In this case, the Hamiltonian h^{el} decomposes into the direct sum of two non-interacting Hamiltonians. Hence, the scattering matrix of $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $s_c = \{h^{el}, h^{el}_0\}$ is diagonal, which immediately yields $J^c_{\rho_0,Q^{el}_\alpha} = 0$. □

5.1.2. Photon induced electron current. It is hopeless to analyze the properties of (4.12) if we make no assumptions concerning ρ^{el} and the scattering operator s_c. The simplest assumptions is that ρ^{el} and s_c commute. In this case, we get $\widehat{\rho}^{el}(\lambda) = \rho^{el}(\lambda)$, $\lambda \in \mathbb{R}$.

Lemma 5.4. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Furthermore, let ρ^{el} be given by* (5.2)*. If one of the cases (E), (S) or (C) of Proposition 5.3 is realized, then the ρ^{el} and s_c commute.*

Proof. If (E) holds, then $\rho^{el} = f(h^{el}_0)$ which yields $[\rho^{el}, s_c] = 0$. If (S) is valid, then the scattering matrix $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ is a scalar function which shows $[\rho^{el}, s_c] = 0$. Finally, if (C) is realized, then the scattering matrix $\{s_c(\lambda)\}_{\lambda\in\mathbb{R}}$ diagonal. Since the ρ^{el} is given by (5.2) we get $[\rho^{el}, s_c] = 0$. □

We will now calculate the current $J^{ph}_{\rho_0,Q^{el}_\alpha}$, see (4.12). Clearly, we have $P_\alpha(\lambda) = \sum_{n\in\mathbb{N}_0} p^{el}_\alpha(\lambda - n\omega)$ and $I_{\mathfrak{h}(\lambda)} = P_l(\lambda) + P_r(\lambda)$, $\lambda \in \mathbb{R}$. We then set:

$$P_{n_\alpha}(\lambda) := P_\alpha(\lambda)P_n(\lambda) = P_n(\lambda)P_\alpha(\lambda) = p^{el}_\alpha(\lambda - n\omega),\ \alpha \in \{l, r\},$$

$n \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$. In the following we use the notation $\widehat{T}_{ph}(\lambda) = \widehat{S}_{ph}(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda \in \mathbb{R}$, where $\{\widehat{T}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the transition matrix and $\{\widehat{S}_{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is given by (3.28). We set:

$$\widehat{T}^{ph}_{k_\alpha m_\varkappa}(\lambda) := P_{k_\alpha}(\lambda)\,\widehat{T}_{ph}(\lambda)P_{m_\varkappa}(\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha, \varkappa \in \{l, r\}, \quad k, m \in \mathbb{N}_0,$$

and:

$$\widehat{\sigma}^{ph}_{k_\alpha m_\varkappa}(\lambda) = \mathrm{tr}(\,\widehat{T}^{ph}_{k_\alpha m_\varkappa}(\lambda)^*\ \widehat{T}^{ph}_{k_\alpha m_\varkappa}(\lambda)\,), \quad \lambda \in \mathbb{R}, \tag{5.5}$$

which is the cross-section between the channels k_α and $m_\varkappa$.

Proposition 5.5. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model.*
(i) *If ρ^{el} commutes with the scattering operator s_c and q^{el}, then:*

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{\substack{m,n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}} \frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} \Big(\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_\varkappa - m\omega)\Big)\ \widehat{\sigma}^{ph}_{n_\alpha m_\varkappa}(\lambda)\,d\lambda. \tag{5.6}$$

(ii) *If in addition $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then*

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{m,n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} \Big(\rho^{ph}(n)f(\lambda - \mu_\alpha - n\omega) - \rho^{ph}(m)f(\lambda - \mu_{\alpha'} - m\omega)\Big)\ \widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda)\,d\lambda, \tag{5.7}$$

$\alpha, \alpha' \in \{l, r\}$, $\alpha \neq \alpha'$.

