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ON THE ROBIN EIGENVALUES OF THE LAPLACIAN IN THE EXTERIOR OF A CONVEX POLYGON

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Let $\Omega\subset\mathbb{R}^2$ be the exterior of a convex polygon whose side lengths are $\ell_1,\dots,\ell_M$. For a real constant α, let H_α^Ω denote the Laplacian in Ω, $u\mapsto-\Delta u$, with the Robin boundary conditions $\partial u/\partial\nu=\alpha u$ at $\partial\Omega$, where ν is the outer unit normal. We show that, for any fixed $m\in\mathbb{N}$, the mth eigenvalue $E_m^\Omega(\alpha)$ of H_α^Ω behaves as $E_m^\Omega(\alpha)=-\alpha^2+\mu_m^D+\mathcal{O}(\alpha^{-1/2})$ as $\alpha\to+\infty$, where μ_m^D stands for the mth eigenvalue of the operator $D_1\oplus\dots\oplus D_M$ and D_n denotes the one-dimensional Laplacian $f\mapsto -f''$ on $(0,\ell_n)$ with the Dirichlet boundary conditions.

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1. Introduction

1.1. Laplacian with Robin boundary conditions

Let $\Omega\subset\mathbb{R}^d$, $d\ge2$, be a connected domain with a compact Lipschitz boundary $\partial\Omega$. For $\alpha>0$, let H_α^Ω denote the Laplacian $u\mapsto-\Delta u$ in Ω with the Robin boundary conditions $\partial u/\partial\nu=\alpha u$ at $\partial\Omega$, where ν stands for the outer unit normal. More precisely, H_α^Ω is the self-adjoint operator in $L^2(\Omega)$ generated by the sesquilinear form:

$$h_\alpha^\Omega(u,u)=\iint\limits_\Omega|\nabla u|^2\,\mathrm{d}x-\alpha\int\limits_{\partial\Omega}|u|^2\,\mathrm{d}\sigma,\quad \mathcal{D}(h_\alpha^\Omega)=W^{1,2}(\Omega).$$

Here and below, σ denotes the $(d-1)$-dimensional Hausdorff measure.

One checks in the standard way that the operator H_α^Ω is semibounded from below. If Ω is bounded (i.e. Ω is an interior domain), then it has a compact resolvent, and we denote by $E_m^\Omega(\beta)$, $m\in\mathbb{N}$, its eigenvalues taken according to their multiplicities and enumerated in the non-decreasing order. If Ω is unbounded (i.e. Ω is an exterior domain), then the essential spectrum of H_α^Ω coincides with $[0,+\infty)$, and the discrete spectrum consists of finitely many eigenvalues which will be denoted again by $E_m^\Omega(\alpha)$, $m\in\{1,\dots,K_\alpha\}$, and enumerated in the non-decreasing order taking into account the multiplicities.

We are interested in the behavior of the eigenvalues $E_m^\Omega(\alpha)$ for large α. It seems that the problem was introduced by Lacey, Ockedon, Sabina [11] when studying a reaction-diffusion system. Giorgi and Smits [6] studied a link to the theory of enhanced surface superconductivity. Recently, Freitas and Krejčiřík [10] and then Pankrashkin and Popoff [15] studied the eigenvalue asymptotics in the context of the spectral optimization.

Let us list some available results. Under various assumptions one showed the asymptotics of the form:

$$E_m^\Omega(\alpha) = -C_\Omega \alpha^2 + o(\alpha^2) \text{ as } \alpha \text{ tends to } +\infty, \tag{1}$$

where $C_\Omega \geq 1$ is a constant depending on the geometric properties of Ω. Lacey, Ockedon, Sabina [11] showed (1) with $m = 1$ for C^4 compact domains, for which $C_\Omega = 1$, and for triangles, for which $C_\Omega = 2/(1 - \cos\theta)$, where θ is the smallest corner. Lu and Zhu [13] showed (1) with $m = 1$ and $C_\Omega = 1$ for compact C^1 smooth domains, and Daners and Kennedy [2] extended the result to any fixed $m \in \mathbb{N}$. Levitin and Parnovski [12] showed (1) with $m = 1$ for domains with piecewise smooth compact Lipschitz boundaries. They proved, in particular, that if Ω is a curvilinear polygon whose smallest corner is θ, then for $\theta < \pi$ there holds $C_\Omega = 2/(1 - \cos\theta)$, otherwise $C_\Omega = 1$. Pankrashkin [14] considered two-dimensional domains with a piecewise C^4 smooth compact boundary and without convex corners, and it was shown that $E_1^\Omega(\alpha) = -\alpha^2 - \gamma\alpha + \mathcal{O}(\alpha^{2/3})$, where γ is the maximum of the signed curvature at the boundary. Exner, Minakov, Parnovski [4] showed that for compact C^4 smooth domains the same asymptotics $E_m^\Omega(\alpha) = -\alpha^2 - \gamma\alpha + \mathcal{O}(\alpha^{2/3})$ holds for any fixed $m \in \mathbb{N}$. Similar results were obtained by Exner and Minakov [3] for a class of two-dimensional domains with non-compact boundaries and by Pankrashkin and Popoff [15] for C^3 compact domains in arbitrary dimensions. Cakoni, Chaulet, Haddar [1] studied the asymptotic behavior of higher eigenvalues.

