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Translation-invariant Gibbs measures for the mixed spin-1/2 and spin-1 Ising model with an external field on a Cayley tree

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ABSTRACT Phase transitions of the mixed spin-1/2 and spin-1 Ising model under the presence of an external field on the general order Cayley tree are investigated within the framework of the tree-indexed Markov chains. We find the conditions that ensure the existence of at least three translation-invariant Gibbs measures for the model on the Cayley tree of order k. We are able to solve the model exactly on the binary tree ($k = 2$) under the specific external field. The main attention is paid to the systematic study of the structure of the set of the Gibbs measures. We find the extremality and non-extremality regions of the disordered phase of the model on the binary tree.

KEYWORDS mixed-spin Ising model; external field; Cayley tree; Gibbs measures.

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1. Introduction

In recent years, the theoretical study of mixed-spin Ising models has received significant attention. Unlike their single-spin counterparts, mixed-spin Ising models possess less translational symmetry, making them primarily subjects of experimental study, with a considerable gap in theoretical research. To the best of the authors' knowledge, the first paper to rigorously study mixed-spin Ising models with mathematical precision is [1]. This paper investigates the model using a measure-theoretic approach and proves the existence of phase transitions. Subsequent papers [2–6] continue the investigation, exploring various properties of mixed-type Ising models on Cayley trees.

The impact of an external field is evident from the outset of the theory. For instance, the classical Ising model on the cubic lattice $\mathbb{Z}^d$ ($d \geq 2$) exhibits a phase transition in the absence of an external field, but no phase transition occurs when a non-zero external field is applied. Introducing an external field to such models typically results in the loss of symmetry, making it more challenging to study the model's properties. In this paper, we employ the exact recursion equations technique to investigate the phase transition of the mixed spin-1/2 and spin-1 Ising model with an external field on the Cayley tree. The aim of this work is to elucidate the influence of the external field on the model's physical properties.

In [7], the mixed spin-$1/2$ and spin-1 Ising model in the absence of an external field on the arbitrary order Cayley tree is studied. It is shown that this particular model exhibits a phase transition phenomena in both the ferromagnetic and antiferromagnetic regions. In that paper, the authors also investigate the extremality of disordered phases employing a Markov chain indexed by a tree on a semi-infinite Cayley tree. Utilizing the Kesten-Stigum condition [8], they delve into the non-extremality aspects of disordered phases by scrutinizing the eigenvalues of the stochastic matrix associated with the (1,1/2) mixed-spin Ising model on Cayley trees with order $k (k \geq 3)$. One of the main contributions of the present paper is to show the existence of a phase transition for the (1,1/2) mixed-spin Ising model under the external field on the general order Cayley tree.

In [9], the author studies the one-dimensional Ising model with mixed spins $(s, \dfrac{2t-1}{2})$ under the influence of nearest-neighbor interactions and an external magnetic field. By analyzing the iterative equations related to the model, the phase transition problem is explored using the cavity approach. Furthermore, various thermodynamic quantities for the model are calculated, and precise formulas are provided to determine the free energy, entropy, magnetization, and susceptibility. For the case $s = 1$ and $t = 1$, our results extend the findings of [9] to higher-order Cayley trees.

Numerous numerical methods also have been applied to the study of mixed-spin models. We mention some of them: One of the earliest, simplest, and most extensively studied mixed-spin Ising models is the spin-1/2 and spin-1 mixed system. This system has been investigated using a variety of techniques, including the renormalization-group technique [10], high-temperature series expansions [11], the free-fermion approximation, the recursion method on the Bethe lattice, and the Bethe-Peierls approximation [12–15]. Additionally, studies have employed the effective-field theory framework [16, 17], the mean-field approximation [18, 19], the finite cluster approximation [20], Monte Carlo simulations [21], the mean-field renormalization-group technique, numerical transfer matrix studies [7], and the cluster variation method in pair approximation [22].

It is known [23–26] that for all $\beta > 0$, the set of Gibbs measures forms a non-empty, convex, and compact subset in the space of probability measures. Moreover, any Gibbs measure can be expressed as an integral over extreme Gibbs measures, known as the extreme decomposition [25]. Consequently, the extreme points are of fundamental importance for describing the entire convex set of Gibbs measures. The extreme disordered phases of models on lattices are particularly significant in the context of information flow theory [27–30]. In the present paper, we provide a non-trivial adaptation of well-known methods, including the Kesten-Stigum criterion [8] for assessing the non-extremality of translation-invariant Gibbs measures, as well as the Martinelli-Sinclair-Weitz method [31] for evaluating the extremality of translation-invariant Gibbs measures.

In this paper, we derive a system of functional equations based on the compatibility condition. We show the presence of a phase transition for the mixed-spin Ising model under the external field on the general order Cayley tree. On the binary tree, solving the model exactly under a constant external field, we demonstrate that the model possesses either one or three Gibbs measures depending on the temperature. Additionally, we investigate the conditions for extremality and non-extremality of the disordered phase of the model.

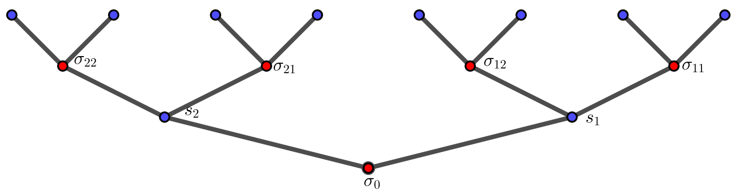


FIG. 1. Some generations of a second order Cayley tree of with a σ_0 spin in the root.