Proof. (i) Let us assume that:

$$q^{el} = \sum_{\varkappa\in\{l,r\}} g_\varkappa(h^{el}_\varkappa),$$

Notice that

$$q^{el}_{ac}(\lambda) = \sum_{\varkappa\in\{l,r\}} g_\varkappa(\lambda)p^{el}_\varkappa(\lambda), \quad \lambda\in\mathbb{R}. \tag{5.8}$$

Inserting (5.8) into (4.12) and using $q^{ph} = I_{\mathfrak{h}^{ph}}$, we obtain the following:

$$J^{ph}_{\rho_0,Q} = \sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m) \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}} \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega)g_\varkappa(\lambda-n\omega)\times$$
$$\mathrm{tr}\left(p^{el}_\alpha(\lambda-m\omega)\left(p^{el}_\varkappa(\lambda-n\omega)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right),$$

where, for simplicity, we have set:

$$\phi_\alpha(\lambda) := f(\lambda-\mu_\alpha), \quad \lambda\in\mathbb{R}, \quad n\in\mathbb{N}_0, \quad \alpha\in\{l,r\}. \tag{5.9}$$

Accordingly, we get:

$$J^{ph}_{\rho_0,Q} = \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}} \rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\varkappa(\lambda-n\omega)g_\varkappa(\lambda-n\omega)\mathrm{tr}\left(p^{el}_\varkappa(\lambda-n\omega)\right) -$$
$$\sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega)g_\varkappa(\lambda-n\omega)\times \tag{5.10}$$
$$\mathrm{tr}\left(p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)p^{el}_\alpha(\lambda-m\omega)\right).$$

Since the scattering matrix $\{\,\widehat{S}^{ph}(\lambda)\}_{\lambda\in\mathbb{R}}$ is unitary, we have:

$$p^{el}_\varkappa(\lambda-n\omega) = \sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{mn}(\lambda)^* p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{mn}(\lambda)p^{el}_\varkappa(\lambda-n\omega), \tag{5.11}$$

for $n\in\mathbb{N}_0$ and $\varkappa\in\{l,r\}$. Inserting (5.11) into (5.10), we find that:

$$J^{ph}_{\rho_0,Q} = \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\varkappa(\lambda-n\omega)g_\varkappa(\lambda-n\omega)\times$$
$$\mathrm{tr}\left(p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{mn}(\lambda)p^{el}_\varkappa(\lambda-n\omega)\right) -$$
$$\sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega)g_\varkappa(\lambda-n\omega)\times$$
$$\mathrm{tr}\left(p^{el}_\alpha(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* p^{el}_\varkappa(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)p^{el}_\alpha(\lambda-m\omega)\right).$$

Using the notation (5.5), we find the following:

$$J^{ph}_{\rho_0,Q} = \sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\varkappa(\lambda-n\omega)g_\varkappa(\lambda-n\omega)\,\widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda) -$$
$$\sum_{\substack{n\in\mathbb{N}_0\\ \varkappa\in\{l,r\}}}\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\ \phi_\alpha(\lambda-m\omega)g_\varkappa(\lambda-n\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)\,.$$

By (3.16), we find that:

$$\sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda) \;=\; \sum_{\substack{m\in\mathbb{N}_0\\ \alpha\in\{l,r\}}} \widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda) \quad \lambda\in\mathbb{R}.$$

Using that, we obtain the following:

$$J^{ph}_{\rho_0,Q} = \sum_{\substack{m,n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}} \frac{1}{2\pi}\int_{\mathbb{R}} \times \tag{5.12}$$

$$\Big(\rho^{ph}(n)\phi_\varkappa(\lambda-n\omega)-\rho^{ph}(m)\phi_\alpha(\lambda-m\omega)\Big)\, g_\varkappa(\lambda-n\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)\,d\lambda.$$

Setting $g_\alpha(\lambda)=-\mathfrak{e}$ and $g_\varkappa(\lambda)\equiv 0$, $\varkappa\neq\alpha$, we obtain (5.6).

(ii) Straightforward computation shows that:

$$\sum_{n,m\in\mathbb{N}_0}\int_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_a-n\omega)-\rho^{ph}(m)f(\lambda-\mu_a-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda =$$

$$\sum_{n,m\in\mathbb{N}_0}\int_{\mathbb{R}}\Big(\rho^{ph}(m)f(\lambda-\mu_a-m\omega)-\rho^{ph}(n)f(\lambda-\mu_a-n\omega)\Big)\,\widehat{\sigma}^{ph}_{m_\alpha n_\alpha}(\lambda)\,d\lambda.$$