1.2. Problem setting and the main result

The computation of further terms in the eigenvalue asymptotics needs more precise geometric assumptions. To our knowledge, such results are available for the two-dimensional case only. Helffer and Pankrashkin [9] studied the tunneling effect for the eigenvalues of a specific domain with two equal corners, and Helffer and Kachmar [8] considered the domains whose boundary curvature has a unique non-degenerate maximum. The machinery of both papers is based on the asymptotic properties of the eigenfunctions: it was shown that the eigenfunctions corresponding to the lowest eigenvalues concentrate near the smallest convex corner at the boundary or, if no convex corners are present, near the point of the maximum curvature, and this is used to obtain the corresponding eigenvalue asymptotics.

The aim of the present note is to consider a new class of two-dimensional domains Ω. Namely, our assumption is as follows:

The domain $\mathbb{R}^2 \setminus \overline{\Omega}$ is a convex polygon (with straight edges).

Such domains are not covered by the above cited works: all the corners are non-convex, and the curvature is constant on the smooth part of the boundary, and it is not clear how the eigenfunctions are concentrated along the boundary. We hope that our result will be of use for the understanding of the role of non-convex corners.

In order to formulate the main result, we first need some notation. We denote the vertices of the polygon $\mathbb{R}^2 \setminus \overline{\Omega}$ by $A_1, \dots, A_M \in \mathbb{R}^2$, $M \geq 3$, and assume that they are enumerated is such a way that the boundary $\partial\Omega$ is the union of the M line segments $L_n := [A_n, A_{n+1}]$, $n \in \{1, \dots, M\}$, where we denote $A_{M+1} := A_1$, $A_0 := A_M$. It is also assumed that there are no artificial vertices, i.e. that $A_n \notin [A_{n-1}, A_{n+1}]$ for any $n \in \{1, \dots, M\}$.

Furthermore, we denote by ℓ_n the length of the side L_n, and by D_n the Dirichlet Laplacian $f \mapsto -f''$ on $(0, \ell_n)$ viewed as a self-adjoint operator in $L^2(0, \ell_n)$. The main result of the present note is as follows:

Theorem 1. *For any fixed $m \in \mathbb{N}$ there holds:*

$$E_m^\Omega(\alpha) = -\alpha^2 + \mu_m^D + \mathcal{O}\Big(\frac{1}{\sqrt{\alpha}}\Big) \text{ as } \alpha \text{ tends to } +\infty,$$

where μ_m^D is the mth eigenvalue of the operator $D_1 \oplus \dots \oplus D_M$.

The proof is based on the machinery proposed by Exner and Post [5] to study the convergence on graph-like manifolds. In reality, our construction appears to be quite similar to that of Post [16], which was used to study decoupled waveguides.

We remark that due to the presence of non-convex corners the domain of the operator H_α^Ω contains singular functions and is not included in $W^{2,2}(\Omega)$, see e.g. Grisvard [7]. This does not produce any difficulties, as our approach is purely variational and is entirely based on analysis of the sesqulinear form.

2. Preliminaries

2.1. Auxiliary operators

For $\alpha > 0$, we denote by T_α the following self-adjoint operator in $L^2(\mathbb{R}_+)$:

$$T_\alpha v = -v'', \quad \mathcal{D}(T_\alpha) = \big\{v \in W^{2,2}(\mathbb{R}_+) : v'(0) + \alpha v(0) = 0\big\}.$$

It is well known that:

$$\operatorname{spec} T_\alpha = \{-\alpha^2\} \cup [0, +\infty), \quad \ker(T + \alpha^2) = \mathbb{C}\varphi_\alpha, \quad \varphi_\alpha(s) := \frac{e^{-\alpha s}}{\sqrt{2\alpha}}. \tag{2}$$

The sesqulinear form t_α for the operator T_α looks as follows:

$$t_\alpha(v, v) = \int_0^\infty \big|v'(s)\big|^2 \mathrm{d}s - \alpha\big|v(0)\big|^2, \quad \mathcal{D}(t_\alpha) = W^{1,2}(\mathbb{R}_+).$$

Lemma 2. *For any $v \in W^{1,2}(\mathbb{R}_+)$, there holds:*

$$\int_0^\infty \big|v(s)\big|^2 \mathrm{d}s - \Big|\int_0^\infty \varphi_\alpha(s) v(s) \mathrm{d}s\Big|^2 \le \frac{1}{\alpha^2}\Big(\int_0^\infty \big|v'(s)\big|^2 \mathrm{d}s - \alpha\big|v(0)\big|^2 + \alpha^2 \int_0^\infty \big|v(s)\big|^2 \mathrm{d}s\Big).$$