2. Preliminaries

Let $\Gamma^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$, with a designated root vertex $x^{(0)}$. In this tree, each vertex has exactly $k+1$ adjacent edges, except for the root $x^{(0)}$, which has only k adjacent edges. The set V represents the vertices of the tree, while L represents the edges.

Two vertices x and y are called *nearest neighbors*, denoted by $l = \langle x, y \rangle$, if there exists an edge in L that connects them. A sequence of edges $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a *path* from the vertex x to the vertex y. The distance $d(x, y)$ between two vertices x and y in the Cayley tree is defined as the length of the shortest path connecting them.

We denote

$$W_n = \{x \in V \mid d(x, x^0) = n\}, V_n = \bigcup_{m=0}^{n} W_m, L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

The set of direct successors of a vertex x is defined as

$$S(x) = \{y \in W_{n+1} : \ d(x, y) = 1\}$$

where $x \in W_n$.

Denote

$$\Gamma_+^k = \{x \in V : \ d(x^0, x) - \text{even}\}, \quad \Gamma_-^k = \{x \in V : \ d(x^0, x) - \text{odd}\}.$$

In this paper, we consider the following spin state spaces: $\Phi = \{-1, 0, 1\}$ and $\Psi = \{-\frac{1}{2}, \frac{1}{2}\}$. The corresponding configuration spaces are defined as $\Omega_+ = \Phi^{\Gamma_+^k}$ and $\Omega_- = \Psi^{\Gamma_-^k}$, where Γ_+^k and Γ_-^k represent two disjoint semi-infinite Cayley trees of order k.

Additionally, the finite-volume configuration spaces are denoted by $\Omega_{+,n} = \Phi^{\Gamma_+^k \cap V_n}$ and $\Omega_{-,n} = \Psi^{\Gamma_-^k \cap V_n}$, where V_n is the set of vertices at distance n from the root.

The overall configuration space of the model is given by $\Xi = \Omega_+ \times \Omega_-$. An element of Ω_+ is denoted by $\sigma(x)$ for $x \in \Gamma^k_+$, and an element of Ω_- is denoted by $s(x)$ for $x \in \Gamma^k_-$.

For the configuration $\xi \in \Xi$, the associated sites are assigned to successive generations of the tree (see Fig. 1). Specifically, at the odd-numbered levels of the tree, the vertices are occupied by spins taking values from the set Ψ. Conversely, at the even-numbered levels, the vertices are occupied by spins taking values from the set Φ. This is formally expressed as follows:

$$\xi(x) = \begin{cases} \sigma(x) \text{ if } x \in \Gamma^k_+; \\ s(x) \text{ if } x \in \Gamma^k_-, \end{cases} \tag{1}$$

where $\sigma(x) \in \Phi = \{-1, 0, 1\}$ and $s(x) \in \Psi = \{-\frac{1}{2}, \frac{1}{2}\}$. This arrangement ensures that spins from Ψ are located at odd levels, while spins from Φ are located at even levels of the tree.

The Hamiltonian of (1/2-1) mixed spin Ising model with an external field is defined by

$$H(\xi) = -J \sum_{\langle x,y \rangle} \xi(x)\xi(y) - \sum_{x \in V} \alpha_{\xi(x)}(x), \ \ \xi \in \Xi \tag{2}$$

where

$$\alpha_{\xi(x)}(x) = \begin{cases} \alpha_{\sigma(x)}(x) \text{ if } x \in \Gamma^k_+; \\ \widetilde{\alpha}_{s(x)}(x) \text{ if } x \in \Gamma^k_-, \end{cases} \tag{3}$$

is the external field.

We denote $\boldsymbol{h} = (\boldsymbol{h}_{\xi(x)}(x))_{x \in \Gamma^k}$, where

$$h_{\xi(x)}(x) = \begin{cases} h_{\sigma(x)}(x), & x \in \Gamma^k_+; \\ \widetilde{h}_{s(x)}(x), & x \in \Gamma^k_-, \end{cases}$$

and $\boldsymbol{h}(x) = (h_{-1}(x), h_0(x), h_{+1}(x)), \widetilde{\boldsymbol{h}}(x) = (\widetilde{h}_{-\frac{1}{2}}(x), \widetilde{h}_{\frac{1}{2}}(x))$.

Now, for each $n \geq 1$, we define the Gibbs measure $\mu_n^{\boldsymbol{h}}$ by

$$\mu_n^{\boldsymbol{h}}(\xi) = \frac{e^{-\beta H_n(\xi) + \sum\limits_{x \in W_n} h_{\xi(x)}(x)}}{Z_n}, \tag{4}$$

where $\xi \in \Xi_n := \Omega_{+,n} \times \Omega_{-,n}$, Z_n is the partition function.
The sequence of measures $\{\mu_n^{\boldsymbol{h}}\}$ is compatible, if for all $n \geq 1$ and $\xi_{n-1} \in \Xi_{n-1}$ one has

$$\sum_{w \in \Xi^{W_n}} \mu_n^{\boldsymbol{h}}(\xi_{n-1} \vee w) = \mu_{n-1}^{\boldsymbol{h}}(\xi_{n-1}), \quad \text{for all} \quad n \geq 1, \tag{5}$$

$$\Xi^{W_n} = \begin{cases} \Phi^{W_n}, & n - \text{even}; \\ \Psi^{W_n}, & n - \text{odd}. \end{cases}$$

Here $\xi_{n-1} \vee w$ is the concatenation of the configurations. In this setting, there is a unique measure μ on Ω such that for all n and $\xi_n \in \Xi_n$

$$\mu(\{\xi | V_n = \xi_n\}) = \mu_n^{\boldsymbol{h}}(\xi_n).$$

Such a measure is called a *splitting Gibbs measure (SGM)* corresponding to the model (2).