Since $\sigma^{ph}_{m_\alpha n_\alpha}(\lambda)=\sigma^{ph}_{n_\alpha m_\alpha}(\lambda)$, $\lambda\in\mathbb{R}$, we obtain the following:

$$\sum_{n,m\in\mathbb{N}_0}\int_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_a-n\omega)-\rho^{ph}(m)f(\lambda-\mu_a-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda =$$

$$-\sum_{n,m\in\mathbb{N}_0}\int_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_a-n\omega)-\rho^{ph}(m)f(\lambda-\mu_a-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda$$

which yields:

$$\sum_{n,m\in\mathbb{N}_0}\int_{\mathbb{R}}\Big(\rho^{ph}(n)f(\lambda-\mu_a-n\omega)-\rho^{ph}(m)f(\lambda-\mu_a-m\omega)\Big)\,\widehat{\sigma}^{ph}_{n_\alpha m_\alpha}(\lambda)\,d\lambda=0.$$

Using that, we immediately obtain the representation (5.7) from (5.6). □

Corollary 5.6. *Let $\mathcal{S}=\{H,H_0\}$ be the JCL-model.*
(i) *If the cases cases (E), (S) or (C) of Proposition 5.3 are realized, then the representation* (5.6) *holds.*
(ii) *If the case (E) of Proposition 5.3 is realized and the system $\mathcal{S}=\{H,H_0\}$ is time reversible symmetric, then*

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{m,n\in\mathbb{N}_0}\frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}}(\rho^{ph}(n)f(\lambda-\mu-n\omega)-\rho^{ph}(m)f(\lambda-\mu-m\omega))\,\widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda)d\lambda \tag{5.13}$$

$n\in\mathbb{N}_0$, $\alpha\in\{l,r\}$ where $\mu:=\mu_l=\mu_r$ and $\alpha\neq\alpha'$.
(iii) *If the case (E) of Proposition 5.3 is realized and the system $\mathcal{S}=\{H,H_0\}$ is time reversible and mirror symmetric, then $J^{ph}_{\rho_0,Q^{el}_\alpha}=0$.*

Proof. (i) The statement follows from Proposition 5.5(i) and Lemma 5.4.

(ii) Setting $\mu_\alpha=\mu_{\alpha'}$ formula (5.13) follows (5.7).

(iii) If $\mathcal{S}=\{H,H_0\}$ is time reversible and mirror symmetric, we obtain from Lemma 2.14 (ii) that $\widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda)=\widehat{\sigma}^{ph}_{n_{\alpha'} m_\alpha}(\lambda)$, $\lambda\in\mathbb{R}$, $n,m\in\mathbb{N}_0$, $\alpha,\alpha'\in\{l,r\}$, $\alpha\neq\alpha'$. Using that, we obtain from (5.13) the following:

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{m,n\in\mathbb{N}_0}\frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}}(\rho^{ph}(n)f(\lambda-\mu-n\omega)-\rho^{ph}(m)f(\lambda-\mu-m\omega))\,\widehat{\sigma}^{ph}_{n_{\alpha'} m_\alpha}(\lambda)d\lambda.$$

Interchanging m and n, we obtain:

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{m,n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} (\rho^{ph}(m)f(\lambda-\mu-m\omega) - \rho^{ph}(n)f(\lambda-\mu-n\omega))\,\widehat{\sigma}^{ph}_{m_{\alpha'}n_\alpha}(\lambda)d\lambda.$$

Using that $\mathcal{S}$ is time reversible symmetric we get from Lemma 2.14 (i) that:

$$J^{ph}_{\rho_0,Q^{el}_\alpha} = -\sum_{m,n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi}\int_{\mathbb{R}} (\rho^{ph}(m)f(\lambda-\mu-m\omega) - \rho^{ph}(n)f(\lambda-\mu-n\omega))\,\widehat{\sigma}^{ph}_{n_\alpha m_{\alpha'}}(\lambda)d\lambda.$$

which shows that $J^{ph}_{\rho_0,Q^{el}_\alpha} = -J^{ph}_{\rho_0,Q^{el}_\alpha}$. Hence $J^{ph}_{\rho_0,Q^{el}_\alpha} = 0$. □

We note that by Proposition 5.3, the contact induced current is zero, i.e. $J^{c}_{\rho_0,Q^{el}_\alpha} = 0$. Hence, if the $\mathcal{S}$ is time reversible and mirror symmetric, then the total current is zero, i.e. $J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha} = 0$.