Proof. We denote by P the orthogonal projector on $\ker(T_\alpha + \alpha^2)$ in $L^2(\mathbb{R}_+)$, then by the spectral theorem, we have:

$$t_\alpha(v, v) + \alpha^2 \|Pv\|^2 = t_\alpha(v - Pv, v - Pv) \ge 0,$$

for any $v \in \mathcal{D}(t_\alpha)$. As φ_α is normalized, there holds

$$\Big|\int_0^\infty \varphi_\alpha(x) v(x) \mathrm{d}x\Big| = \|Pv\|,$$

and we arrive at the conclusion. □

Another important estimate is as follows, see Lemmas 2.6 and 2.8 in [12]:

Lemma 3. *Let $\Lambda \subset \mathbb{R}^2$ be an infinite sector of opening $\theta \in (0, 2\pi)$, then for any $\varepsilon > 0$ and any function $v \in W^{1,2}(\Lambda)$ there holds:*

$$\int_{\partial\Lambda} |v|^2 \mathrm{d}s \le \varepsilon \iint_\Lambda |\nabla v|^2 \mathrm{d}x + \frac{C_\theta}{\varepsilon} \iint_\Lambda |v|^2 \mathrm{d}x \quad \text{with} \quad C_\theta = \begin{cases} \dfrac{2}{1 - \cos\theta}, & \theta \in (0, \pi), \\ 1, & \theta \in [\pi, 2\pi). \end{cases} \tag{3}$$

2.2. Decomposition of Ω

Let us proceed with a decomposition of the domain Ω which will be used through the proof. Let $n \in \{1,\dots,M\}$. We denote by S_n^1 and S_n^2 the half-lines originating respectively at A_n and A_{n+1}, orthogonal to L_n and contained in Ω. By Π_n, we denote the half-strip bounded by the half-lines S_n^1 and S_n^2 and the line segment L_n, and by Λ_n we denote the infinite sector bounded by the lines S_{n-1}^2 and S_n^1 and contained in Ω. The constructions are illustrated in Fig. 1. We note that the $2M$ sets Λ_n and Π_n, $n \in \{1,\dots,M\}$, are non-intersecting and that $\overline{\Omega} = \bigcup_{n=1}^{M} \overline{\Lambda}_n \cup \bigcup_{n=1}^{M} \overline{\Pi}_n$.

We deduce from Lemma 3:

Lemma 4. *There exists a constant $C > 0$ such that for any $\varepsilon > 0$, any $n \in \{1,\dots,M\}$ and any $v \in W^{1,2}(\Lambda_n)$ there holds*

$$\int\limits_{\partial\Lambda_n} |v|^2 \mathrm{d}\sigma \le C\varepsilon\Big(\iint\limits_{\Lambda_n} |\nabla v|^2 \mathrm{d}x + \frac{1}{\varepsilon^2} \iint\limits_{\Lambda_n} |v|^2 \mathrm{d}x \Big).$$

Furthermore, for each $n \in \{1,\dots,M\}$ we denote by Θ_n the uniquely defined isometry $\mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$A_n = \Theta_n(0,0) \quad \text{and} \quad \Pi_n = \Theta_n\big((0,\ell_n) \times \mathbb{R}_+\big) :$$

We remark that due to the spectral properties of the above operator T_α, see (2), we have, for any $u \in W^{1,2}(\Pi_n)$,

$$\int_0^{\ell_n}\int_0^{\infty} \Big|\frac{\partial}{\partial s} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t - \alpha \int_0^{\ell_n} \Big|u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}t + \alpha^2 \int_0^{\ell_n}\int_0^{\infty} \Big|u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t$$
$$= \int_0^{\ell_n} \Big(\int_0^{\infty} \Big|\frac{\partial}{\partial s} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s - \alpha \Big|u\big(\Theta_n(t,0)\big)\Big|^2 + \alpha \int_0^{\infty} \Big|u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s \Big) \mathrm{d}t \ge 0,$$

which implies, in particular, the following:

$$\begin{aligned} 0 \le \int_0^{\ell_n}\int_0^{\infty} &\Big|\frac{\partial}{\partial t} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t \\ &\le \int_0^{\ell_n}\int_0^{\infty} \Big|\frac{\partial}{\partial t} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t + \int_0^{\ell_n}\int_0^{\infty} \Big|\frac{\partial}{\partial s} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t \\ &\quad - \alpha \int_0^{\ell_n} \Big|u\big(\Theta_n(t,0)\big)\Big|^2 \mathrm{d}t + \alpha^2 \int_0^{\ell_n}\int_0^{\infty} \Big|u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t \\ &= \iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \alpha \int\limits_{L_n} |u|^2 \mathrm{d}\sigma + \alpha^2 \iint\limits_{\Pi_n} |u|^2 \mathrm{d}x. \end{aligned} \tag{4}$$

2.3. Eigenvalues and identification maps

We will use an eigenvalue estimate which is based on the max-min principle and is just a suitable reformulation of Lemma 2.1 in [5] or of Lemma 2.2 in [16]:

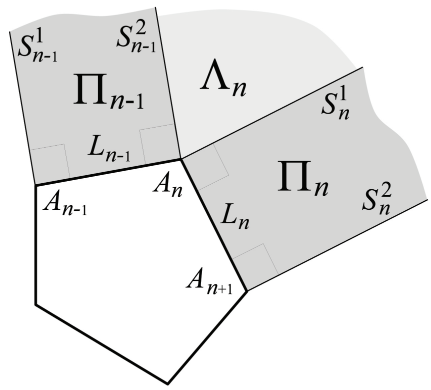

FIG. 1. Decomposition of the domain

Proposition 5. *Let B and B' be non-negative self-adjoint operators acting respectively in Hilbert space $\mathcal{H}$ and $\mathcal{H}'$ and generated by sesqulinear form b and b'. Choose $m \in \mathbb{N}$ and assume that the operator B has at least m eigenvalues $\lambda_1 \le \cdots \le \lambda_m < \inf \operatorname{spec}_{\text{ess}} B$ and that the operator B' has a compact resolvent. If there exists a linear map $J : \mathcal{D}(b) \to \mathcal{D}(b')$ (identification map) and two constants $\delta_1, \delta_2 > 0$ such that $\delta_1 \le (1+\lambda_m)^{-1}$, and that for any $u \in \mathcal{D}(b)$ there holds:*

$$\|u\|^2 - \|Ju\|^2 \le \delta_1\Big(b(u,u) + \|u\|^2\Big),$$
$$b'(Ju,Ju) - b(u,u) \le \delta_2\Big(b(u,u) + \|u\|^2\Big),$$

then

$$\lambda'_m \le \lambda_m + \frac{(\lambda_m\delta_1 + \delta_2)(1+\lambda_m)}{1-(1+\lambda_m)\delta_1},$$

where λ'_m is the mth eigenvalue of the operator B'.

3. Proof of Theorem 1

3.1. Dirichlet-Neumann bracketing

Consider the following sesqulinear form:

$$h^{\Omega,D}_\alpha(u,u) = \sum_{n=1}^M \iint\limits_{\Lambda_n} |\nabla u|^2 \mathrm{d}x + \sum_{n=1}^M \Bigg(\iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \alpha \int\limits_{L_n} |u|^2 \mathrm{d}\sigma \Bigg),$$
$$\mathcal{D}(h^{\Omega,D}_\alpha) = \bigoplus_{n=1}^M W^{1,2}_0(\Lambda_n) \oplus \bigoplus_{n=1}^M \widetilde{W}^{1,2}_0(\Pi_n),$$
$$\widetilde{W}^{1,2}_0(\Pi_n) := \big\{ f \in W^{1,2}(\Pi_n) : f = 0 \text{ at } S^1_n \cup S^2_n \big\},$$

and denote by $H^{\Omega,D}_\alpha$ the associated self-adjoint operator in $L^2(\Omega)$. Clearly, the form $h^{\Omega,D}_\alpha$ is a restriction of the initial form h^Ω_α, and due to the max-min principle we have:

$$E^\Omega_m(\alpha) \le E^{\Omega,D}_m(\alpha),$$

where $E_m^{\Omega,D}(\alpha)$ is the mth eigenvalue of $H_\alpha^{\Omega,D}$ (as soon at it exists). On the other hand, we have the decomposition:

$$H_\alpha^{\Omega,D} = \bigoplus_{n=1}^{M}\big(-\Delta_n^D\big) \oplus \bigoplus_{n=1}^{M} G_{n,\alpha}^D,$$

where $(-\Delta_n^D)$ is the Dirichlet Laplacian in $L^2(\Lambda_n)$ and $G_{n,\alpha}^D$ is the self-adjoint operator in $L^2(\Pi_n)$ generated by the sesquilinear form:

$$g_{n,\alpha}^D(u,u) = \iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \alpha \int\limits_{L_n} |u|^2 \mathrm{d}\sigma, \quad \mathcal{D}(g_{n,\alpha}^D) = \widetilde{W}_0^{1,2}(\Pi_n).$$

Consider the following unitary maps:

$$U_n : L^2(\Pi_n) \to L^2\big((0,\ell_n)\times\mathbb{R}_+\big), \quad U_n f := f\circ\Theta_n, \quad n\in\{1,\dots,M\},$$

then it is straightforward to check that $U_n G_{n,\alpha}^D U_n^* = D_n\otimes 1 + T_\alpha\otimes 1$. As the operators $(-\Delta_n^D)$ are non-negative, it follows that $\operatorname{spec}_{\mathrm{ess}} H_\alpha^{\Omega,D} = [0,+\infty)$ and that $E_m^{\Omega,D}(\alpha) = -\alpha^2 + \mu_m^D$, which gives the majoration:

$$E_m^\Omega(\alpha) \le -\alpha^2 + \mu_m^D, \tag{5}$$

for all m with $\mu_m^D < \alpha^2$. In particular, the inequality (5) holds for any fixed m as α tends to $+\infty$.