The following result describes the condition on $\boldsymbol{h}$ ensuring that the sequence $\{\mu_n^{\boldsymbol{h}}\}$ is compatible.

Theorem 1. The sequence of measures $\{\mu_n^{\boldsymbol{h}}\}$, $n = 1, 2, ...$ given by (5) is compatible if and only if for any $x \in V$ the following equations hold:

$$e^{h_{-1}(x) - h_0(x)} = \prod_{y \in S(x)} \left(\frac{e^{\frac{1}{2}J\beta + \beta\widetilde{\alpha}_{-\frac{1}{2}}(y) + \widetilde{h}_{-\frac{1}{2}}(y)} + e^{-\frac{1}{2}J\beta + \beta\widetilde{\alpha}_{\frac{1}{2}}(y) + \widetilde{h}_{\frac{1}{2}}(y)}}{e^{\beta\widetilde{\alpha}_{-\frac{1}{2}}(y) + \widetilde{h}_{-\frac{1}{2}}(y)} + e^{\beta\widetilde{\alpha}_{\frac{1}{2}}(y) + \widetilde{h}_{\frac{1}{2}}(y)}} \right), \tag{6}$$

$$e^{h_{1}(x) - h_0(x)} = \prod_{y \in S(x)} \left(\frac{e^{-\frac{1}{2}J\beta + \beta\widetilde{\alpha}_{-\frac{1}{2}}(y) + \widetilde{h}_{-\frac{1}{2}}(y)} + e^{\frac{1}{2}J\beta + \beta\widetilde{\alpha}_{\frac{1}{2}}(y) + \widetilde{h}_{\frac{1}{2}}(y)}}{e^{\beta\widetilde{\alpha}_{-\frac{1}{2}}(y) + \widetilde{h}_{-\frac{1}{2}}(y)} + e^{\beta\widetilde{\alpha}_{\frac{1}{2}}(y) + \widetilde{h}_{\frac{1}{2}}(y)}} \right), \tag{7}$$

$$e^{\widetilde{h}_{\frac{1}{2}}(x) - \widetilde{h}_{-\frac{1}{2}}(x)} = \prod_{y \in S(x)} \left(\frac{e^{-\frac{1}{2}J\beta + \beta\alpha_{-1}(y) + h_{-1}(y)} + e^{\beta\alpha_0(y) + h_0(y)} + e^{\frac{1}{2}J\beta + \beta\alpha_1(y) + h_1(y)}}{e^{\frac{1}{2}J\beta + \beta\alpha_{-1}(y) + h_{-1}(y)} + e^{\beta\alpha_0(y) + h_0(y)} + e^{-\frac{1}{2}J\beta + \beta\alpha_1(y) + h_1(y)}} \right). \tag{8}$$

Proof. The proof can be carried out using the standard argument presented in [7].

3. Translation-invariant Gibbs measures

In this section, we deal with the existence of translation-invariant splitting Gibbs measures (TISGMs) corresponding to the Ising model with mixed spin-1 and spin-1/2 by analyzing the the equations (6)-(8). Recall that the vector-valued functions $\widetilde{\boldsymbol{h}} = \{\widetilde{h}_{-\frac{1}{2}}(x), \widetilde{h}_{\frac{1}{2}}(x)\}$ and $\boldsymbol{h}(x) = (h_{-1}(x), h_0(x), h_{+1}(x))$ are called *translation-invariant* if $\widetilde{h}_i(x) = \widetilde{h}_i(y) =: \widetilde{h}_i$ and $h_j(x) = h_j(y) =: h_j$ for all $y \in S(x)$ (see [7]). The measures corresponding to the vector valued functions $\widetilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ are called TISGMs. We assume that external field $\alpha(x)$ is also translation-invariant, i.e., $\alpha(x) := \alpha \, \forall x \in \Gamma^k$.

Denote $\boldsymbol{h}_j := \boldsymbol{h}_j(x)$ for all $x \in \Gamma^k_+$, $j \in \Phi$, $\widetilde{\boldsymbol{h}}_i := \widetilde{\boldsymbol{h}}_i(x)$, $x \in \Gamma^k_-$, $i \in \Psi$. Introducing the notations $U_1 = h_{-1} - h_0$, $U_2 = h_1 - h_0$, $V = \widetilde{h}_{\frac{1}{2}} - \widetilde{h}_{-\frac{1}{2}}$, $T_1 = \beta(\alpha_{-1} - \alpha_0)$, $T_2 = \beta(\alpha_1 - \alpha_0)$, $F = \beta(\alpha_{\frac{1}{2}} - \alpha_{-\frac{1}{2}})$, we have the following

$$e^{U_1} = \prod_{y \in S(x)} \left(\frac{\theta^2 + e^V \cdot e^F}{\theta \cdot (1 + e^V \cdot e^F)} \right), \tag{9}$$

$$e^{U_2} = \prod_{y \in S(x)} \left(\frac{1 + \theta^2 \cdot e^V \cdot e^F}{\theta \cdot (1 + e^V \cdot e^F)} \right), \tag{10}$$

$$e^{V} = \prod_{y \in S(x)} \left(\frac{e^{U_1} \cdot e^{T_1} + \theta^2 \cdot e^{U_2} \cdot e^{T_2} + \theta}{\theta^2 \cdot e^{U_1} \cdot e^{T_1} + e^{U_2} \cdot e^{T_2} + \theta} \right). \tag{11}$$