Remark 5.7. Let the case (E) of Proposition 5.3 be realized, that is, $\mu_l = \mu_r$. Moreover, we assume for simplicity that $0 =: v_r \leqslant v := v_l$.

(i) If $\beta = \infty$, then $\rho^{ph}(n) = \delta_{0n}$, $n \in \mathbb{N}_0$. From (5.6), we immediately obtain that $J^{ph}_{\rho^{el},Q^{el}_\alpha} = 0$. That means, if the temperature is zero, then the photon-induced electron current is zero.

(ii) The photon-induced electron current might be zero even if $\beta < \infty$. Indeed, let $\mathcal{S} = \{H, H_0\}$ be time reversible symmetric and let the case (E) be realized. If $\omega \geqslant v+4$ and $\mathfrak{h}^{el}(\lambda) := \mathfrak{h}^{el}_n(\lambda) = \mathfrak{h}^{el}(\lambda - n\omega)$, $n \in \mathbb{N}_0$. Hence, one always has $n = m$ in formula (5.13), which immediately yields $J^{ph}_{\rho_0,Q^{el}_\alpha} = 0$.

(iii) The photon-induced electron current might be different than zero. Indeed, let $\mathcal{S} = \{H, H_0\}$ be time reversible symmetric and let $v = 2$ and $\omega = 4$, then one sees that to calculate the $J^{ph}_{\rho_0,Q^{el}_l}$, one has to consider $m = n+1$ in formula (5.13). Therefore, we find that:

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi}\times$$

$$\int_{\mathbb{R}} d\lambda \left(\rho^{ph}(n)f(\lambda-\mu-n\omega) - \rho^{ph}(n+1)f(\lambda-\mu-(n+1)\omega)\right)\widehat{\sigma}^{ph}_{n_l\,(n+1)_r}(\lambda).$$

If ρ^{ph} is given by (5.3) and $f(\lambda) = f_{FD}(\lambda)$, cf. (3.21), then one easily verifies that

$$\frac{\partial}{\partial x}\rho^{ph}(x)f_{FD}(\lambda-\mu-x\omega) < 0, \quad x,\mu,\lambda \in \mathbb{R}.$$

Hence, $\rho^{ph}(n)f_{FD}(\lambda-\mu-n\omega)$ is decreasing in $n \in \mathbb{N}_0$ for $\lambda,\mu \in \mathbb{R}$ which yields $\left(\rho^{ph}(n)f(\lambda-\mu-n\omega) - \rho^{ph}(n+1)f(\lambda-\mu-(n+1)\omega)\right) \geqslant 0$. Therefore, $J^{ph}_{\rho_0,Q^{el}_l} \leqslant 0$ which means that the photon-induced current leaves the left-hand side and enters the right-hand side. In fact, $J^{ph}_{\rho_0,Q^{el}_l} = 0$ implies that $\widehat{\sigma}^{ph}_{n_l\,(n+1)_r}(\lambda) = 0$ for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, which means that there is no scattering from the left-hand side to the right one and vice versa which can be excluded generically.

5.2. Photon current

The photon current is related to the charge by equation:

$$Q := Q^{ph} = -I_{\mathfrak{h}^{el}} \otimes \mathfrak{n},$$

where $\mathfrak{n} = \mathrm{d}\Gamma(1) = b^*b$ is the photon number operator on $\mathfrak{h}^{ph} = \mathfrak{F}_+(\mathbb{C})$, which is self-adjoint and commutes with h^{ph}. It follows that Q^{ph} is also self-adjoint and commutes with H_0. It is not bounded, but since $\mathrm{dom}(\mathfrak{n}) = \mathrm{dom}(h^{ph})$, it is immediately obvious that $Q^{ph}(H_0+\theta)^{-1}$ is bounded,

when $\mathfrak{N}$ is a tempered charge. Its charge matrix with respect to the spectral representation $\Pi(H_0^{ac})$ of Lemma 2.12 is given by:

$$Q_{ac}^{ph}(\lambda) = -\bigoplus_{n\in\mathbb{N}_0} nP_n(\lambda).$$

We recall that $P_n(\lambda)$ is the orthogonal projection from $\mathfrak{h}(\lambda)$ onto $\mathfrak{h}_n(\lambda) = \mathfrak{h}^{el}(\lambda - n\omega)$, $\lambda \in \mathbb{R}$. We will now calculate the photon current or, as it is also known, the photon production rate.