Similarly, we introduce the following sesquilinear form:

$$h_\alpha^{\Omega,N}(u,u) = \sum_{n=1}^{M} \iint\limits_{\Lambda_n} |\nabla u|^2 \mathrm{d}x + \sum_{n=1}^{M}\Bigg(\iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \alpha\int\limits_{L_n} |u|^2\mathrm{d}\sigma\Bigg),$$

$$\mathcal{D}(h_\alpha^{\Omega,N}) = \bigoplus_{n=1}^{M} W^{1,2}(\Lambda_n) \oplus \bigoplus_{n=1}^{M} W^{1,2}(\Pi_n),$$

and denote by $H_\alpha^{\Omega,N}$ the associated self-adjoint operator in $L^2(\Omega)$. Clearly, the initial form h_α^Ω is a restriction of the form $h_\alpha^{N,\Omega}$, and due to the max-min principle we have:

$$E_m^{\Omega,N}(\alpha) \le E_m^\Omega(\alpha),$$

where $E_m^{\Omega,N}(\alpha)$ is the mth eigenvalue of $H_\alpha^{\Omega,N}$, and the inequality holds for those m for which $E_m^\Omega(\alpha)$ exists. On the other hand, we have the decomposition:

$$H_\alpha^{\Omega,N} = \bigoplus_{n=1}^{M}\big(-\Delta_n^N\big) \oplus \bigoplus_{n=1}^{M} G_{n,\alpha}^N,$$

where $(-\Delta_n^N)$ denotes the Neumann Laplacian in $L^2(\Lambda_n)$ and $G_{n,\alpha}^N$ is the self-adjoint operator in $L^2(\Pi_n)$ generated by the sesquilinear form

$$g_{n,\alpha}^N(u,u) = \iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \alpha \int\limits_{L_n} |u|^2 \mathrm{d}\sigma, \quad \mathcal{D}(g_{n,\alpha}^N) = W^{1,2}(\Pi_n).$$

There holds $U_n G_{n,\alpha}^N U_n^* = N_n \otimes 1 + T_\alpha \otimes 1$, where N_n is the operator $f \mapsto -f''$ on $(0,\ell_n)$ with the Neumann boundary condition viewed as a self-adjoint operator in the Hilbert space $L^2(0,\ell_n)$, $n\in\{1,\dots,M\}$. The operators $(-\Delta_n^N)$ are non-negative, and we have $\operatorname{spec}_{\mathrm{ess}} H_\alpha^{\Omega,N} = [0,+\infty)$ and $E_m^{\Omega,N}(\alpha) = -\alpha^2 + \mu_m^N$, where μ_m^N is the mth eigenvalue of the operator $N_1\oplus\dots\oplus N_M$. Thus, we obtain the minorations:

$$H_\alpha^\Omega \ge -\alpha^2 \text{ and } E_m^\Omega(\alpha) \ge -\alpha^2 + \mu_m^N, \tag{6}$$

which holds for any fixed m as α tends to $+\infty$. By combining the inequalities (5) and (6) we also obtain the rough estimate:

$$E_m^\Omega(\alpha) = -\alpha^2 + \mathcal{O}(1) \quad \text{for any fixed } m \text{ and for } \alpha \text{ tending to } +\infty. \tag{7}$$

3.2. Construction of an identification map

In order to conclude the proof of Theorem 1, we will apply Proposition 5 to the operators:

$$B = H_\alpha^\Omega + \alpha^2, \quad B' = D_1 \oplus \cdots \oplus D_n,$$

which will allow us to obtain another inequality between the quantities:

$$\lambda_m = E_m^\Omega(\alpha) + \alpha^2, \quad \lambda'_m = \mu_m^D.$$

Note that for any fixed $m \in \mathbb{N}$, one has $\lambda_m = \mathcal{O}(1)$ for large α, see (7). Therefore, it is sufficient to construct an identification map $J = J_\alpha$ as in Proposition 5 with $\delta_1 + \delta_2 = \mathcal{O}(\alpha^{-1/2})$. Recall that the respective forms b and b' in our case are given by:

$$b(u,u) = h_\alpha^\Omega(u,u) + \alpha^2 \|u\|^2, \quad \mathcal{D}(b) = \mathcal{D}(h_\alpha^\Omega) = W^{1,2}(\Omega),$$

$$b'(f,f) = \sum_{n=1}^M \int_0^{\ell_n} \big|f'_n(t)\big|^2 \mathrm{d}t, \quad \mathcal{D}(b') = \big\{f = (f_1, \dots, f_M) : f_n \in W_0^{1,2}(0,\ell_n)\big\}.$$

Here and below, by $\|u\|$ we mean the usual norm in $L^2(\Omega)$. The positivity of b' is obvious, and the positivity of b follows from (6).