Denoting $e^{U_1} = X$, $e^{U_2} = Y$, $e^V = Z$, $e^{T_1} = M$, $e^{T_2} = N$, $e^F = L$, we obtain the following system of equations:

$$\begin{cases} X = \left(\dfrac{\theta^2 + LZ}{\theta(1 + LZ)} \right)^k, \\ Y = \left(\dfrac{1 + \theta^2 LZ}{\theta\,(1 + LZ)} \right)^k, \\ Z = \left(\dfrac{MX + \theta^2 NY + \theta}{\theta^2 MX + NY + \theta} \right)^k. \end{cases} \tag{12}$$

3.1. Stability of a fixed point

We consider the system of equations (12). For simplicity, we assume that $L = 1, M = N = m$. In this case, the model possesses the disordered phase [32], i.e., $Z = 1, X = Y = \left(\dfrac{\theta^2 + 1}{2\theta} \right)^k$ is always a solution to the system (12). We study the stability of this solution.

To investigate the dynamics of (12), we find the eigenvalues of the following Jacobian matrix J_F:

$$J_F = \begin{pmatrix} 0 & 0 & -\dfrac{k(\theta^2-1)}{4\theta} \cdot (\dfrac{\theta^2+1}{2\theta})^{k-1} \\ 0 & 0 & \dfrac{k(\theta^2-1)}{4\theta} \cdot (\dfrac{\theta^2+1}{2\theta})^{k-1} \\ -\dfrac{k \cdot m \cdot (\theta^2 - 1)}{m \cdot \theta^2 (\frac{\theta^2+1}{2\theta})^2 + m \cdot (\frac{\theta^2+1}{2\theta})^2 + \theta} & \dfrac{k \cdot m \cdot (\theta^2 - 1)}{m \cdot \theta^2 (\frac{\theta^2+1}{2\theta})^2 + m \cdot (\frac{\theta^2+1}{2\theta})^2 + \theta} & 0 \end{pmatrix} \tag{13}$$

After some algebraic manipulations, we obtain that the eigenvalues of the matrix are: $\lambda_1 = 0$,

$$\lambda_2 = \frac{2\sqrt{(\theta^8 m + 4\theta^6 m + 4\theta^5 + 6\theta^4 m + 4\theta^3 + 4\theta^2 m + m)2^{-k}(\frac{\theta^2+1}{\theta})^k m}(\theta+1)(\theta-1)\theta k}{\theta^8 m + 4\theta^6 m + 4\theta^5 + 6\theta^4 m + 4\theta^3 + 4\theta^2 m + m},$$

$$\lambda_3 = -\frac{2\sqrt{(\theta^8 m + 4\theta^6 m + 4\theta^5 + 6\theta^4 m + 4\theta^3 + 4\theta^2 m + m)2^{-k}(\frac{\theta^2+1}{\theta})^k m}(\theta+1)(\theta-1)\theta k}{\theta^8 m + 4\theta^6 m + 4\theta^5 + 6\theta^4 m + 4\theta^3 + 4\theta^2 m + m}.$$

Remark 1. From Fig. 2, it can be seen that, at some values of parameters θ and k, we have

$$|\lambda_3(\theta, k, m)| > 1,$$

which shows that the fixed point $\left(\left(\dfrac{\theta^2+1}{2\theta} \right)^2, \left(\dfrac{\theta^2+1}{2\theta} \right)^2, 1 \right)$ is saddle [33]. This fact indicates that there is a phase transition.

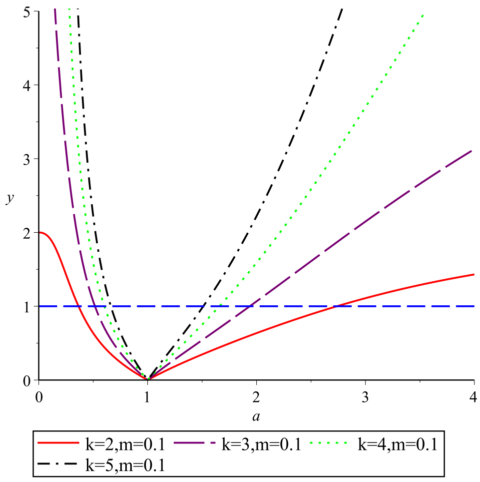


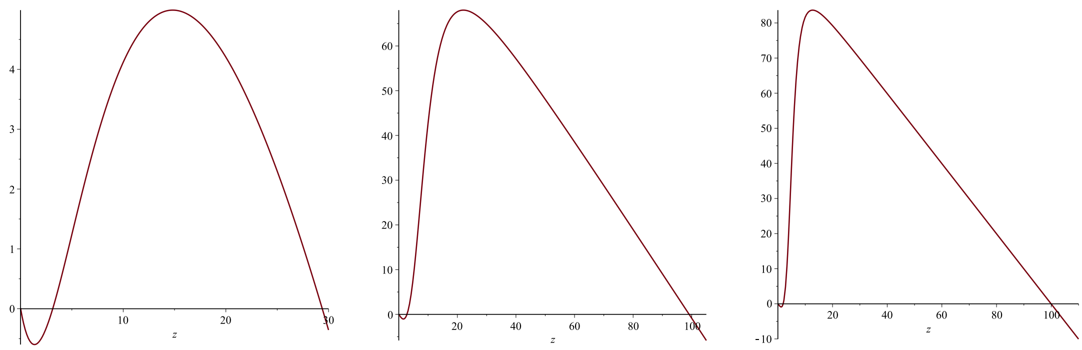
FIG. 2. Plots of the function $|\lambda_3(\theta,k,m)|$ for $k=2,3,4,5$ and $m=0,1$.