5.2.1. Contact induced photon current. The following proposition is, in fact, in accordance with the physical intuition.

Proposition 5.8. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model. Then $J^c_{\rho_0,Q^{ph}} = 0$.*

Proof. We note that $q^{el}_{ac}(\lambda) = I_{\mathfrak{h}^{el}(\lambda)}$, $\lambda \in \mathbb{R}$. Inserting this into (3.25), we obtain $J^{s_c}_{\rho^{el},q^{el}} = 0$. Applying Proposition 4.2 we prove $J^c_{\rho_0,Q^{ph}} = 0$. □

The result reflects the fact that the lead contact does not contribute to the photon current, which is plausible from the physical point of view.

5.2.2. Photon current. From Proposition 5.8, we see that only the photon-induced photon current $J^{ph}_{\rho_0,Q^{ph}}$ contributes to the photon current $J^{\mathcal{S}}_{\rho_0,Q^{ph}}$. Since $J^{\mathcal{S}}_{\rho_0,Q^{ph}} = J^{ph}_{\rho_0,Q^{ph}}$, we call $J^{ph}_{\rho_0,Q^{ph}}$ simply the *photon current*.

Using the notation $\widehat{T}^{ph}_{nm}(\lambda) := P_n(\lambda)\,\widehat{T}_{ph}(\lambda) \restriction \mathfrak{h}^{el}(\lambda - m\omega)$, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}_0$. We set:

$$\widetilde{T}^{ph}_{nm}(\lambda) = \widehat{T}^{ph}_{nm}(\lambda) s_c(\lambda - m\omega), \quad \lambda \in \mathbb{R}, \quad m, n \in \mathbb{N}_0, \tag{5.14}$$

and

$$\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda) := P_{n_\varkappa}(\lambda)\widetilde{T}^{ph}_{nm}(\lambda) \restriction \mathfrak{h}^{el}_\alpha(\lambda - m\omega), \quad \lambda \in \mathbb{R}, \tag{5.15}$$

$m, n \in \mathbb{N}_0$, $\alpha, \varkappa \in \{l, r\}$, as well as $\widetilde{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda) := \mathrm{tr}(\widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda)^* \widetilde{T}^{ph}_{n_\varkappa m_\alpha}(\lambda))$, $\lambda \in \mathbb{R}$.

Proposition 5.9. *Let $\mathcal{S} = \{H, H_0\}$ be the JCL-model.*
(i) *Then:*

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0 \\ \alpha,\varkappa\in\{l,r\}}} (n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega)\widetilde{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda. \tag{5.16}$$

(ii) *If ρ^{el} commutes with s_c, then:*

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0 \\ \alpha,\varkappa\in\{l,r\}}} (n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda - \mu_\alpha - m\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda. \tag{5.17}$$

(iii) *If ρ^{el} commutes with s_c and $\mathcal{S} = \{H, H_0\}$ is time reversible symmetric, then:*

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{\substack{m,n\in\mathbb{N}_0, n>m \\ \varkappa,\alpha\in\{l,r\}}} \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \times \tag{5.18}$$

$$(n-m)\left(\rho^{ph}(m)f(\lambda - \mu_\alpha - m\omega) - \rho^{ph}(n)f(\lambda - \mu_\varkappa - n\omega)\right)\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda),$$

where $\alpha' \in \{l, r\}$ and $\alpha' \neq \alpha$.

Proof. (i) From (4.12) we get

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m,n\in\mathbb{N}_0} n\rho^{ph}(m)\times$$
$$\frac{1}{2\pi}\int_{\mathbb{R}} d\lambda\,\mathrm{tr}\left(\widehat{\rho}^{\,el}_{ac}(\lambda-m\omega)\,\left(P_n(\lambda)\delta_{mn} - \widehat{S}^{ph}_{nm}(\lambda)^* q^{el}_{ac}(\lambda-n\omega)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right).$$

Hence:

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m\in\mathbb{N}_0} m\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\widehat{\rho}^{\,el}_{ac}(\lambda-m\omega)\,\left(P_m(\lambda) - \widehat{S}^{ph}_{mm}(\lambda)^* P_m(\lambda)\,\widehat{S}^{ph}_{mm}(\lambda)\right)\right)d\lambda +$$
$$\sum_{\substack{m,n\in\mathbb{N}_0\\ m\neq n}} n\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\widehat{\rho}^{\,el}_{ac}(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* P_n(\lambda)\,\widehat{S}^{ph}_{nm}(\lambda)\right)d\lambda.$$

Using the relation $P_m(\lambda) = I_{\mathfrak{h}(\lambda)} - \sum_{n\in\mathbb{N}_0, m\neq n} P_n(\lambda)$, $\lambda\in\mathbb{R}$, we obtain the following:

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{\substack{m,n\in\mathbb{N}_0\\ m\neq n}} m\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\widehat{\rho}^{\,el}_{ac}(\lambda-m\omega)\,\left(\widehat{S}^{ph}_{nm}(\lambda)^* P_n(\lambda)\,\widehat{S}^{ph}_{nm}(\lambda)\right)\right)d\lambda +$$
$$\sum_{\substack{m,n\in\mathbb{N}_0\\ m\neq n}} n\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\widehat{\rho}^{\,el}_{ac}(\lambda-m\omega)\,\widehat{S}^{ph}_{nm}(\lambda)^* P_n(\lambda)\,\widehat{S}^{ph}_{nm}(\lambda)\right)d\lambda.$$

Since $\widehat{T}_{ph}(\lambda) = \widehat{S}_{ph}(\lambda) - I_{\mathfrak{h}(\lambda)}$, $\lambda\in\mathbb{R}$, we find

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m,n\in\mathbb{N}_0}(m-n)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\widehat{\rho}^{\,el}_{ac}(\lambda-m\omega)\,\widehat{T}^{ph}_{nm}(\lambda)^*\,\widehat{T}^{ph}_{nm}(\lambda)\right)d\lambda.$$

Using (4.13) and definition (5.14) one readily sees that:

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{m,n\in\mathbb{N}_0}(m-n)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{tr}\left(\rho^{el}_{ac}(\lambda-m\omega)\widetilde{T}^{ph}_{nm}(\lambda)^*\widetilde{T}^{ph}_{nm}(\lambda)\right)d\lambda\,.$$

Since $\rho^{el}_{ac} = \rho^{el}_l \oplus \rho^{el}_r$ where ρ^{el}_α is given by (5.2), we find the following:

$$J^{ph}_{\rho_0,Q^{ph}} = -\sum_{\substack{m,n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}}(m-n)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda-\mu_\alpha-m\omega)\mathrm{tr}\left(\widetilde{T}^{ph}_{n\varkappa m_\alpha}(\lambda)^*\widetilde{T}^{ph}_{n\varkappa m_\alpha}(\lambda)\right)d\lambda,$$

where we have used (5.15). Using $\widetilde{\sigma}^{ph}_{n\varkappa m_\alpha}(\lambda) = \mathrm{tr}(\widetilde{T}^{ph}_{n\varkappa m_\alpha}(\lambda)^*\widetilde{T}^{ph}_{n\varkappa m_\alpha}(\lambda))$, we prove (5.16).

(ii) If ρ^{el}_{ac} commutes with s_c, then $\widehat{\rho}^{\,el}_{ac}(\lambda) = \rho^{el}_{ac}(\lambda)$, $\lambda\in\mathbb{R}$, which yields that one can replace $\widetilde{\sigma}^{ph}_{n\varkappa m_\alpha}(\lambda)$ by $\widehat{\sigma}^{ph}_{n\varkappa m_\alpha}(\lambda)$, $\lambda\in\mathbb{R}$. Therefore, (5.17) holds.