Consider the maps:

$$P_{n,\alpha} : W^{1,2}(\Pi_n) \to L^2(0,\ell_n), \quad (P_{n,\alpha}u)(t) = \int_0^\infty \varphi_\alpha(s) u\big(\Theta_n(t,s)\big) \mathrm{d}s, \quad n \in \{1, \dots, M\}.$$

If $u \in W^{1,2}(\Omega)$, then $u \in W^{1,2}(\Pi_n)$ for any $n \in \{1, \dots, M\}$, and one can estimate, using the Cauchy-Schwarz inequality:

$$\begin{aligned}\big|(P_{n,\alpha}u)(0)\big|^2 + \big|(P_{n,\alpha}u)(\ell_n)\big|^2 &\le \int_0^\infty \big|u\big(\Theta_n(0,s)\big)\big|^2 \mathrm{d}s + \int_0^\infty \big|u\big(\Theta_n(\ell_n,s)\big)\big|^2 \mathrm{d}s \\ &= \int_{S_n^1} |u|^2 \mathrm{d}\sigma + \int_{S_n^2} |u|^2 \mathrm{d}\sigma.\end{aligned}$$

As $S_{n-1}^2 \cup S_n^1 = \partial\Lambda_n$, we can use Lemma 4 with $\varepsilon = \alpha^{-1}$, which gives:

$$\begin{aligned}\sum_{n=1}^M \Big(\big|(P_{n,\alpha}u)(0)\big|^2 + \big|(P_{n,\alpha}u)(\ell_n)\big|^2\Big) &\le \sum_{n=1}^M \Big(\int_{S_n^1} |u|^2 \mathrm{d}\sigma + \int_{S_n^2} |u|^2 \mathrm{d}\sigma\Big) \\ &= \sum_{n=1}^M \int_{\partial\Lambda_n} |u|^2 \mathrm{d}\sigma \le \frac{C}{\alpha} \sum_{n=1}^M \Big(\iint_{\Lambda_n} |\nabla u|^2 \mathrm{d}x + \alpha^2 \iint_{\Lambda_n} |u|^2 \mathrm{d}x\Big).\end{aligned} \tag{8}$$

For each $n \in \{1, \dots, M\}$, we introduce a map:

$$\pi_n : (0,\ell_n) \to \{0,\ell_n\}, \quad \pi_n(t) = 0 \text{ for } t < \frac{\ell_n}{2}, \quad \pi_n(t) = \ell_n \text{ otherwise},$$

and choose a function: $\rho_n \in C^\infty\big([0,\ell_n]\big)$ with $\rho_n(0)=\rho_n(\ell_n)=1$ and $\rho_n\Big(\dfrac{\ell_n}{2}\Big)=0$.

Finally, we define:

$$J_\alpha : W^{1,2}(\Omega) \to \bigoplus_{n=1}^{M} L^2(0,\ell_n), \quad (J_\alpha u)_n(t) = (P_{n,\alpha}u)(t) - (P_{n,\alpha}u)\big(\pi_n(t)\big)\rho_n(t).$$

We remark that $(J_\alpha u)_n \in W_0^{1,2}(0,\ell_n)$ for any $u \in W^{1,2}(\Omega)$ and $n \in \{1,\dots,M\}$, i.e. J_α maps $\mathcal{D}(b)$ into $\mathcal{D}(b')$ and will be used as an identification map.

3.3. Estimates for the identification map

Take any $\delta > 0$. Using the following inequality:

$$(a_1+a_2)^2 \ge (1-\delta)a_1^2 - \frac{1}{\delta}\,a_2^2, \quad a_1, a_2 \ge 0,$$

we estimate:

$$\|u\|^2 - \|J_\alpha u\|^2 = \sum_{n=1}^{M} \iint\limits_{\Lambda_n} |u|^2 \mathrm{d}x + \sum_{n=1}^{M} \Big(\iint\limits_{\Pi_n} |u|^2 \mathrm{d}x - \int_0^{\ell_n} \Big|\big(P_{n,\alpha}u\big)(t) - \big(P_{n,\alpha}u\big)\big(\pi(t)\big)\rho(t)\Big|^2 \mathrm{d}t \Big)$$

$$\le \sum_{n=1}^{M} \iint\limits_{\Lambda_n} |u|^2 \mathrm{d}x + \sum_{n=1}^{M} \Big(\iint\limits_{\Pi_n} |u|^2 \mathrm{d}x - (1-\delta)\int_0^{\ell_n} \big|\big(P_{n,\alpha}u\big)(t)\big|^2 \mathrm{d}t + \frac{1}{\delta}\int_0^{\ell_n} \big|\big(P_{n,\alpha}u\big)\big(\pi(t)\big)\rho(t)\big|^2 \mathrm{d}t \Big)$$

$$= \sum_{n=1}^{M} \iint\limits_{\Lambda_n} |u|^2 \mathrm{d}x + \sum_{n=1}^{M} \Big(\iint\limits_{\Pi_n} |u|^2 \mathrm{d}x - \int_0^{\ell_n} \big|\big(P_{n,\alpha}u\big)(t)\big|^2 \mathrm{d}t \Big)$$

$$+\delta \sum_{n=1}^{M} \int_0^{\ell_n} \big|\big(P_{n,\alpha}u\big)(t)\big|^2 \mathrm{d}t + \frac{1}{\delta} \sum_{n=1}^{M} \int_0^{\ell_n} \Big|\big(P_{n,\alpha}u\big)\big(\pi(t)\big)\rho_n(t)\Big|^2 \mathrm{d}t$$

$$=: I_1 + I_2 + I_3 + I_4.$$

We have the trivial inequality:

$$I_1 \le \frac{1}{\alpha^2} \sum_{n=1}^{M} \Big(\iint\limits_{\Lambda_n} |\nabla u|^2 \mathrm{d}x + \alpha^2 \iint\limits_{\Lambda_n} |u|^2 \mathrm{d}x \Big).$$