3.2. The existence of the phase transition

On substituting the first and second equations of (12) into the third equation, we obtain

$$Z=\left(\frac{M(\theta^2+LZ)^k+(1+\theta^2LZ)^k+\theta^{k+1}(1+LZ)^k}{\theta^2M(\theta^2+LZ)^k+N(1+\theta^2LZ)^k+\theta^{k+1}(1+LZ)^k}\right)^k=F(Z). \tag{14}$$

It follows from (14) that solving (12) is reduced to finding the fixed points of the function $F(Z)$. It is clear that the function $F(Z)$ is an increasing, bounded function with $F(0)>0$ and $F(\infty)<\infty$. It follows from properties of the function $F(Z)$ that the function has at least one fixed point, say, Z^*. We have



(a)	(b)	(c)
$F(Z)-Z$ when	$F(Z)-Z$ when	$F(Z)-Z$ when
$\theta=0.1,$	$\theta=0.1,$	$\theta=0.1,$
$M=0.5,$	$M=2,$	$M=3,$
$N=50,$	$N=5,$	$N=5,$
$L=7,$	$L=0.7,$	$L=0.7,$
$k=2,$	$k=3,$	$k=4,$
$Z=0,...,1.$	$Z=0,...,105.$	$Z=0,...,110.$

FIG. 3. The plots of the function $F(Z)-Z$ at some values of parameters

Theorem 2. For the mixed-spin Ising model with external field on the Cayley tree of order $k\geq 2$ if the condition $F'(Z^*)>1$ is satisfied, then there exist at least three distinct SGMs, i.e. the phase transition occurs.

Proof. The condition $F'(Z^*)>1$ implies the existence of at least three solutions of Equation (14). Let Z^* be the fixed point of $F(Z)$. When $F'(Z^*)>1$, Z^* is unstable. Consequently, there exists a small neighborhood $(Z^*-\varepsilon,Z^*+\varepsilon)$ of Z^* such that for $Z\in(Z^*-\varepsilon,Z^*)$ $F(Z)<Z$, and for $Z\in(Z^*,Z^*+\varepsilon)$ $F(Z)>Z$. Since $F(0)>0$, there exists a solution Z^*_- between 0 and Z^*. Similarly, since $F(+\infty) < +\infty$, there is another solution Z^*_+ between Z^* and $+\infty$. Given that there is a bijection between the solution of Eq.(14) and SGMs, it follows that there exist at least three SGMs, which implies the existence of a phase transition.

Remark 2. Note that the set of parameters which satisfy $F'(Z^*) > 1$ is not empty, e.g., see Fig. 3.

3.3. The case $k = 2$

In what follows, we restrict ourselves to the case $k = 2, L = 1, M = N = m$ in (12). Then the system of equations (12) can be reduced to the following equation:

$$(Z-1)(AZ^4 + BZ^3 + CZ^2 + BZ + A) = 0 \tag{15}$$

where

$$A = \theta^8 m^2 + 2\theta^7 m + 2\theta^6 m^2 + \theta^6 + 2\theta^5 m + \theta^4 m^2,$$
$$B = -\theta^{12}m^2 - 2\theta^9 m + 5\theta^8 m^2 + 10\theta^7 m + 8\theta^6 m^2 + 4\theta^6 + 10\theta^5 m + 5\theta^4 m^2 - 2\theta^3 m - m^2,$$
$$C = -\theta^{12}m^2 - 2\theta^{10}m^2 - 4\theta^9 m + 7\theta^8 m^2 + 16\theta^7 m + 16\theta^6 m^2 + 6\theta^6 + 16\theta^5 m + 7\theta^4 m^2 - 4\theta^3 m - 2\theta^2 m^2 - m^2.$$

We obtain that $Z = 1$ is a solution of (15) independent of remaining parameters, and we denote it by z_1. We consider the second factor in (15). After some algebraic operations, we have

$$A(Z^2 + \frac{1}{Z^2}) + B(Z + \frac{1}{Z}) + C = 0.$$

Introducing the new variable

$$t = Z + \frac{1}{Z}, \tag{16}$$

we have

$$f(t) = At^2 + Bt + (C - 2A) = 0. \tag{17}$$

The solutions of (17) are

$$t_1 = \frac{-B - \sqrt{8A^2 + B^2 - 4AC}}{2A}, \quad t_2 = \frac{-B + \sqrt{8A^2 + B^2 - 4AC}}{2A}. \tag{18}$$

Taking into account $t = Z + \dfrac{1}{Z}$, we consider the following two cases: $t_1 < 2 < t_2$ or $2 < t_1 < t_2$.

Case 1. Let $t_1 < 2 < t_2$. In this case, the parabola defined in (17) should satisfy $Af(2) < 0$. Since $A > 0$ we have that

$$2(A + B) + C < 0$$

or

$$m(\theta^2 + 1)(\theta^2 - 3)(\theta^2 - \frac{1}{3}) > \frac{4}{3}\theta^3. \tag{19}$$

Taking into account that $\theta > 0$, we solve the inequality (19) with respect to the parameter m:

$$m > \frac{4\theta^3}{3(\theta^2 + 1)(\theta^2 - 3)(\theta^2 - \frac{1}{3})}$$

where $\theta \in (0; \dfrac{1}{\sqrt{3}}) \cup (\sqrt{3}; \infty)$. Under these conditions, we obtain the solutions of the second factor of the equation (15):

$$z_2 = \frac{t_2 + \sqrt{t_2^2 - 4}}{2}, \quad z_3 = \frac{t_2 - \sqrt{t_2^2 - 4}}{2}.$$