(iii) Clearly, we have:

$$J^{ph}_{\rho_0,Q^{ph}} = \tag{5.19}$$
$$\sum_{\substack{m,n\in\mathbb{N}_0, n>m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{n\varkappa m_\alpha}(\lambda)d\lambda +$$
$$\sum_{\substack{m,n\in\mathbb{N}_0, n<m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{n\varkappa m_\alpha}(\lambda)d\lambda\,.$$

Moreover, a straightforward computation shows the following:

$$\sum_{\substack{m,n\in\mathbb{N}_0,n<m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda=$$

$$\sum_{\substack{m,n\in\mathbb{N}_0,n>m\\ \alpha,\varkappa\in\{l,r\}}}(m-n)\rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\varkappa-n\omega)\,\widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda)d\lambda.$$

Since $\mathcal{S}=\{H,H_0\}$ is time reversible symmetric, we find the following:

$$\sum_{\substack{m,n\in\mathbb{N}_0,n<m\\ \alpha,\varkappa\in\{l,r\}}}(n-m)\rho^{ph}(m)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha-m\omega)\,\widehat{\sigma}^{ph}_{m_\alpha n_\varkappa}(\lambda)d\lambda= \tag{5.20}$$

$$\sum_{\substack{m,n\in\mathbb{N}_0,n>m\\ \alpha,\varkappa\in\{l,r\}}}(m-n)\rho^{ph}(n)\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\varkappa-n\omega)\,\widehat{\sigma}^{ph}_{n_\varkappa m_\alpha}(\lambda)d\lambda.$$

Inserting (5.20) into (5.19), we obtain (5.18). □

Corollary 5.10. *Let $\mathcal{S}=\{H,H_0\}$ be the JCL-model and let $f=f_{FD}$. If case (E) of Proposition 5.3 is realized and $\mathcal{S}=\{H,H_0\}$ is time reversible symmetric, then $J^{ph}_{\rho_0,Q^{ph}}\geqslant 0$.*

Proof. We set $\mu:=\mu_l=\mu_r$. One has:

$$\rho^{ph}(m)f(\lambda-\mu-m\omega)-\rho^{ph}(n)f(\lambda-\mu-n\omega)=$$
$$e^{-m\beta\omega}(1-e^{-(n-m)\beta\omega})f_{FD}(\lambda-\mu-m\omega)f_{FD}(\lambda-\mu-n\omega)\geqslant 0,$$

for $n>m$. From (5.18), we see that $J^{ph}_{\rho_0,Q^{ph}}\geqslant 0$. □

Remark 5.11. We will now comment the results. If $J^{ph}_{\rho_0,Q^{ph}}\geqslant 0$, then system $\mathcal{S}$ is called light emitting. Similarly, if $J^{ph}_{\rho_0,Q^{ph}}\leqslant 0$, then we call it light absorbing. Of course if $\mathcal{S}$ is light emitting and absorbing, then $J^{ph}_{\rho_0,Q^{ph}}=0$.

(i) If $\beta=\infty$, then $\rho^{ph}(m)=\delta_{0m}$, $m\in\mathbb{N}_0$. Inserting this into (5.16), we get:

$$J^{ph}_{\rho_0,Q^{ph}}=\sum_{\substack{n\in\mathbb{N}_0\\ \alpha,\varkappa\in\{l,r\}}}n\frac{1}{2\pi}\int_{\mathbb{R}}f(\lambda-\mu_\alpha)\widetilde{\sigma}^{ph}_{n_\varkappa 0_\alpha}(\lambda)d\lambda\geqslant 0$$

Hence, the system $\mathcal{S}$ is light emitting.

(ii) Let us show $\mathcal{S}$ might be light emitting even if $\beta<\infty$. We consider the case (E) of Proposition 5.3. If $\mathcal{S}$ is time reversible symmetric, then it follows from Corollary 5.10 that the system is light emitting.

If the system $\mathcal{S}$ is time reversible and mirror symmetric, then $J^{ph}_{\rho_0,Q^{el}_\alpha}=0$, $\alpha\in\{l,r\}$, by Corollary 5.6(iii) . Since $J^{c}_{\rho_0,Q^{el}}=0$ by Proposition 5.3, we get that $J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha}=0$ but the photon current is larger than zero. So our JCL-model is light emitting by a zero total electron current $J^{\mathcal{S}}_{\rho_0,Q^{el}_\alpha}$.

Let $v_r=0$, $v_l=2$ and $\omega=4$. Hence $\mathcal{S}$ is not mirror symmetric. Then, we get from Remark 5.7(iii) that $J^{ph}_{\rho_0,Q^{el}_l}=-J^{ph}_{\rho_0,Q^{el}_r}\leqslant 0$. Hence, there is an electron current from the left to the right lead. Notice that by Proposition 5.3 $J^{c}_{\rho_0,Q^{el}_l}=0$. Hence, $J^{\mathcal{S}}_{\rho_0,Q^{el}_l}\leqslant 0$.