To estimate the term I_2, we use Lemma 2 and then (4):

$$I_2 = \sum_{n=1}^{M} \int_0^{\ell_n} \Big(\int_0^{\infty} \big|u\big(\Theta_n(t,s)\big)\big|^2 \mathrm{d}s - \Big| \int_0^{\infty} \varphi_\alpha(s) u\big(\Theta_n(t,s)\big) \mathrm{d}s \Big|^2 \Big) \mathrm{d}t$$

$$\le \frac{1}{\alpha^2} \sum_{n=1}^{M} \int_0^{\ell_n} \Big(\int_0^{\infty} \Big| \frac{\partial}{\partial s} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s - \alpha\big|u\big(\Theta_n(t,0)\big)\big|^2 + \alpha^2 \int_0^{\infty} \big|u\big(\Theta_n(t,s)\big)\big|^2 \mathrm{d}s \Big) \mathrm{d}t$$

$$\le \frac{1}{\alpha^2} \sum_{n=1}^{M} \Big(\iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \int\limits_{L_n} |u|^2 \mathrm{d}\sigma + \alpha^2 \iint\limits_{\Pi_n} |u|^2 \mathrm{d}x \Big),$$

which gives:

$$I_1 + I_2 \le \frac{1}{\alpha^2}\Big(h_\alpha^\Omega(u,u) + \alpha^2\|u\|^2\Big).$$

Furthermore, with the help of the Cauchy-Schwarz inequality, we have:

$$I_3 \le \delta \sum_{n=1}^{M} \int_0^{\ell_n}\int_0^{\infty} \Big|u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t = \delta\sum_{n=1}^{M}\iint_{\Pi_n} |u|^2\mathrm{d}x \le \delta\|u\|^2,$$

To estimate the last term, I_4, we introduce the following constant:

$$R := \max\Big\{\int_0^{\ell_n} \big|\rho_n(t)\big|^2\mathrm{d}t : \, n\in\{1,\dots,M\}\Big\},$$

then, using first the estimate (8), and then the inequality (4),

$$\begin{aligned}
I_4 &\le \frac{R}{\delta}\sum_{n=1}^{M} \sup_{t\in(0,\ell_n)} \Big|\big(P_{n,\alpha}u\big)\big(\pi_n(t)\big)\Big|^2\\
&\le \frac{R}{\delta}\sum_{n=1}^{M}\Big(\big|(P_{n,\alpha}u)(0)\big|^2 + \big|(P_{n,\alpha}u)(\ell_n)\big|^2\Big)\\
&\le \frac{RC}{\delta\alpha}\sum_{n=1}^{M}\Big(\iint_{\Lambda_n}|\nabla u|^2\mathrm{d}x + \alpha^2\iint_{\Lambda_n}|u|^2\mathrm{d}x\Big)\\
&\le \frac{RC}{\delta\alpha}\Big[\sum_{n=1}^{M}\Big(\iint_{\Lambda_n}|\nabla u|^2\mathrm{d}x + \alpha^2\iint_{\Lambda_n}|u|^2\mathrm{d}x\Big)\\
&\qquad + \sum_{n=1}^{M}\Big(\iint_{\Pi_n}|\nabla u|^2\mathrm{d}x - \int_{L_n}|u|^2\mathrm{d}\sigma + \alpha^2\iint_{\Pi_n}|u|^2\mathrm{d}x\Big)\Big]\\
&= \frac{RC}{\delta\alpha}\Big(h_\alpha^\Omega(u,u) + \alpha^2\|u\|\Big).
\end{aligned}$$

Choosing $\delta = \alpha^{-1/2}$ and summing up the four terms, we see that:

$$\|u\|^2 - \|J_\alpha u\|^2 \le \frac{c_1}{\sqrt{\alpha}}\Big(h_\alpha^\Omega(u,u) + \alpha^2\|u\|^2 + \|u\|^2\Big) \equiv \frac{c_1}{\sqrt{\alpha}}\Big(b(u,u) + \|u\|^2\Big),$$

with a suitable constant $c_1 > 0$.