Case 2. $2 < t_1 < t_2$. In this case, it suffices to consider $\dfrac{1}{\sqrt{3}} < \theta < \sqrt{3}$. Since $A > 0$ from properties of the (17), we have $f(2) > 0$. It follows that $-\dfrac{B}{2A} > 2$ or $B + 4A < 0$ which is equivalent to

$$m(\theta^2 + 1)(\theta^4 - 4\theta^2 + 1) > 2\theta^3.$$

Solving this equality, we have

$$m > \frac{2\theta^3}{(\theta^2 + 1)(\theta^4 - 4\theta^2 + 1)}$$

where $\theta \in (0; \sqrt{2 - \sqrt{3}}) \cup (\sqrt{2 + \sqrt{3}}; \infty)$. However, the obtained solution does not satisfy the condition $\dfrac{1}{\sqrt{3}} < \theta < \sqrt{3}$, thus, in this case, we do not obtain any solution.

Introduce

$$\Upsilon_1 = \{(\theta, m) : m(\theta^2 + 1)(\theta^4 - 4\theta^2 + 1) > 2\theta^3 \text{ and } \theta \in (0; \sqrt{2 - \sqrt{3}}) \cup (\sqrt{2 + \sqrt{3}}; \infty)\}$$

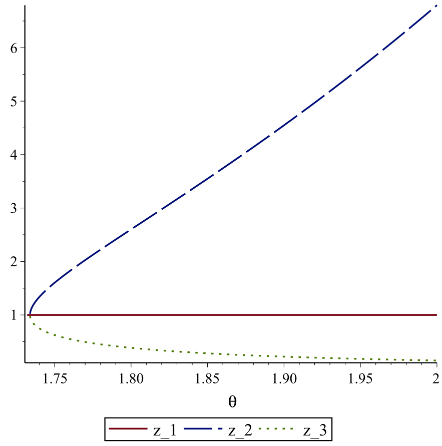
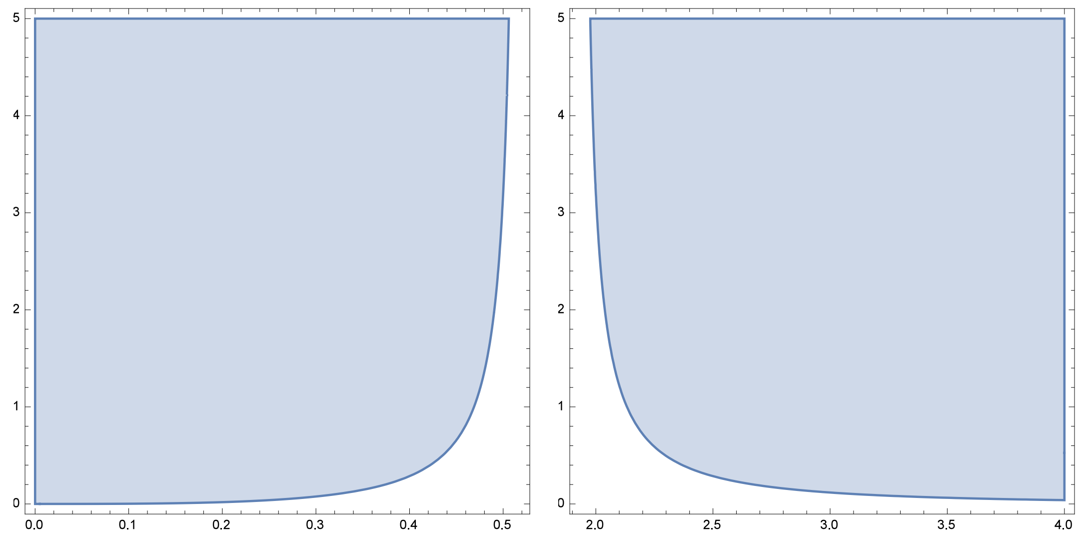


FIG. 4. The plot of the solutions z_1, z_2, z_3 at $m = 100$ and $\theta \in$ [1.75;2].

(see e.g., Fig. 5).

Theorem 3. Assume that $J \neq 0$. If $(\theta, m) \in \Upsilon_1$ then there is a phase transition for the mixed-spin Ising model on the Cayley tree of order two.



(a)The plot of Υ_1 is drawn for $\theta \in (0, \sqrt{2-\sqrt{3}})$ and $m \in (0, 5)$.

(b)The plot of Υ_1 is drawn for $\theta \in (\sqrt{2+\sqrt{3}}, 4)$ and $m \in (0, 5)$.

FIG. 5. The plot of Υ_1 is drawn for $\theta \in (0, \sqrt{2-\sqrt{3}}) \cup (\sqrt{2+\sqrt{3}}, 4)$ and $m \in (0, 5)$.

4. Extremality of disordered phase

In this section, we check the non-extremality of the obtained Gibbs measures:
Let us consider the following stochastic matrix $\mathbb{P} = (P_{ij})$:

$$P_{ij} = \frac{e^{ij\beta J + \widetilde{h}_j + \beta \widetilde{\alpha}_j}}{\sum\limits_{u=\mp\frac{1}{2}} e^{iu\beta J + \widetilde{h}_u + \beta \widetilde{\alpha}_u}}$$

where $i \in \{-1, 0, 1\}$ va $j \in \{-\frac{1}{2}, \frac{1}{2}\}$.