(iii) To realize a light absorbing situation, we consider the case (S) of Proposition 5.3 and assume that $\mathcal{S}$ is time reversible symmetric. Notice that by Lemma 5.4, s_c commutes with ρ^{el}. We make the following choices:

$$v_r = 0, \quad v_l \geqslant 4, \quad \omega = v_l, \quad \mu_l = 0, \quad \mu_r = \omega = v_l.$$

It follows out that with respect to the representation (5.18) one has only to $m = n - 1$, $\varkappa = r$ and $\alpha = l$. Hence,

$$J^{ph}_{\rho_0,Q^{ph}} = \sum_{n\in\mathbb{N}} \frac{1}{2\pi} \times \int_{\mathbb{R}} d\lambda \left(\rho^{ph}(n-1) f(\lambda - (n-1)\omega) - \rho^{ph}(n) f(\lambda - (n+1)\omega)\right) \widehat{\sigma}^{ph}_{n_l(n-1)_r}(\lambda)$$

Since, $f(\lambda) = f_{FD}(\lambda)$, we find:

$$\begin{aligned} &\rho^{ph}(n-1) f(\lambda - (n-1)\omega) - \rho^{ph}(n) f(\lambda - (n+1)\omega) = \\ &\quad \rho^{ph}(n-1) f(\lambda - (n-1)\omega) f(\lambda - (n+1)\omega) \times \\ &\quad \left(1 + e^{\beta(\lambda - (n+1)\omega)} - e^{-\beta\omega}(1 + e^{\beta(\lambda - \omega(n-1))})\right), \end{aligned}$$

or

$$\begin{aligned} &\rho^{ph}(n-1) f(\lambda - (n-1)\omega) - \rho^{ph}(n) f(\lambda - (n+1)\omega) = \\ &\quad \rho^{ph}(n-1) f(\lambda - (n-1)\omega) f(\lambda - (n+1)\omega)(1 - e^{-\beta\omega})(1 - e^{\beta(\lambda - \omega n)}). \end{aligned}$$

Since $\lambda - n\omega \geqslant 0$ we find $\rho^{ph}(n-1) f(\lambda - (n-1)\omega) - \rho^{ph}(n) f(\lambda - (n+1)\omega) \leqslant 0$ which yields $J^{ph}_{\rho_0,Q^{ph}} \leqslant 0$.

To calculate $J^{ph}_{\rho_0,Q^{el}_l}$, we use formula (5.7). Setting $\alpha = l$, we obtain $\alpha' = r$, which yields

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{m,n\in\mathbb{N}_0} \frac{\mathfrak{e}}{2\pi} \times \int_{\mathbb{R}} d\lambda \left(\rho^{ph}(n) f(\lambda - \mu_r - n\omega) - \rho^{ph}(m) f(\lambda - \mu_l - m\omega)\right) \widehat{\sigma}^{ph}_{n_l m_r}(\lambda)\,.$$

One verifies that $\widehat{\sigma}^{ph}_{0_l 0_r}(\lambda) = 0$ and $\widehat{\sigma}^{ph}_{n_l m_r}(\lambda) = 0$ for $m \neq n+1$, $n \in \mathbb{N}$. Hence,

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{n\in\mathbb{N}} \frac{\mathfrak{e}}{2\pi} \times \int_{\mathbb{R}} d\lambda \left(\rho^{ph}(n) f(\lambda - \mu_r - n\omega) - \rho^{ph}(n-1) f(\lambda - \mu_l - (n+1)\omega)\right) \widehat{\sigma}^{ph}_{n_l(n+1)_r}(\lambda)\,,$$

Since $\mu_r = \omega$ and $\mu_l = 0$, we find:

$$J^{ph}_{\rho_0,Q^{el}_l} = -\sum_{n\in\mathbb{N}} \frac{\mathfrak{e}}{2\pi} \times \int_{\mathbb{R}} f(\lambda - (n+1)\omega) \rho^{ph}(n-1)(1 - e^{-\beta\omega})\, \widehat{\sigma}^{ph}_{n_l(n+1)_r}(\lambda)\, d\lambda \leqslant 0.$$

Hence, there is electron current flowing from the left to right induced by photons. We recall that $J^{c}_{\rho_0,Q^{el}_l} = 0$.

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