Now, we need to compare $b'(J_\alpha u, J_\alpha u)$ and $b(u,u)$. Take $\delta\in(0,1)$ and use the inequality:

$$(a_1+a_2)^2 \le (1+\delta)a_1^2 + \frac{2}{\delta}\,a_2^2, \quad a_1,a_2\ge 0,$$

Then:

$$\begin{aligned}
b'(J_\alpha u, J_\alpha u) - b(u,u) &= \sum_{n=1}^{M} \int_0^{\ell_n} \Big|(P_{n,\alpha}u)' - {\rho'}_n\big[(P_{n,\alpha}u)\circ\pi_n\big]\Big|^2 \mathrm{d}t - \Big(h_\alpha^\Omega(u,u) + \alpha^2\|u\|^2\Big) \\
&\le (1+\delta)\sum_{n=1}^{M} \int_0^{\ell_n} \Big|(P_{n,\alpha}u)'\Big|^2 \mathrm{d}t + \frac{2}{\delta}\sum_{n=1}^{M}\int_0^{\ell_n} \Big|{\rho'}_n\big[(P_{n,\alpha}u)\circ\pi_n\big]\Big|^2 \mathrm{d}t \\
&\quad - \sum_{n=1}^{M}\Big(\iint\limits_{\Lambda_n} |\nabla u|^2 \mathrm{d}x + \alpha^2 \iint\limits_{\Lambda_n} |u|^2 \mathrm{d}x\Big) \\
&\quad - \sum_{n=1}^{M}\Big(\iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \int\limits_{L_n} |u|^2 \mathrm{d}\sigma + \alpha^2 \iint\limits_{\Pi_n} |u|^2 \mathrm{d}x\Big) \qquad (9) \\
&\le (1+\delta)\sum_{n=1}^{M} \int_0^{\ell_n} \Big|(P_{n,\alpha}u)'\Big|^2 \mathrm{d}t + \frac{2}{\delta}\sum_{n=1}^{M}\int_0^{\ell_n} \Big|{\rho'}_n\big[(P_{n,\alpha}u)\circ\pi_n\big]\Big|^2 \mathrm{d}t \\
&\quad - \sum_{n=1}^{M}\Big(\iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \int\limits_{L_n} |u|^2 \mathrm{d}\sigma + \alpha^2 \iint\limits_{\Pi_n} |u|^2 \mathrm{d}x\Big).
\end{aligned}$$

Using first the Cauchy-Schwarz inequality and then inequality (4), we have:

$$\int_0^{\ell_n} \big|(P_{n,\alpha}u)'\big|^2 \mathrm{d}t \le \int_0^{\ell_n}\int_0^{\infty} \Big|\frac{\partial}{\partial t} u\big(\Theta_n(t,s)\big)\Big|^2 \mathrm{d}s\,\mathrm{d}t \le \iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \int\limits_{L_n} |u|^2 \mathrm{d}\sigma + \alpha^2 \iint\limits_{\Pi_n} |u|^2 \mathrm{d}x.$$

Substituting the last inequality into (9), we arrive at:

$$\begin{aligned}
b'(J_\alpha u, J_\alpha u) - b(u,u) \le \delta \sum_{n=1}^{M}\Big(\iint\limits_{\Pi_n} |\nabla u|^2 \mathrm{d}x - \int\limits_{L_n} |u|^2 \mathrm{d}\sigma + \alpha^2 \iint\limits_{\Pi_n} |u|^2 \mathrm{d}x\Big) \\
+ \frac{2}{\delta}\sum_{n=1}^{M}\int_0^{\ell_n} \Big|\rho'_n\big[(P_{n,\alpha}u)\circ\pi_n\big]\Big|^2 \mathrm{d}t.
\end{aligned} \qquad (10)$$

Furthermore, using the constant:

$$R' := \max\Big\{\int_0^{\ell_n} \big|\rho'_n(t)\big|^2 \mathrm{d}t : \, n \in \{1,\dots,M\}\Big\},$$

and the inequality (8), we have:

$$\begin{aligned}
\sum_{n=1}^{M}\int_0^{\ell_n} \Big|\rho'_n\big[(P_{n,\alpha}u)\circ\pi_n\big]\Big|^2 \mathrm{d}t \le R' \sum_{n=1}^{M} \sup_{t\in(0,\ell_n)} \big|(P_{n,\alpha}u)\big(\pi_n(t)\big)\big|^2 \\
\le R' \sum_{n=1}^{M}\Big(\big|(P_{n,\alpha}u)(0)\big|^2 + \big|(P_{n,\alpha}u)(\ell_n)\big|^2\Big) \le \frac{R'C}{\alpha}\sum_{n=1}^{M}\Big(\iint\limits_{\Lambda_n} |\nabla u|^2 \mathrm{d}x + \alpha^2 \iint\limits_{\Lambda_n} |u|^2 \mathrm{d}x\Big).
\end{aligned}$$

The substitution of this inequality into (10) and the choice $\delta = \alpha^{-1/2}$ then leads to:

$$b'(Ju, Ju) - b(u,u) \le \frac{c_2}{\sqrt{\alpha}}\Big(h_\alpha^\Omega(u,u) + \alpha^2\|u\|^2\Big) \le \frac{c_2}{\sqrt{\alpha}}\Big(b(u,u) + \|u\|^2\Big),$$

with a suitable constant $c_2 > 0$. By Proposition 5, for any fixed $m \in \mathbb{N}$ and for large α, we have the estimate $\mu_m^D \le E_m^\Omega(\alpha) + \alpha^2 + \mathcal{O}(\alpha^{-1/2})$. The combination with (5) gives the result.

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