Using our notations $Z = e^{\widetilde{h}_{\frac{1}{2}} - \widetilde{h}_{-\frac{1}{2}}}$ and $L = e^{\beta(\widetilde{\alpha}_{\frac{1}{2}} - \widetilde{\alpha}_{-\frac{1}{2}})}$, introduced in Section 2, we have the following matrix

$\mathbb{P} = (P_{ij})$:

$$\mathbb{P} = \begin{pmatrix} P_{(-1,-\frac{1}{2})} & P_{(-1,\frac{1}{2})} \\ P_{(0,-\frac{1}{2})} & P_{(0,\frac{1}{2})} \\ P_{(1,-\frac{1}{2})} & P_{(1,\frac{1}{2})} \end{pmatrix} = \begin{pmatrix} \frac{\theta^2}{\theta^2+LZ} & \frac{LZ}{\theta^2+LZ} \\ \frac{1}{1+LZ} & \frac{LZ}{1+LZ} \\ \frac{1}{1+\theta^2 LZ} & \frac{\theta^2 LZ}{1+\theta^2 LZ} \end{pmatrix} \tag{20}$$

Similarly, we introduce the following stochastic matrix $\mathbb{Q} = (Q_{ij})$:

$$Q_{ij} = \frac{\exp(ij\beta J + h_j + \beta\alpha_j)}{\sum\limits_{u=\{-1,0,1\}} \exp(iu\beta J + h_u + \beta\alpha_u)}$$

where $i \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $j \in \{-1, 0, 1\}$.

Using the notations $e^{h_{-1}-h_0} = X, \quad e^{h_1-h_0} = Y, \quad e^{\beta(\alpha_{-1}-\alpha_0)} = M$ and $e^{\beta(\alpha_1-\alpha_0)} = N$, we have $\mathbb{Q} = (Q_{ij})$:

$$\mathbb{Q} = \begin{pmatrix} Q_{(-\frac{1}{2},-1)} & Q_{(-\frac{1}{2},0)} & Q_{(-\frac{1}{2},1)} \\ Q_{(\frac{1}{2},-1)} & Q_{(\frac{1}{2},0)} & Q_{(\frac{1}{2},1)} \end{pmatrix} = \begin{pmatrix} \frac{\theta^2 MX}{\theta^2 MX+\theta+NY} & \frac{\theta}{\theta^2 MX+\theta+NY} & \frac{NY}{\theta^2 MX+\theta+NY} \\ \frac{MX}{MX+\theta+\theta^2 NY} & \frac{\theta}{MX+\theta+\theta^2 NY} & \frac{\theta^2 NY}{MX+\theta+\theta^2 NY} \end{pmatrix} \tag{21}$$

For the solution $Z = 1, X = Y = \left(\frac{\theta^2+1}{2\theta}\right)^2$, the matrices $\mathbb{P}$ and $\mathbb{Q}$ have the following forms:

$$\mathbb{P} = \begin{pmatrix} \frac{\theta^2}{1+\theta^2} & \frac{1}{1+\theta^2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{1+\theta^2} & \frac{\theta^2}{1+\theta^2} \end{pmatrix}, \tag{22}$$

$$\mathbb{Q} = \begin{pmatrix} \frac{m\theta^2(\theta^2+1)^2}{4\theta^3+m(\theta^2+1)^3} & \frac{4\theta^3}{4\theta^3+m(\theta^2+1)^3} & \frac{m(\theta^2+1)^2}{4\theta^3+m(\theta^2+1)^3} \\ \frac{m(\theta^2+1)^2}{4\theta^3+m(\theta^2+1)^3} & \frac{4\theta^3}{4\theta^3+m(\theta^2+1)^3} & \frac{m\theta^2(\theta^2+1)^2}{4\theta^3+m(\theta^2+1)^3} \end{pmatrix}. \tag{23}$$

It is easy to see that $\mathbb{P} \cdot \mathbb{Q}$ is again a stochastic matrix:

$$\mathbb{H} = \mathbb{P} \cdot \mathbb{Q} = \frac{1}{4\theta^3+m(\theta^2+1)^3} \begin{pmatrix} m(\theta^2+1)(\theta^4+1) & 4\theta^3 & 2m\theta^2(\theta^2+1) \\ \frac{m(\theta^2+1)^3}{2} & 4\theta^3 & \frac{m(\theta^2+1)^3}{2} \\ 2m\theta^2(\theta^2+1) & 4\theta^3 & m(\theta^2+1)(\theta^4+1) \end{pmatrix}. \tag{24}$$

The eigenvalues of the stochastic matrix $\mathbb{H}$ are:

$$\{0, \frac{m(\theta^6-\theta^4-\theta^2+1)}{\theta^6 m+3\theta^4 m+4\theta^3+3\theta^2 m+m}, 1\}.$$

After some calculation, one can show that the second eigenvalue in terms of the absolute value is

$$\lambda_{max} = \frac{m(\theta^6-\theta^4-\theta^2+1)}{\theta^6 m+3\theta^4 m+4\theta^3+3\theta^2 m+m} \tag{25}$$

According to the Kesten-Stigum criterition [8], in order to check the non-extremality of the measure, we should consider the following inequality

$$2\lambda_{max}^2 - 1 > 0.$$

$$2\lambda_{max}^2 - 1 = 2\Big(\frac{m(\theta^6-\theta^4-\theta^2+1)}{\theta^6 m+3\theta^4 m+4\theta^3+3\theta^2 m+m}\Big)^2 - 1 = \frac{A\cdot m^2+B\cdot m+C}{(\theta^6 m+3\theta^4 m+4\theta^3+3\theta^2 m+m)^2}$$

$$A := \theta^{12} - 10\theta^{10} - 17\theta^8 - 12\theta^6 - 17\theta^4 - 10\theta^2 + 1,$$

$$B := -8\theta^9 - 24\theta^7 - 24\theta^5 - 8\theta^3, \quad C := -16\theta^6.$$

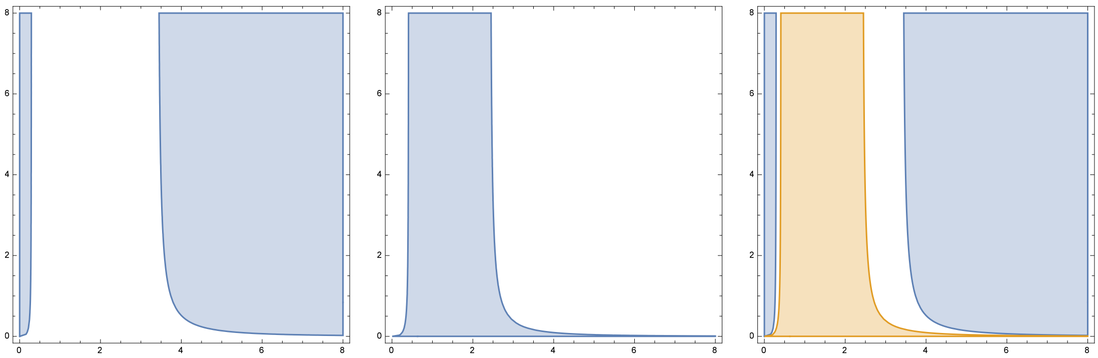
Denote

$$K(\theta, m) = A\cdot m^2 + B\cdot m + C. \tag{26}$$

Thus, the inequality $2\lambda_{max}^2 - 1 > 0$ is reduced to $K(\theta, m) > 0$.

Theorem 4. If $K(\theta, m) > 0$ then the disordered phase is non-extreme.

Remark 3. Note that the set $\Upsilon_2 = \{(\theta, m) \in \mathbb{R}_+^2 : K(\theta, m) > 0\}$ is not empty, see, for example Fig. 4(a).



(a)The non-extremality regions of the disordered Gibbs measures is plotted for $\theta \in (0,8)$ and $m \in (0,8)$.

(b)The extremality regions of the disordered Gibbs measures is plotted for $\theta \in (0,8)$ and $m \in (0,8)$.

(c)(a) and (b) Figures are plotted jointly to compare results.

FIG. 6. The extremality vs non-extremality regions of the obtained Gibbs measures are plotted.

In the subsection, we check the extremality of the measures.

Definition. [31] For a set of Gibbs distributions $\mu^s_{\tau_x}$, the quantities $\kappa \equiv \kappa(\{\mu^s_{\tau_x}\})$ and $\gamma \equiv \gamma(\{\mu^s_{\tau_x}\})$ are defined by

(1) $\kappa = \sup_{z\in\Gamma^k} \max_{z,s,s'} \|\mu^s_{T_z} - \mu^{s'}_{T_z}\|_z$

(2) $\gamma = \sup_{z\subset\Gamma^k} \max_{z,s,s'} \|\mu^{\eta^{y,s}}_A - \mu^{\eta^{y,s'}}_A\|_z$, where the maximum is taken over all boundary conditions η, all sites $y \in \partial A$, all neighbors $x \in A$ of y, and all spins $s, s' \in \{-1,0,1\}$.

It is known [1,7,31] that to check the extremality of the translation-invariant Gibbs measures, we should consider the following inequality:

$$2\kappa\gamma < 1, \tag{27}$$

where $\kappa = \sqrt{\tau_{\mathbb{P}}\tau_{\mathbb{Q}}}$ and $\tau_{\mathbb{H}} = \frac{1}{2}\max_{i,j}\{\sum_{l=1}^{3} \mid H_{i,l} - H_{j,l} \mid\}$. From (22) and (23), we have

$$\tau_{\mathbb{P}} = \frac{\mid \theta^2 - 1 \mid}{\theta^2+1}, \quad \tau_{\mathbb{Q}} = \frac{m(\theta^2+1)^2 \mid \theta^2 - 1 \mid}{4\theta^3 + m(\theta^2+1)^3}.$$

As in [1,7,34] we assume that $\kappa = \gamma$. Then,

$$\kappa^2 = \frac{m(\theta^2-1)^2(\theta^2+1)}{4\theta^3 + m(\theta^2+1)^3},$$

$$2\kappa^2 - 1 = \frac{2m(\theta^2-1)^2(\theta^2+1)}{4\theta^3 + m(\theta^2+1)^3} - 1 < 0. \tag{28}$$

Simplifying the above expression, we have

$$m(\theta^6 - 5\theta^4 - 5\theta^2 + 1) < 4\theta^3.$$

Introduce

$$\Upsilon_3 = \{(\theta, m) : m(\theta^6 - 5\theta^4 - 5\theta^2 + 1) < 4\theta^3 \text{ and } \theta \in (0, \sqrt{2}-1) \cup (\sqrt{2}+1, \infty)\}.$$

We can deduce that

Theorem 5. 1. If $\theta^6 - 5\theta^4 - 5\theta^2 + 1 \leq 0$ or $(\theta, m) \in \Upsilon_3$, then the disordered phase is extreme.

Remark 4. Note that the set Υ_3 is not empty, see, for example Fig. 4(b).

5. Conclusion

In the present work, we have investigated the phase transition of the mixed type Ising model on the Cayley tree under the non-zero external field. We showed that under some conditions on parameters the model exhibits a phase transition on the general order Cayley tree. On the binary tree, we solved the model exactly under the specific external field, i.e., we find all regions where the phase transition occurs. Moreover, we checked the extremality and non-extremality of one of obtained translation-invariant Gibbs measures on the binary tree.

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